Math 132 - Week 2
Textbook sections: 2.3-2.5, 7.3
Topics covered:

- Complex analytic functions
- Real and imaginary parts of analytic functions
- Harmonic functions
- Harmonic conjugates
- Complex maps
- Translation, Rotation, Dilation, Inversion
- Mobius transforms

Differentiability vs analyticity

- Last week, we defined what it meant for a complex function $w = f(z)$ to be complex differentiable at a point z_0. If the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist and are continuous at z_0, then it turns out that $f(z)$ is complex differentiable at z_0 if and only if the Cauchy-Riemann equations

$$\frac{\partial f}{\partial x}(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}(z_0)$$

  hold.

- In many cases the Cauchy-Riemann equations only hold for a handful of points z_0, so that the function f is mostly non-differentiable. For instance, consider the function $f(x + iy) = x^2 + y^2$. The Cauchy-Riemann equations read

$$2x = \frac{1}{i}2y,$$

  which is only satisfied when $x = y = 0$. Thus this function is only differentiable at the origin.

- A derivative $f'(z)$ is pretty useless when it is only defined on one or two points. In order to use even the most basic results in calculus - Fundamental Theorem of Calculus, Mean Value Theorem, Taylor expansion, etc. - one needs a derivative to be defined on a larger set.

- Definition: A function $f(z)$ is *complex analytic* (or just *analytic*) at z_0 if there exists a disk $\{z : |z - z_0| < r\}$ such that f is complex differentiable at every point on the disk. ("disk" is just another word for "ball").

- In other words, we define a function to be analytic at z_0 if it is differentiable not only at z_0, but also at all points near z_0.

- Thus analyticity is a stronger condition than differentiability, and it turns out to be more useful (you can say more things about analytic functions than about merely differentiable functions).

- The adjectives "regular" or "holomorphic" are sometimes used instead of "analytic".

- If a function is analytic on the entire complex plane, it is said to be *entire*. Entire functions are the very best class of complex functions.

Examples

- $f(x+iy) = x^2 + y^2$ is differentiable at the origin, but not analytic at the origin because it is not differentiable on any ball around the origin.

- Let $f(x+iy) = x^2 + iy^2$. The Cauchy-Riemann equations are
$$2x = \frac{1}{i}(2iy).$$
Thus f is differentiable on the line $x = y$, but is not analytic anywhere because there is no ball on which f is differentiable.

- Let $f(z) = z^2$, so $f(x+iy) = x^2 - y^2 + 2ixy$. The Cauchy-Riemann equations are
$$2x + 2iy = \frac{1}{i}(-2y + 2ix),$$
which holds for all x and y. So f is differentiable everywhere, and analytic everywhere. In other words, $f(z) = z^2$ is an entire function.

- In general, the set on which a function is analytic is the *interior* of the set on which a function is differentiable.

- In particular, if D is a domain (an open connected set), then f is analytic on D if and only if f is differentiable at every point in D.

A remark on notation

- In real analysis, there is also a notion of a function being analytic, which looks quite different from the definition just given. Namely, a function $y = f(x)$ is said to be *real analytic*, (or just *analytic*) at x_0 if one can write $f(x)$ as a power series

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots$$

  which converges on some interval $\{x : |x - x_0| \leq r\}$.

- Real analytic functions are always differentiable (in fact, they can be differentiated as many times as one wishes), but not every differentiable function is real analytic. They are the best kind of real function.

- Later on in the course, we will show that the notions of complex analytic and real analytic are not as different as they seem. In fact:

- **Theorem:** A complex function $w = f(z)$ is complex analytic at z_0 if and only if it can be expanded as a power series

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

  which converges on some disk $\{z : |z - z_0| < r\}$.

- (We'll define what it means for a power series to converge later in the course).

Real and imaginary parts of analytic functions

- Let D be a domain, and let f be an analytic function on D. We can break f into real and imaginary parts:

$$f(x+iy) = u(x+iy) + iv(x+iy)$$

- In order for f to be analytic, the real part u and imaginary part v have to be related by the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- These equations have an interesting consequence: if you only know the real part u of an analytic function f, you can deduce the imaginary part v (up to a constant) by integrating the Cauchy-Riemann equations.

- Example: suppose we have a function $f(z)$ which is entire (i.e. analytic on $\mathbf{C}$) and we know the real part is $u(x+iy) = xy$. Let's work out what the imaginary part is.

- From the Cauchy-Riemann equations we have

$$\frac{\partial v}{\partial y} = y, \quad \frac{\partial v}{\partial x} = -x.$$

- Let's look at the first equation. If we integrate it in y we get

$$v(x+iy) = \frac{1}{2}y^2 + c(x)$$

where $c(x)$ is some unknown function which can depend on x but not on y. If we then substitute this back into the other equation we get

$$c'(x) = -x$$

so

$$c(x) = -\frac{1}{2}x^2 + C$$

where C is a constant that doesn't depend on either x or y. Thus we have

$$v(x+iy) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + C$$

so

$$f(x+iy) = xy + i(\frac{1}{2}y^2 - \frac{1}{2}x^2) + iC,$$

which can be written in complex form as

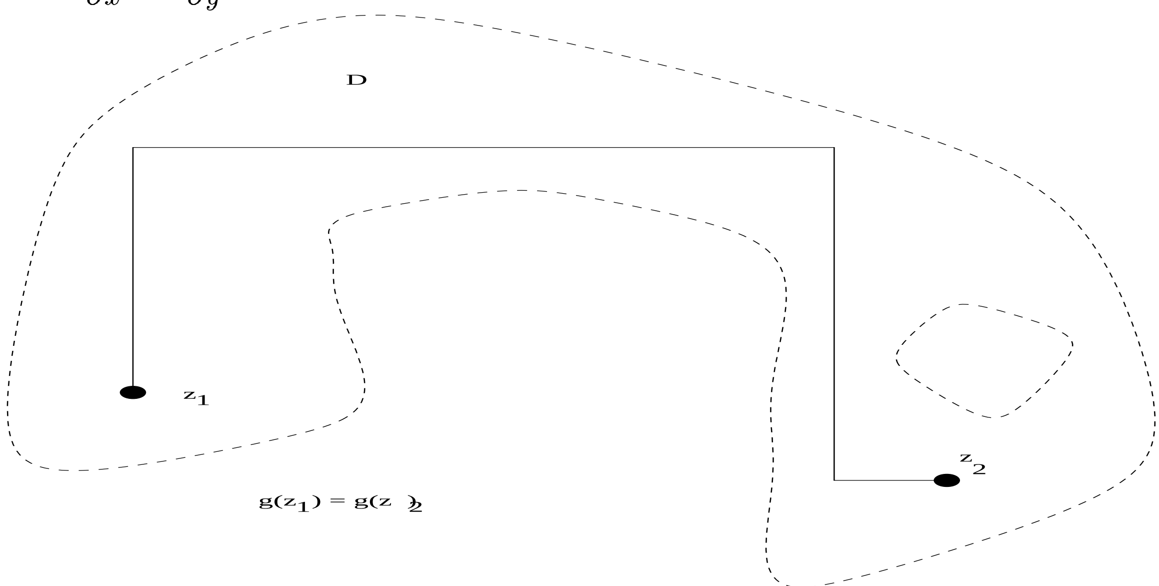
$$f(z) = -\frac{iz^2}{2} + iC,$$

which makes the analyticity obvious.

- In general, given u, one can find v up to a constant:

- **Theorem** If $f_1 = u + iv_1$ and $f_2 = u + iv_2$ are two analytic functions on the same domain D which have the same real part u, then $v_1 = v_2 + C$ for some constant C.

- **Lemma** If g is a function on a domain D such that $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$ on D, then g is a constant.



- **Proof of lemma** Since $\frac{\partial g}{\partial x} = 0$, g is constant on every horizontal line segment in D (by the fundamental theorem of calculus). Since $\frac{\partial g}{\partial y} = 0$, g is constant on every vertical line segment in D. Since every two points in D can be connected by vertical and horizontal line segments, g is constant.

- **Proof of theorem** From the Cauchy-Riemann equations we have

$$\frac{\partial u}{\partial x} = \frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v_1}{\partial x} = -\frac{\partial v_2}{\partial x}$$

so

$$\frac{\partial (v_1 - v_2)}{\partial y} = 0 = \frac{\partial (v_1 - v_2)}{\partial x}.$$

By the lemma, $v_1 - v_2 = C$. QED

- If $u + iv$ is an analytic function, then v is called a harmonic conjugate of u, and vice versa. The above theorem then says that every function u has only one harmonic conjugate v (up to a constant).

Harmonic functions

- Not every function has a harmonic conjugate. For instance, consider the function $u(x+iy) = x^2$. If we had a function $v(x+iy)$ such that $u+iv$ was analytic, then we would have
$$2x = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
and
$$0 = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
From the first equation we get $v(x+iy) = 2xy+c(x)$. But if we insert this equation back into the second one we get $0 = -2y - c'(x)$. We can't solve this for $c(x)$, because $2y$ is a function of y rather than x. So x^2 cannot be the real part of an analytic function.

- To see what's going on, take the Cauchy-Riemann equations
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
and differentiate the first one with respect to x and the second one with respect to y:
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.$$

Adding the two equations together, the v's cancel, and we get

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0.$$

This equation is known as *Laplace's equation*. Functions which satisfy this equation are called *harmonic*.

- What we've just shown is that in order for u to be the real part of an analytic function, it must be harmonic. If u is not harmonic, it cannot possibly be the real part of an analytic function.

- Example: the function $u(x + iy) = x^2$ is not harmonic for any x, y, and so cannot be the real part of an analytic function.

- Conversely, if u is harmonic in a domain D, does this mean that u is the real part of an analytic function? Usually the answer is yes. For instance, this is always true when D is the entire complex plane. (When D has some "holes" there are some cases of harmonic functions which are not the real part of any analytic function, but we won't dwell on this topic here).

- Everything we said above also applies to the imaginary part v. Thus both the real and imaginary parts of an analytic function are harmonic.

Physical interpretation of harmonic functions

- A function $u(x, y)$ of two variables is harmonic if it obeys Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- Similarly, a function $u(x, y, z)$ of three variables is harmonic if we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

- These functions occur often in applications, especially in physics. Generally speaking, any steady-state field will be harmonic in the absence of external influences. Steady-state temperature fields, electrostatic fields, and gravitational fields fall into this category.

- As a first approximation, the surface of soap films (or any elastic surface) is the graph of a harmonic function.

- The study of harmonic functions is a field in itself, and is beyond the scope of this course.

Singularities

- If z_0 is not an analytic point of $w = f(z)$, but it is the limit of a sequence of analytic points, then it is said to be a *singular point* of $w = f(z)$, and f is said to have a *singularity* at z_0.

- For instance, if $f(z) = 1/z$, then f is analytic everywhere except at zero (because one can differentiate f everywhere except at zero). So 0 is not an analytic point of $f(z)$, but every nearby point is. So $1/z$ has a singularity at 0.

- The function $f(x+iy) = x^2+y^2$ is not analytic anywhere. So it has no singularities (i.e. points which border on the region of analyticity, but aren't actually contained inside it).

- We'll study singularities in great detail much later in the course. For now, we'll just be satisfied with the definition of a singularity.

Complex maps

- Let's now start looking at some specific complex functions (aka complex maps) $w = f(z)$.
- As mentioned before, complex functions are difficult to graph directly. A more mundane way to depict such these functions to display the z plane and w plane side-by-side, and describe how points on the z-plane map to points in the w-plane.
- We'll illustrate this with some very basic maps:

  Translations: $w = z + c$;

  Dilations/rotations: $w = kz$

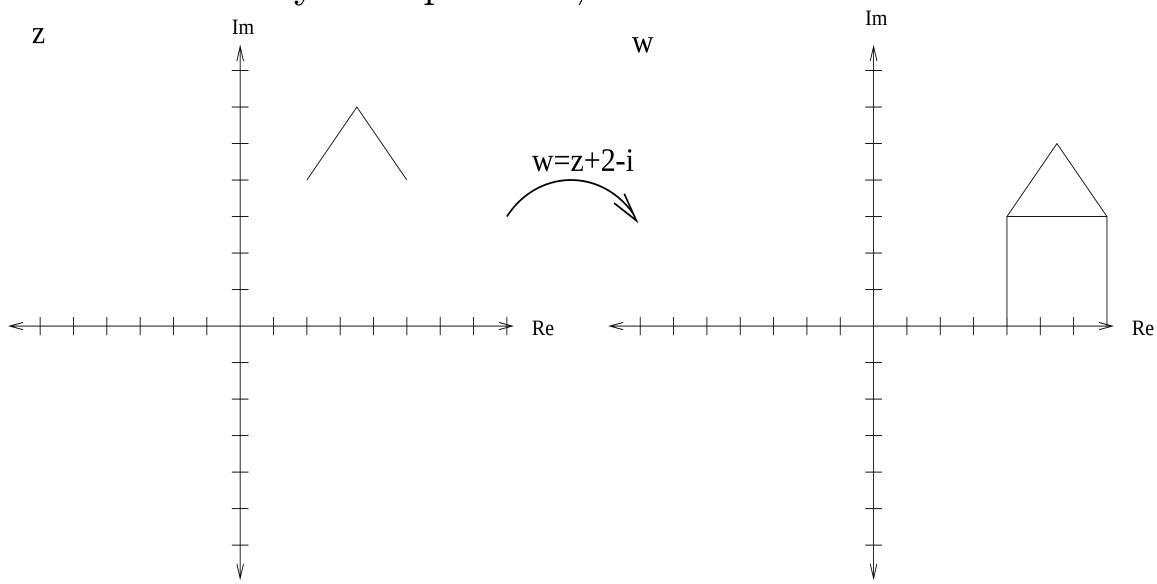
  Rigid motions: $w = kz + c$

  Inversion: $w = 1/z$

  Here k and c are complex numbers. Then we'll look at a wider class of maps which contain these four examples, namely the *Möbius transforms* $w = (az+b)/(cz+d)$.

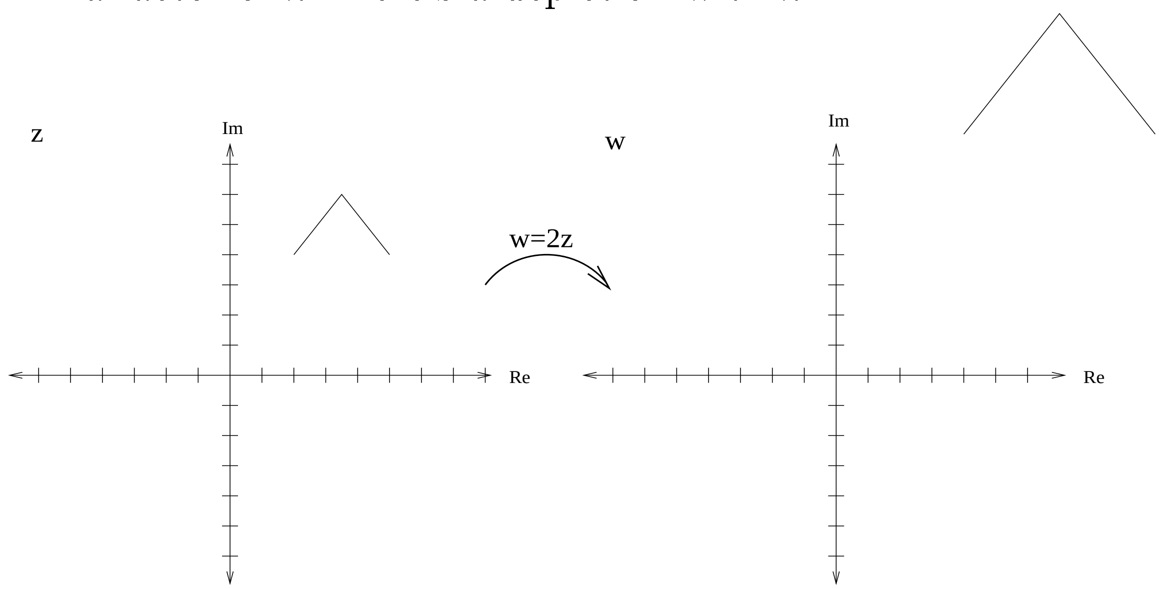
Translations

- Let $c = a+bi$ be a complex number, and consider the map $w = z + c$. This map adds a to the real part of z and adds b to the imaginary part. Thus, this map is a translation to the right by a and upwards by b. Here's a way to depict this, with $c = 2 - i$:



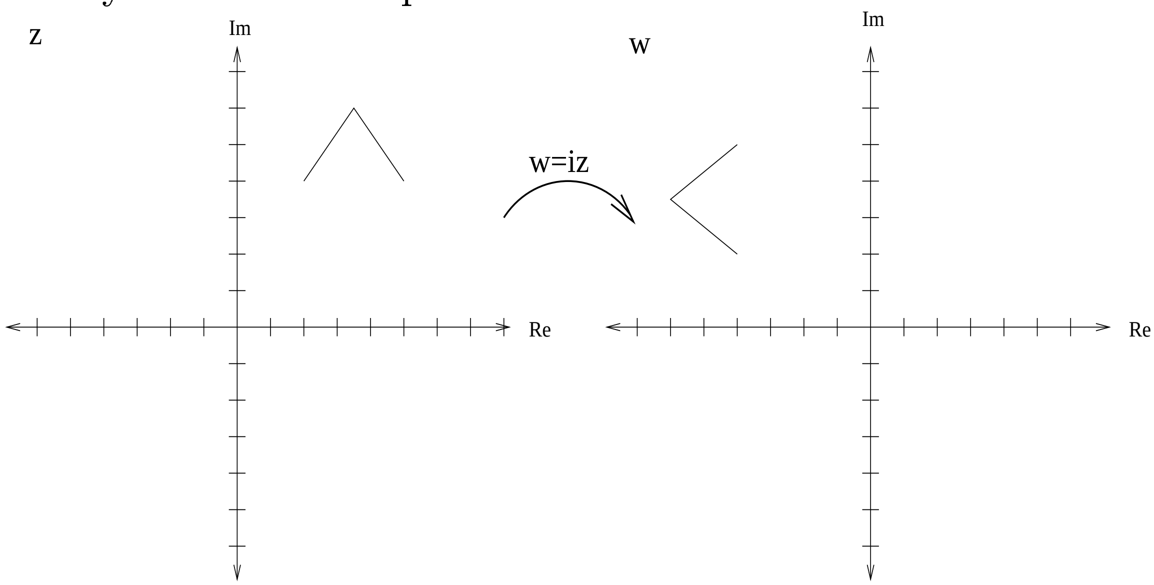
Dilations

- Now consider the map $w = kz$, where k is a positive real number. This multiplies both the real and imaginary parts of z by k, and is called a dilation by a factor of k. Here's a depiction with $k = 2$:



Rotations

- Now consider the map $w = kz$, where $k = e^{i\alpha}$. This is best understood in polar co-ordinates. If $z = re^{i\theta}$, then $w = kz = re^{i(\theta+\alpha)}$. Thus this map does not change the magnitude of z, but adds α to the phase. In other words, this is a counter-clockwise rotation by α. Here's a depiction with $k = e^{i\pi/2} = i$.



- If $k = -i$, we would rotate *clockwise* by $\pi/2$. If $k = -1$, then we would rotate by π. (This gives a nice interpretation of the relationship $i^2 = -1$).

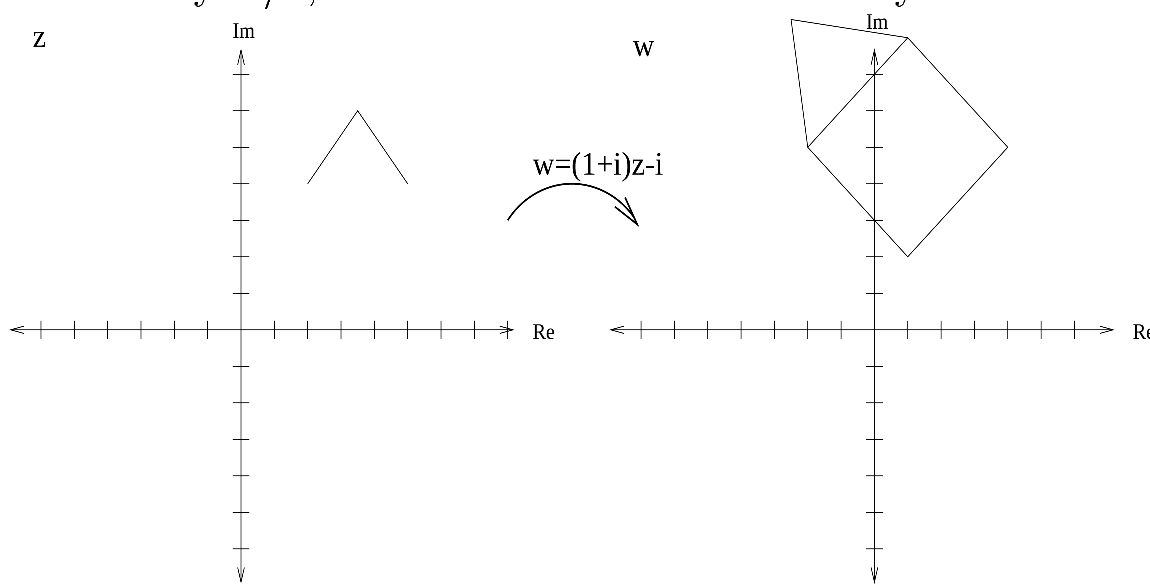
- For more general k, the map $w = kz$ is a combination of dilation and rotation. E.g. the map $w = 2iz$ dilates by 2 and then rotates counterclockwise by

$\pi/2$.

Rigid motion

- A map of the form $w = kz + c$ is called a *similarity*, or *affine transformation*. (It's also called a linear transformation, but this is slightly inaccurate). Clearly it is composed of a dilation, rotation, and translation.

- For instance, the map $w = (1+i)z - i = \sqrt{2}e^{i\pi/4}z - i$ consists of a dilation by $\sqrt{2}$, a counterclockwise rotation by $\pi/4$, and then a downward shift by i:



Inversion

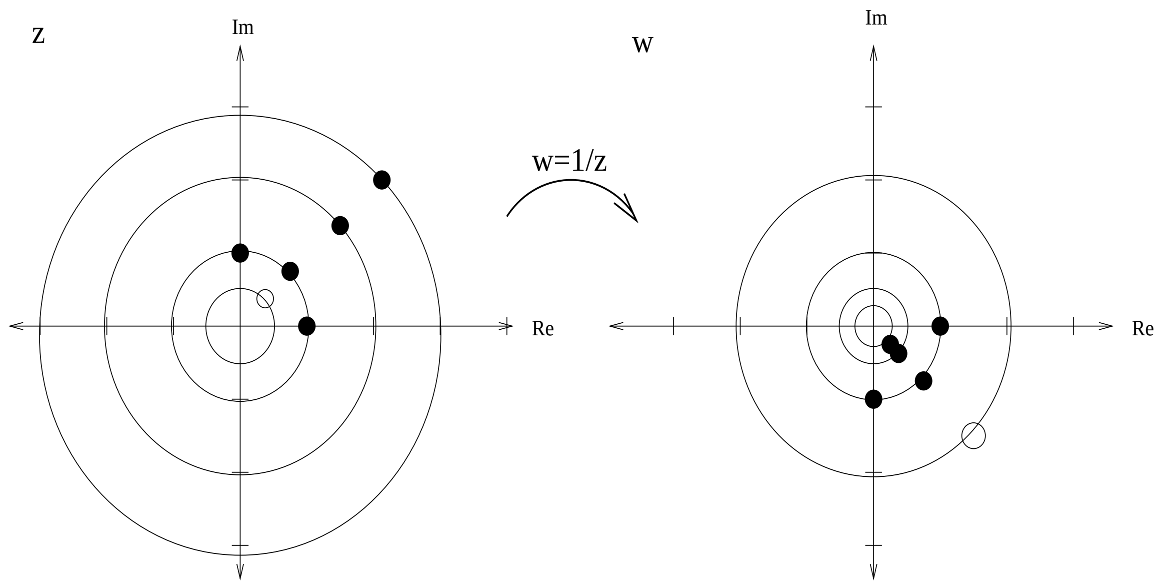
- Now let's look at a more interesting map, the inversion map $w = 1/z$.
- In Cartesian co-ordinates, the map looks intimidating: if $z = x + iy$, then

  $$w = 1/(x+iy) = (x-iy)/(x^2+y^2) = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i.$$

- Let's look at polar instead. If $z = re^{i\theta}$, then

  $$w = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}.$$

  In other words, the magnitude of w is the reciprocal of the magnitude of z, while the argument of w is the negative of that of z. (Inversion turns the complex plane inside-out and upside-down).
- Qualitatively: if you move z closer to the origin, then w moves further away, and vice versa; if you move z clockwise around the origin, then w moves anti-clockwise, and vice versa.



- In particular if you take a line through the origin in the z-plane and invert it, you get a line through the origin in the w plane which is the reflection of the original line through the real axis. (But with the origin removed).

- Let's try to invert some simple geometric objects, starting with the line $\{z : \mathrm{Re}(z) = 1\}$. What is the image of this line under the map $w = 1/z$?

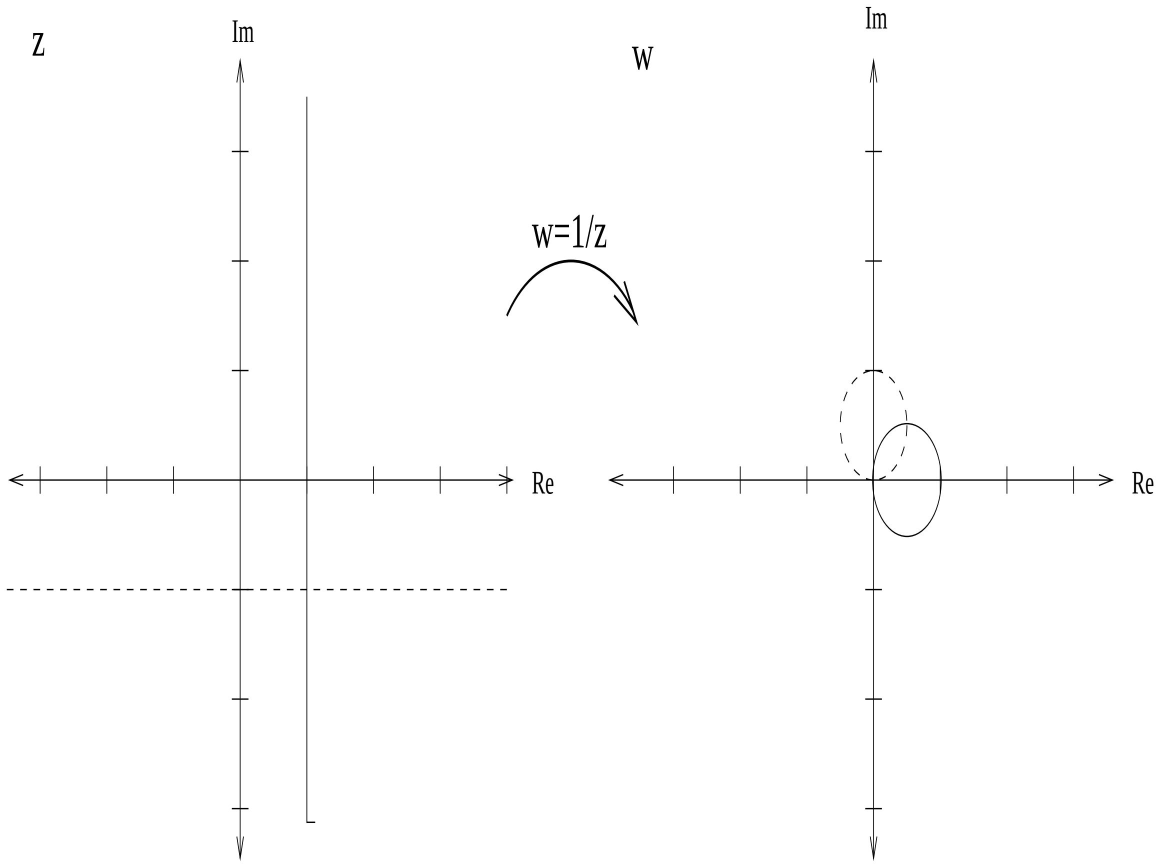
- In other words, we ask what are all the w such that $w = 1/z$ and $\mathrm{Re}(z) = 1$.

- Saying $w = 1/z$ is the same as saying $z = 1/w$, so we're looking for those w such that $\mathrm{Re}(1/w) = 1$.

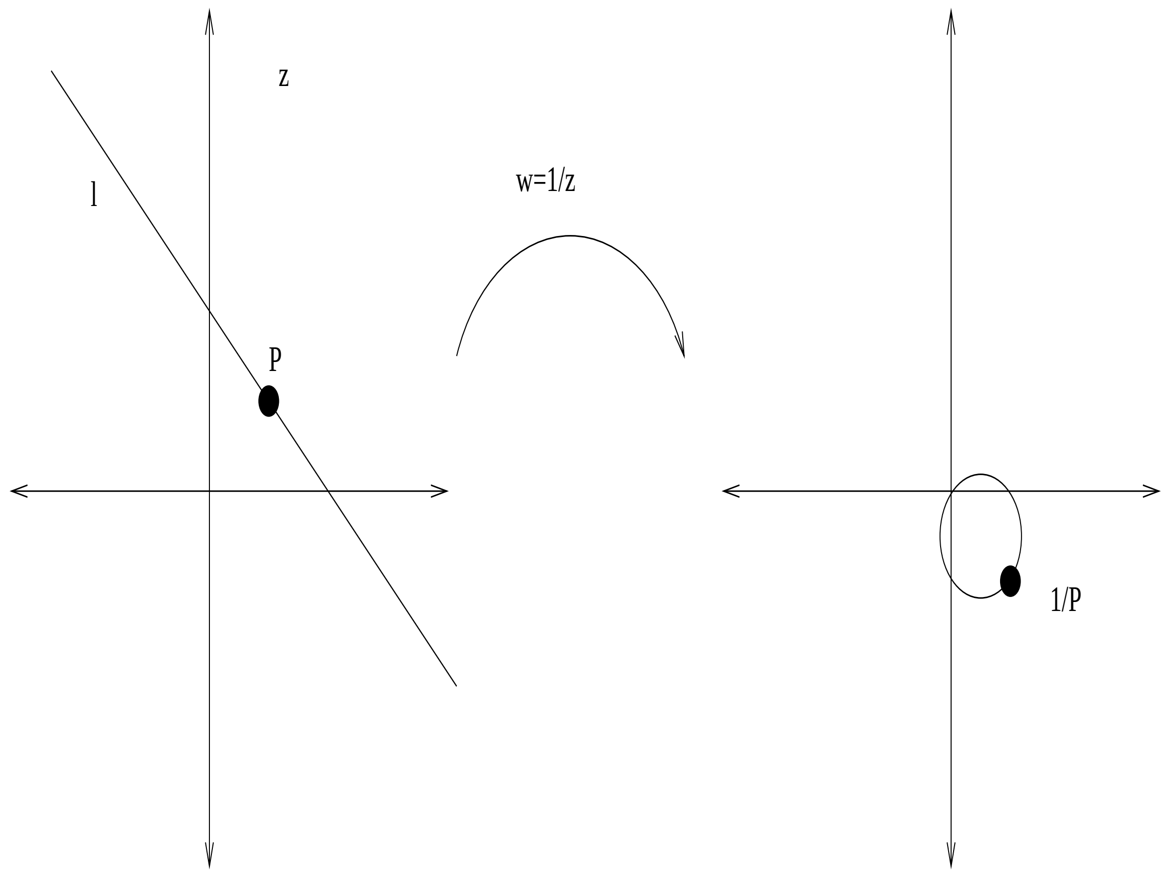
- Write $w = x + yi$. Then $1/w = x/(x^2 + y^2) + yi/(x^2 + y^2)$, so we're looking for those w such that $x/(x^2 + y^2) = 1$.

- This simplifies to $x^2 - x + y^2 = 0$ (unless $x = y = 0$). We complete the square by adding $1/4$ to each side, obtaining
$$(x - 1/2)^2 + y^2 = 1/4.$$
This is the equation for the circle of radius $1/2$ and center $1/2$. Thus the inverse of the line is a circle (with the origin removed).

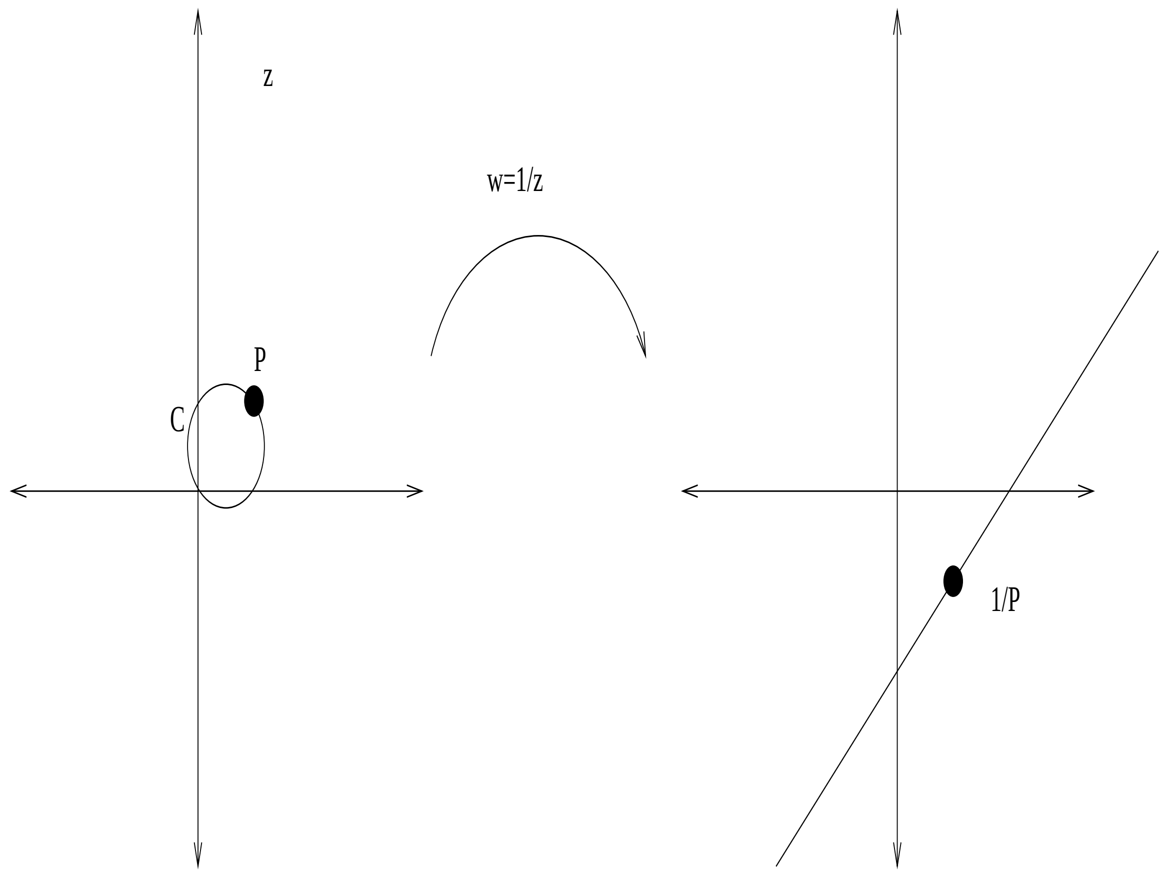
- One can do a similar procedure for other simple objects. It turns out that whenever you invert a line or circle, you always get another line or circle (but the origin is always removed, since 0 is not the reciprocal of anything).



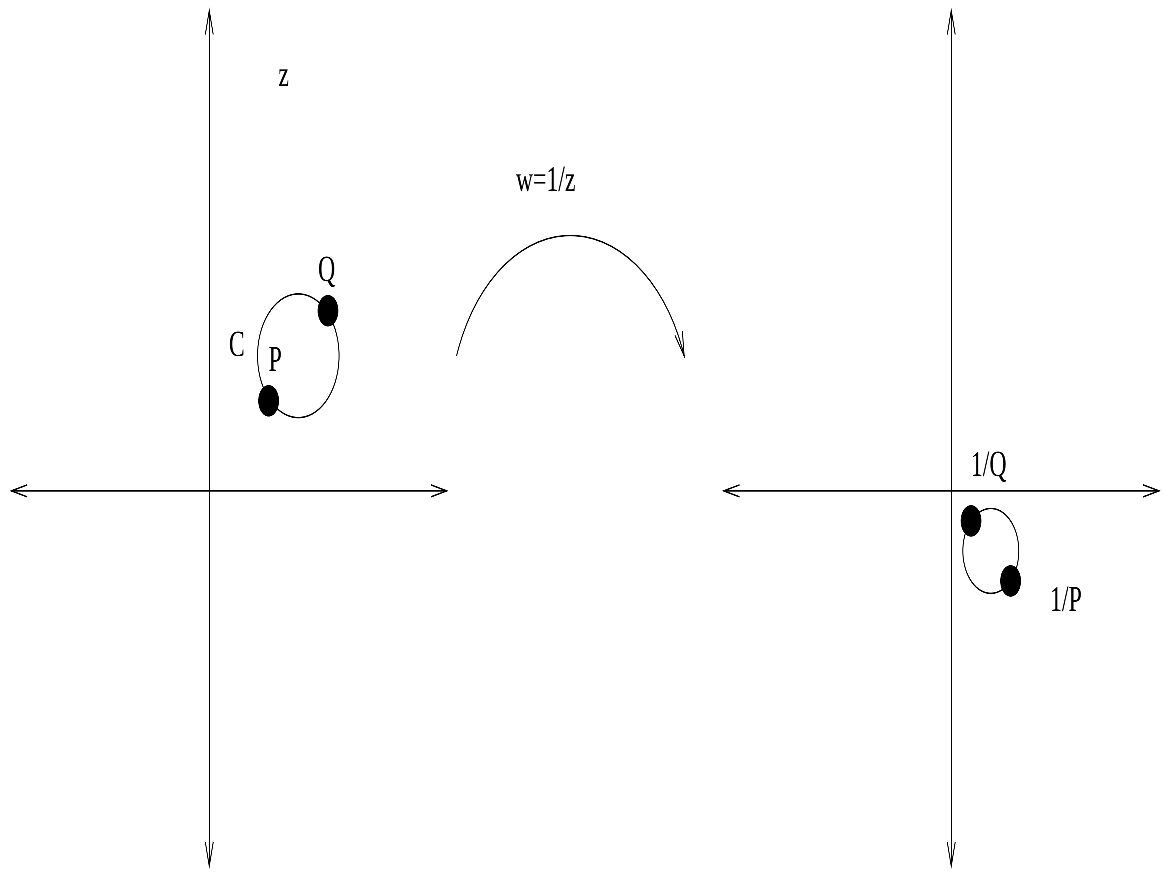
- For instance, if l is a line not passing through the origin, whose closest approach to the origin is at a point P, then the inverse of l will be a circle through the origin, whose *furthest* point from the origin is $1/P$.



- Conversely, if C is a circle through the origin with furthest point from the origin P, then the inverse of C is a line not passing through the origin with nearest point $1/P$.



- Finally, if C is a circle not passing through the origin, but with nearest point to the origin P and furthest point to the origin Q, then the inverse of C is another circle not passing through the origin with furthest point $1/P$ and nearest point $1/Q$.



- There is a Java applet on the class web page at
`www.math.ucla.edu/~tao/java/Mobius.html`
which displays these operations graphically.

- Translations, dilations, rotations, and inversions are all special cases of *Möbius transformations* (due to August Möbius, 1790-1868, most famous for his strip). A Möbius transformation is defined as any transformation of the form

$$w = \frac{az + b}{cz + d}$$

where a, b, c, d are complex numbers such that $ad - bc \neq 0$. This last condition is intended to avoid silly maps such as

$$w = \frac{3z + 6}{z + 2} = 3$$

which send everything to a constant. Note that every map described above is a Möbius transform; for instance $w = z + 2 - i$ is

$$w = \frac{z + 2 - i}{0z + 1}.$$

We'll study Möbius transforms in more detail next week.