Math 132 - Week 3
Textbook sections: 7.4, 3.1, 3.2
Topics covered:

- Mobius transforms
- Exponential functions
- Trig and hyperbolic functions
- The complex logarithm

Möbius transforms

- A Möbius transform is any mapping of the form

$$w = \frac{az+b}{cz+d}$$

where a, b, c, d are complex numbers such that $ad - bc \neq 0$. Examples:

$$w = \frac{z+i}{z-i}$$
$$w = \frac{iz+3}{2}$$
$$w = \frac{z+i}{2z+1}$$

- There are some other names for these transformations, such as fractional linear transformations or bilinear transformations, but we won't use that terminology in this course.

- This transform is defined everywhere except when $z = -d/c$. When $z = -d/c$ we say that $w = \infty$. Conversely, we say that $w = a/c$ when $z = \infty$, e.g. the transform $\frac{z+i}{2z+i}$ takes the value of $1/2$ when $z = \infty$. The point $-d/c$ is called the *singularity* or *pole* of the Möbius transform. For instance, $\frac{z+i}{z-i}$ has its pole at $z = i$.

- These transforms are useful for converting bounded regions into unbounded regions and vice versa; for instance, you can turn a half-plane into a disk with these transformations. We'll see some examples of this later.

- If $c = 0$, then Möbius transforms are just glide transforms of the type discussed last week. If $c \neq 0$, a Möbius transform can be analyzed by rewriting it as a vulgar fraction:

$$w = \frac{z+i}{z-i} = 1 + \frac{2i}{z-i}$$

$$w = \frac{z+i}{2z+1} = \frac{1}{2} + \frac{i - \frac{1}{2}}{2z+1}$$

$$w = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d}$$

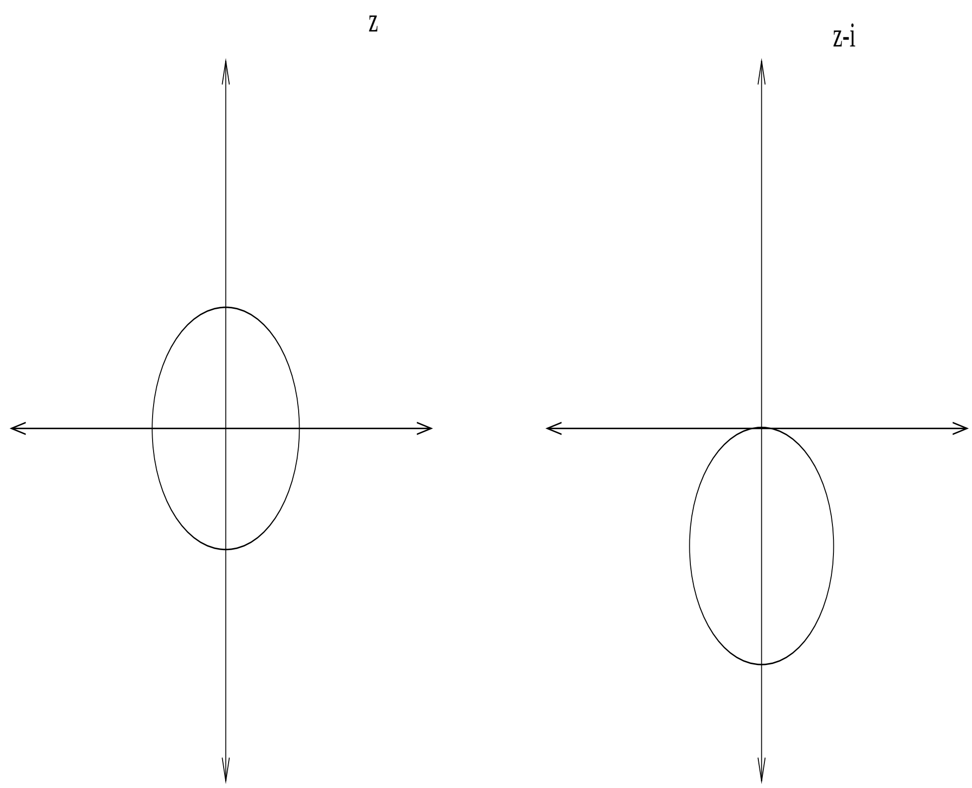
The advantage of writing a Möbius transform this way is that it can be broken up into simpler components such as translations, inversions, and so on. For instance, the transform $w = \frac{z+i}{z-i}$ can now be broken up into a translation, followed by an inversion, followed by a dilation and rotation, followed by a translation:

$$z \mapsto z - i \mapsto \frac{1}{z-i} \mapsto \frac{2i}{z-i} \mapsto 1 + \frac{2i}{z-i} = w.$$

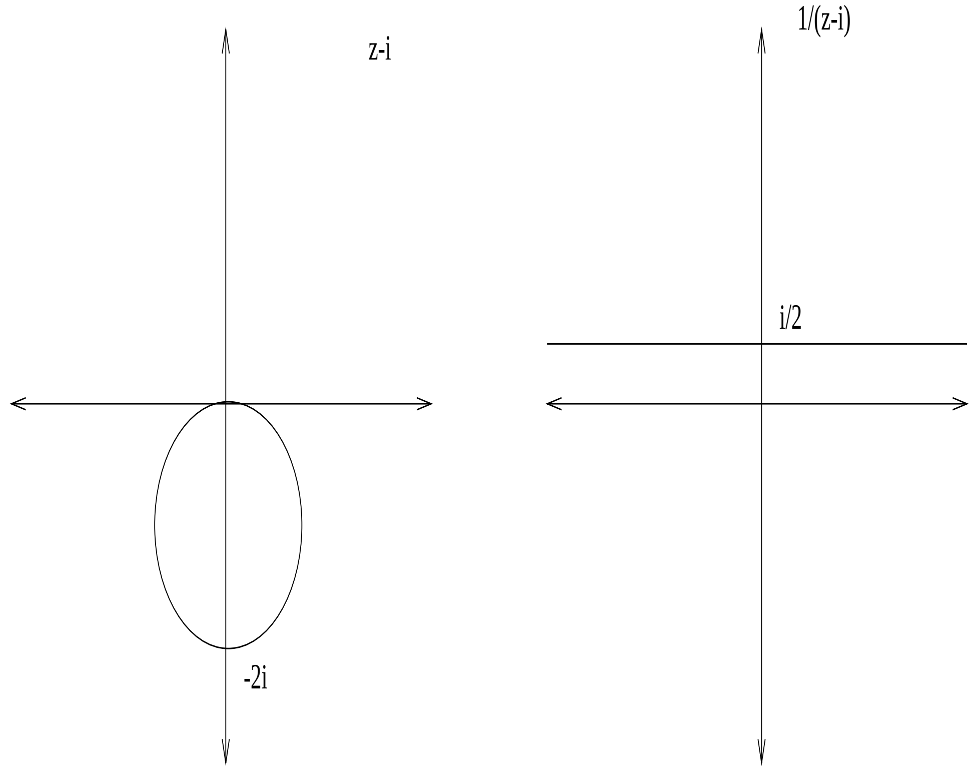
Example

- For instance, suppose we want to find out what happens to the unit circle $\{z : |z| = 1\}$ via the transform $w = \frac{z+i}{z-i}$. To do this directly is somewhat difficult because the transform is so complicated. However, if we break it up into components as on the previous page, then it becomes much simpler to handle.

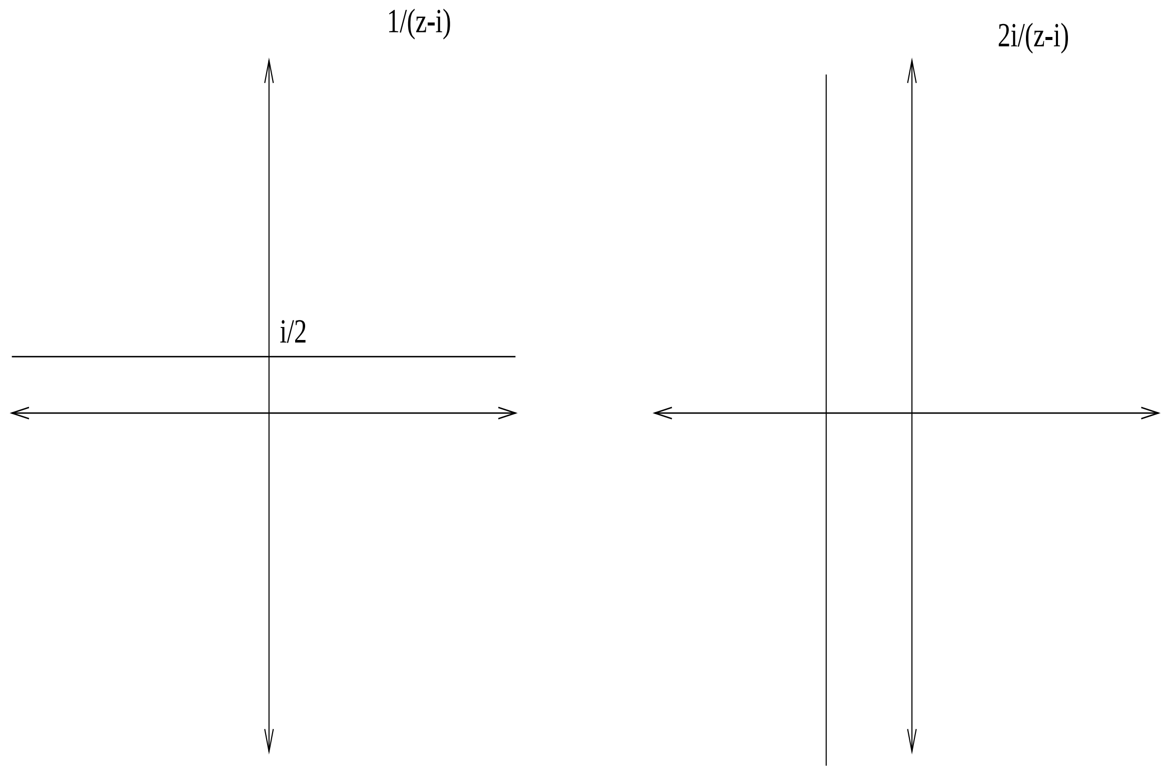
- We begin with $z - i$. Subtracting an i lowers everything by one unit. If z is on the unit circle centered at the origin, then $z - i$ is on the circle of radius 1 centered at $-i$.



- Now we do $1/(z-i)$. The circle of radius 1 centered at $-i$ is a circle through the origin with furthest point $-2i$. So the inverse of this circle is a straight line with closest approach to the origin at $1/(-2i) = i/2$. Thus $1/(z-i)$ lives on the horizontal line going through $i/2$.



- Now we do $2i/(z-i)$. Multiplying by $2i$ increases the magnitude by a factor of 2, and rotates the phase by $+\pi/2$. Thus, $2i/(z-i)$ lives on the vertical line going through -1.



- Finally, we do $w = 1 + 2i/(z - i)$. Adding by 1 shifts everything to the right, hence w lives on the vertical line going through 0, i.e. the imaginary axis.
- To summarize, we have shown that the transform $z \mapsto \frac{z+i}{z-i}$ maps the unit circle $\{z : |z| = 1\}$ to the imaginary axis $\{w : \mathrm{Re}(w) = 0\}$.
- The same procedure - breaking up a complicated transform into simpler pieces - can be used to find the transform of any circle or line under any Mobius transform.

- Once one can do circles and lines, it is not hard to work out more complicated objects like arcs, line segments, disks, etc. For instance, suppose we want to know how the semi-circle

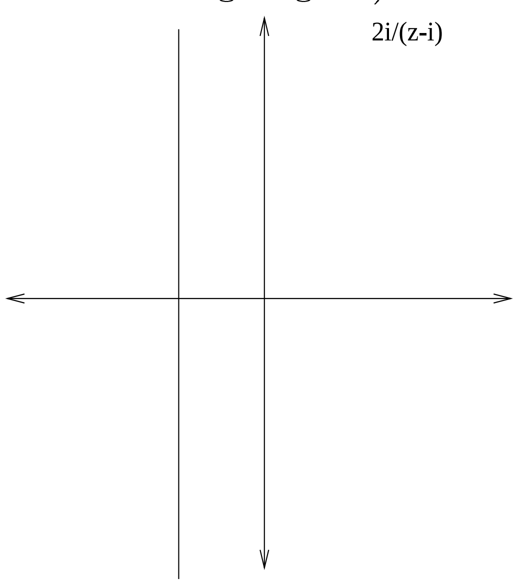
$$\{z : |z| = 1, \mathrm{Im}(z) \geq 0\}$$

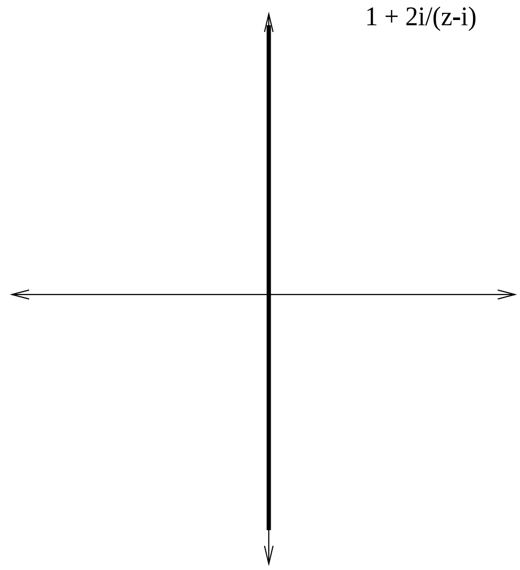
transforms under the Möbius transformation $w = \frac{z+i}{z-i}$. This semi-circle is part of the unit circle $\{z : |z| = 1\}$, so we know that the image of the semi-circile will be some subset of the line $\{w : \mathrm{Re}(w) = 0\}$ that we've just computed.

- Now let's look at the endpoints of the semi-circle, namely +1 and -1. When $z = +1$, $w = \frac{1+i}{1-i} = i$; when $z = -1$, $w = \frac{-1+i}{-1-i} = -i$. So it seems that the semi-circle is mapping to the line segment between i and $-i$.

- However, that isn't quite correct. Let's test some interior values of the semi-circle. The most obvious value to test is $z = i$, but this is the pole of the transform, and $w = \infty$. This already tells us that something odd is going on. Let's look at another value, say $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$. A calculation shows that $w = (1 + \frac{\sqrt{2}}{2})i$. Or, if $z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, then $w = -(1 + \frac{\sqrt{2}}{2})i$.

- What's going on is that as z moves along the upper semi-circle from 1 to -1, w starts off at i and moves *upwards*. When z reaches i, w has moved all the way upwards to ∞, and then comes back on the other side. When z reaches -1, w has moved upwards back to $-i$. (Think about how the graph of $y = 1/x$ behaves as x moves from -1 to 1 to get an idea of what's going on).





- One can also find the image of the upper semi-circle by breaking up $w = (z+i)/(z-i)$ into pieces as we did before. Note that the lower semi-circle $\{z : \text{Im}(z) \leq 0\}$, being the complement of the upper semi-circle, will move along the line segment i to $-i$.

- Now let's look at what happens to the disk $\{z : |z| < 1\}$ under the same transformation $w = \frac{z+i}{z-i}$. Since the boundary of this disk maps to the imaginary axis, the disk will map to either the left half or the right half of the imaginary axis. To find out which half it is, the easiest way is to take a single test point in the disk (e.g. $z = 0$) and see where that goes. When $z = 0$, $w = \frac{0+i}{0-i} = -1$, which is to the left of the imaginary axis. Thus the disk maps to the left half-plane $\{w : \mathrm{Re}(w) < 0\}$. Similarly, the complement $\{z : |z| > 1\}$ of the disk will map to the right half-plane $\{w : \mathrm{Re}(w) > 0\}$.

Some properties of Möbius transforms

- The translations, dilations, rotations, and inversion are all special cases of Möbius transforms.

- Möbius transforms map circles and lines to circles and lines. In other words, the image of a circle under a Möbius transform is always either a circle or a line (and never something like an ellipse, square, etc.). It is sometimes convenient to think of lines as really big circles with an infinite radius, and a center infinitely far away.

- All Möbius transforms are invertible, and the inverse of a Möbius transform is another Möbius transform. For instance, to invert the transform
$$w = \frac{z+i}{z-i}$$
we solve for z:
$$\begin{aligned} w(z-i) &= z+i \\ wz - iw &= z+i \\ wz - z &= iw + i \\ z &= \frac{iw+i}{w-1}. \end{aligned}$$
Thus the inverse transformation is given by $w \mapsto \frac{iw+1}{w-1}$.

- When you compose two Möbius transforms you always get another Möbius transforms. (Combined with the previous property, this means that the set of Möbius transforms form a *group*). For instance, if $f(z) = \frac{z+i}{z-i}$ and $g(z) = 2z$, then

$$f \circ g(z) = f(g(z)) = \frac{2z+i}{2z-i}$$

  and

$$g \circ f(z) = g(f(z)) = 2\frac{z+i}{z-i} = \frac{2z+2i}{z-i}$$

  are also Möbius transforms. Note that $f \circ g$ is, in general, not the same as $g \circ f$.

- Möbius transforms are complex differentiable at every point except at their pole - this is just because of the quotient rule. For instance, the transform $f(z) = \frac{z+i}{z-i}$ is differentiable for all $z \neq i$, and its derivative is given by

$$f'(z) = \frac{(z+i)'(z-i) - (z+i)(z-i)'}{(z-i)^2} = \frac{-2i}{(z-i)^2}.$$

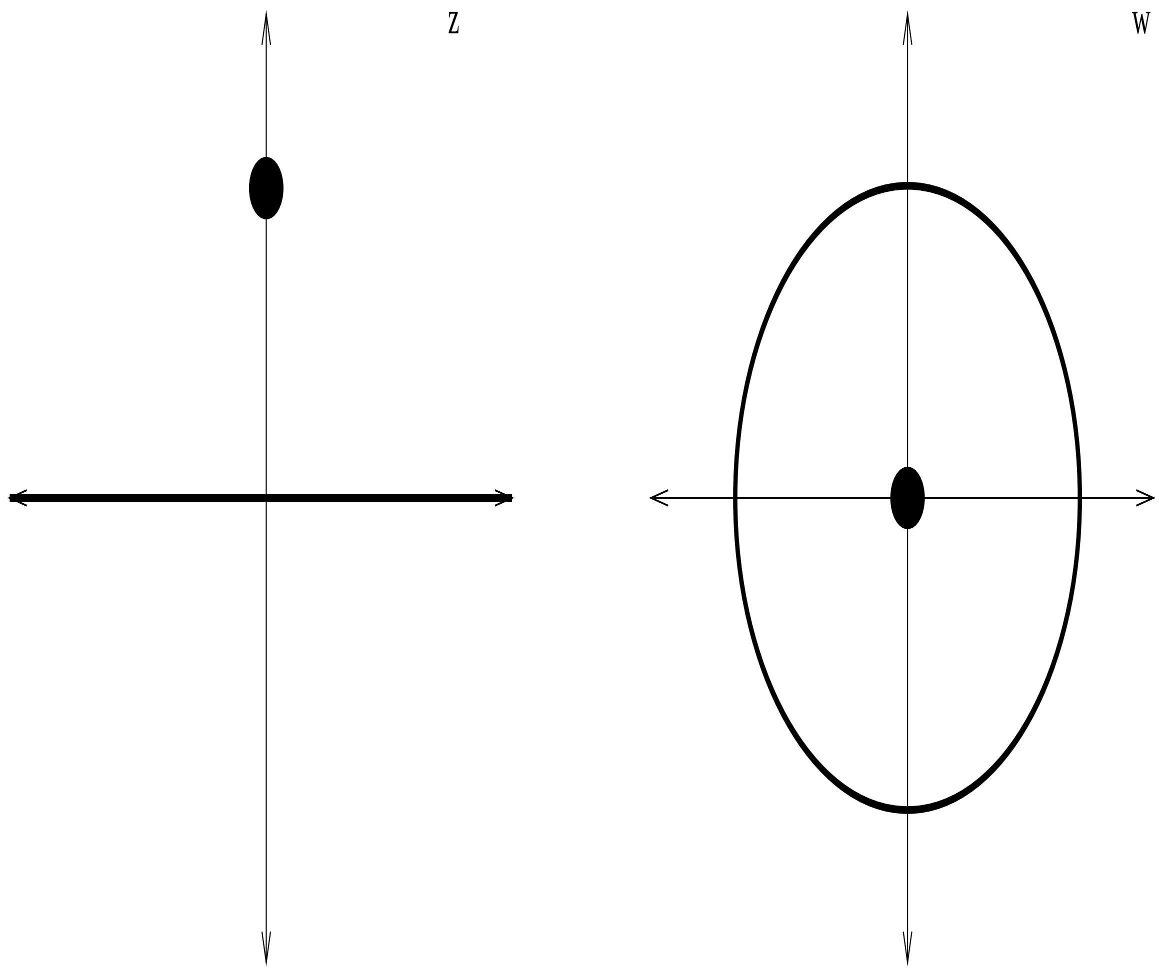
- Möbius transforms preserve orientation: a clock whose hands rotate clockwise will, after a transformation remain a clock whose hands rotate clockwise (although the clock may be seriously distorted by this). This is unlike, say, a reflection, which reverses orientation (so a reflected clock's hands rotate anti-clockwise).

- They also preserve angle. If two curves intersect at an angle θ, when one transforms them using a Möbius transform their images also intersect at an angle of θ. So angles are not distorted, only distances are. (Another way of stating the above two properties are that Möbius transforms are *conformal*).

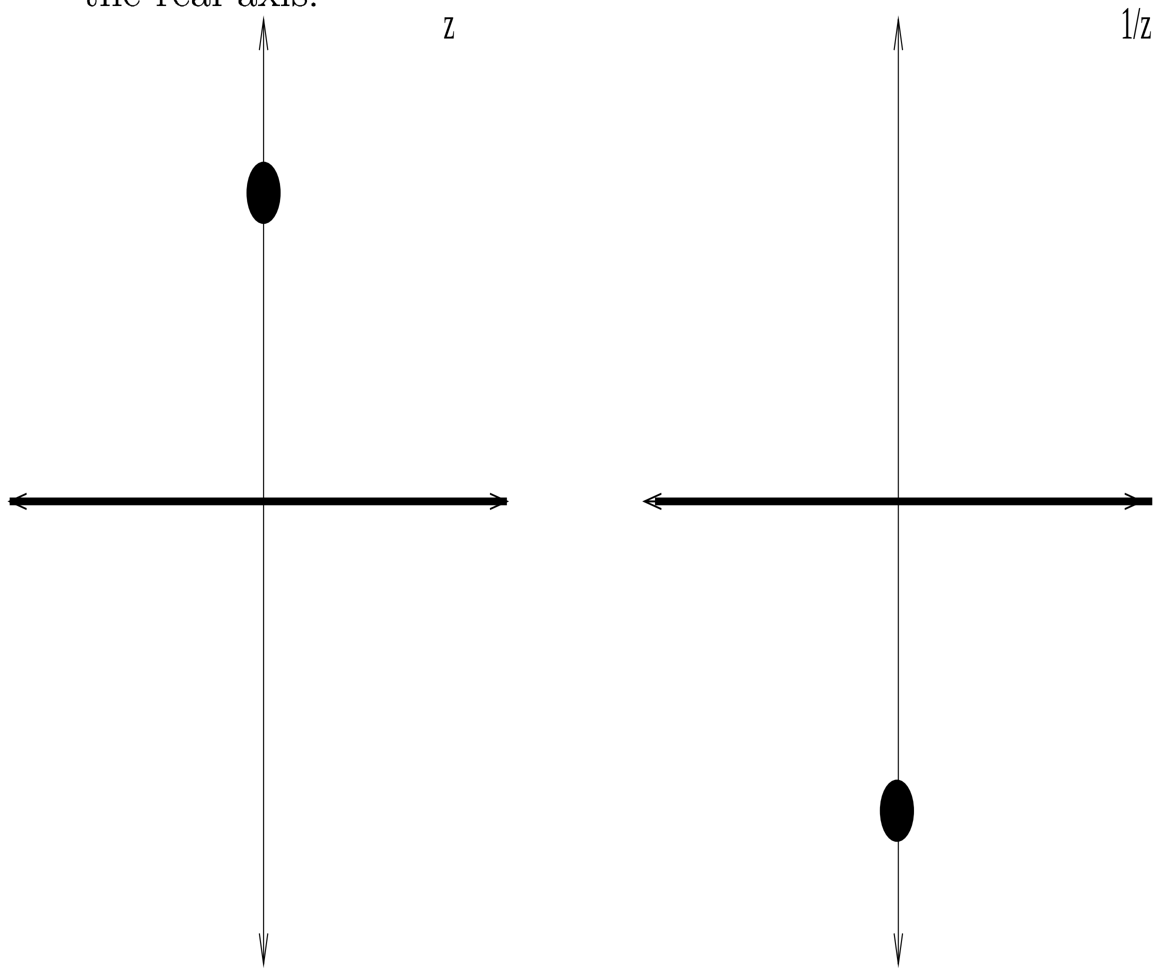
Constructing a Möbius transform with specific properties

- Sometimes one wants to construct a Möbius transform that does a specific task, such as map one object X to another object Y. The easiest way to do this is by a trial-and-error process, with each trial making X look more and more like Y.

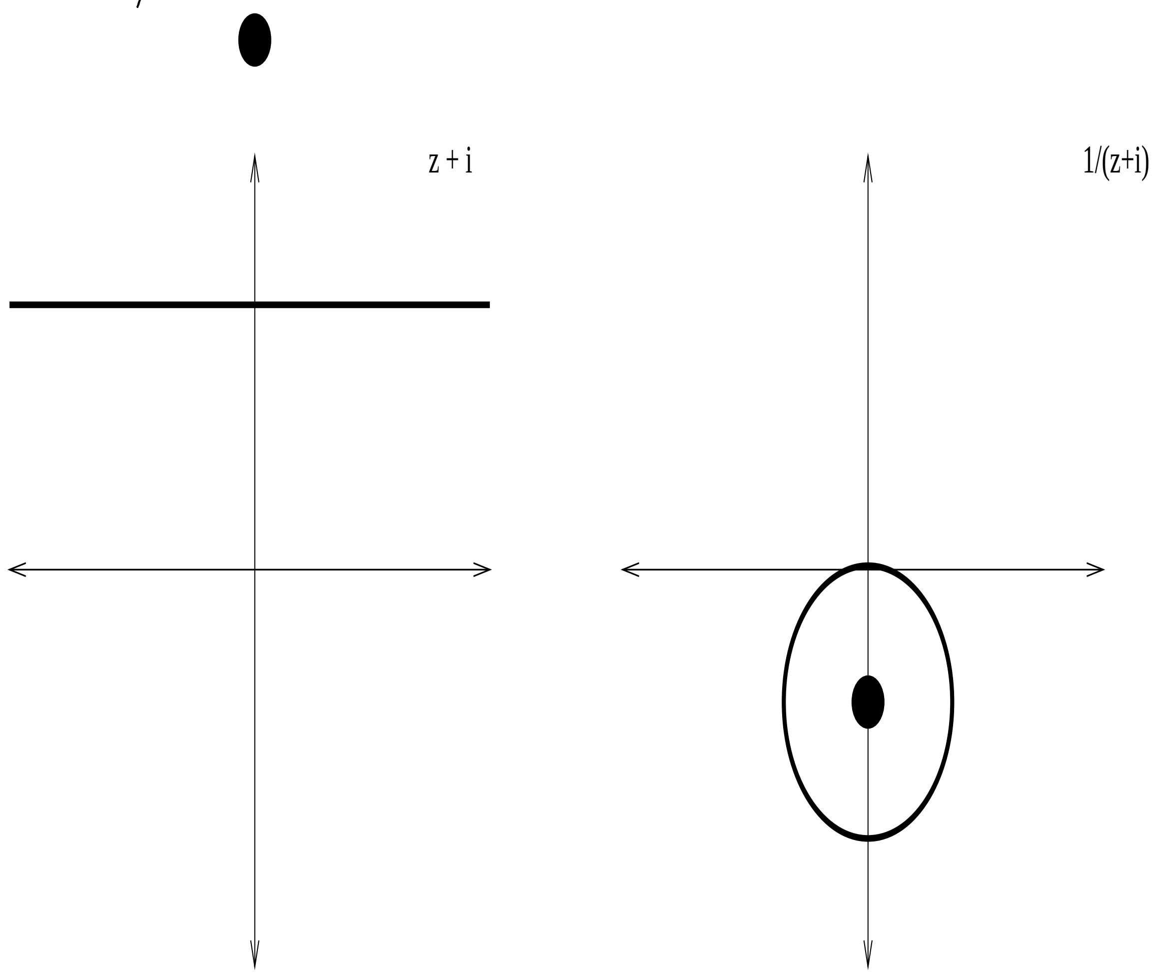
- As an example, suppose we want a transform which maps the upper half-plane $\{z : \mathrm{Im}(z) > 0\}$ to the disk $\{w : |w| < 1\}$, and maps the point i to the point 0:



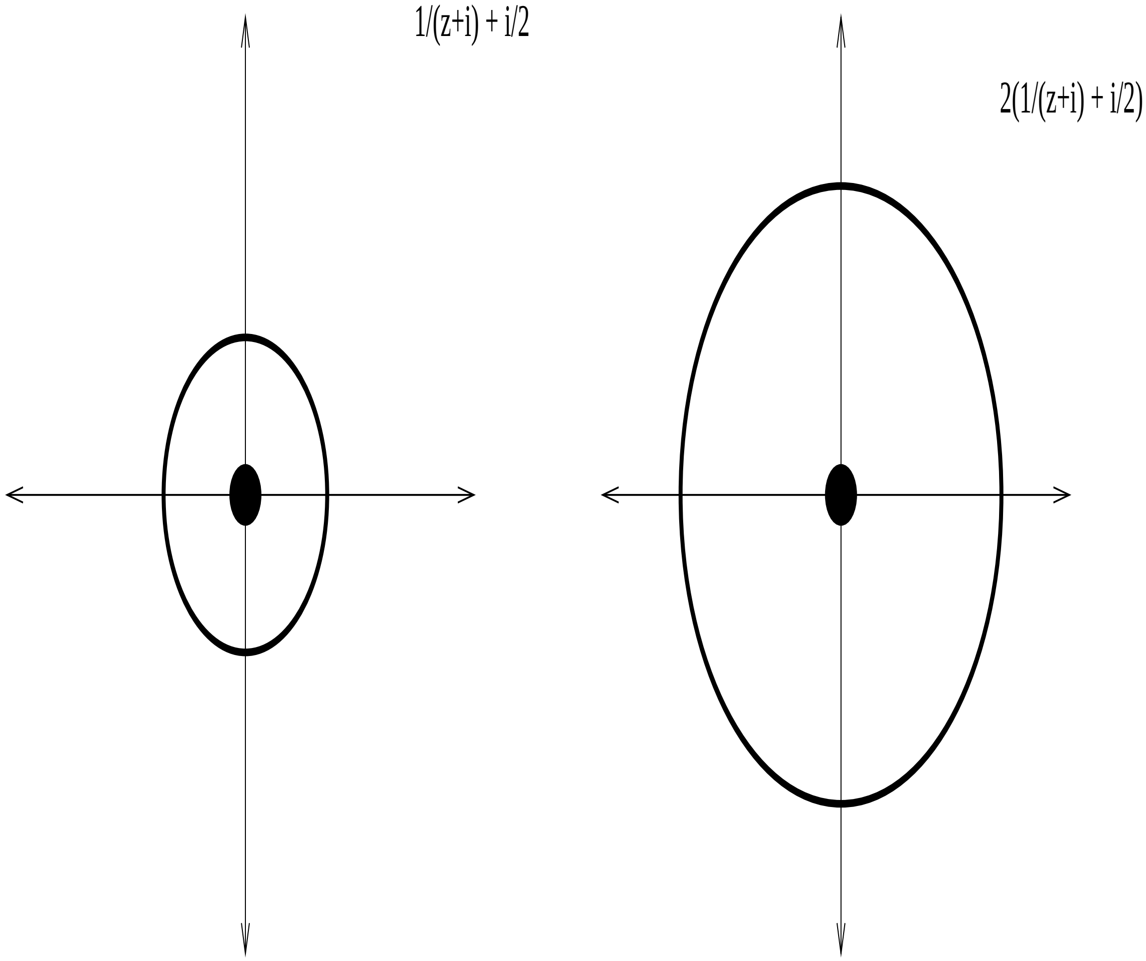
- We first try the elementary Möbius transforms to convert the left image to the right image. Translations, dilations, and rotations do not appear to make the left image look anything like the right image. An inversion doesn't seem to help either, because when one inverts the real line one just gets the real line, and when one inverts i one just gets $-i$. So inversion does nothing apart from flipping the set across the real axis.



- However, if you first do a translation, then an inversion does something interesting. If we shift upwards by i, mapping z to $z+i$, and *then* invert to $1/(z+i)$, the real line maps first to the vertical line through i, and then to the circle through the origin with furthest point $-i$, while i maps first to $2i$ and then to $-i/2$:



- This is almost what we want. To move the point $-i/2$ to the origin we can shift upwards by $i/2$, to $1/(z+i)+i/2$. Then to get the unit circle instead of the half-unit circle we multiply by 2, to $2(1/(z+i)+i/2)$:



- Thus the Möbius transform

$$w = 2(\frac{1}{z+i} + \frac{i}{2}) = \frac{iz+1}{z+i}$$

will map the upper half-plane to the unit circle and map i to 0. (There are a couple other transformations which will also work, such as $w = -\frac{iz+1}{z+i}$. Why does this also work?)

- In the special case that you want to map three points z_1, z_2, z_3 to three other points w_1, w_2, w_3 there is a nice formula for the Möbius transform. Namely, write down the equation

  $$\frac{(z - z_1)(z_2 - z_3)}{(z - z_2)(z_1 - z_3)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_2)(w_1 - w_3)}$$

  and solve for w. For instance, to map $-1, 0, 1$ to $-1, i, 1$ respectively, we solve

  $$\frac{(z + 1)(0 - 1)}{(z - 0)(-1 - 1)} = \frac{(w + 1)(i - 1)}{(w - i)(-1 - 1)}$$

  $$\frac{-z - 1}{-2z} = \frac{(i - 1)w + (i - 1)}{-2(w - i)}$$

  $$(-z - 1)(w - i) = (i - 1)wz + (i - 1)z$$

  $$(-z - 1 - iz + z)w = -zi - i + iz - z$$

  $$w = \frac{-z - i}{-iz - 1}.$$

  (Exercise: why does this trick work?)

Elementary complex functions

- In the next few lectures, we'll discuss some basic complex functions, such as

$$e^z, \cos(z), \sin(z), \tan(z), \cosh(z), \sinh(z),$$

$$\log(z), z^{1/2}, z^i, \ldots$$

- In some ways, these functions are very similar to their real-variable counterparts. For instance, $\sin(z)$ is differentiable and its derivative is $\cos(z)$. But there will be some surprises. For instance, $\sin(z)$ is not restricted to between -1 and 1; it can take the value of 2, or i, or any other complex number!

The complex exponential

- We've already defined the complex exponential e^z or $\exp(z)$ as

$$e^{x+iy} = \exp(x+iy) = e^x \cos y + ie^x \sin y.$$

- We have the usual exponentiation laws:

$$e^{z+w} = e^z e^w, \quad e^{z-w} = e^z / e^w, \quad (e^z)^n = e^{nz}$$

  for all integers n and complex numbers z and w. (Exercise!)

- The function e^z is entire, with derivative

$$\frac{d}{dz} e^z = e^z.$$

- Unlike the real exponential, the complex exponential is *periodic*:

$$e^{z+2\pi i} = e^z.$$

  So if $e^z = e^w$, one cannot conclude that $z = w$. The best one can say is that $z = w + 2k\pi$ for some integer k.

- e^{x+iy} has magnitude e^x and phase y. Thus increasing (decreasing) x causes e^{x+iy} to increase (decrease) in magnitude, whereas increasing (decreasing) y causes e^{x+iy} to rotate anti-clockwise (clockwise).

- Also unlike the real exponential, the complex exponential does not have to be positive. For instance, $e^{\pi i} = -1$.

- In fact, any non-zero number can be written as the exponential of some complex number. For instance, to solve the equation $e^z = 1 + i$, we write z in Cartesian and $1 + i$ in polar:

$$e^x e^{iy} = e^{x+iy} = \sqrt{2} e^{\pi i/4}$$

  so $e^x = \sqrt{2}$ and $y = \pi/4 + 2k\pi$. Thus

$$z = \ln \sqrt{2} + i(\pi/4 + 2k\pi).$$

- However, e^z can never be zero, because the magnitude e^x is never zero.

- Incidentally, the way Euler arrived at his formula

$$e^{iy} = \cos(y) + i \sin(y)$$

  was by considering the Taylor series

$$e^{iy} = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \ldots$$

  and replacing y by iy:

$$e^y = 1 + iy + \frac{-y^2}{2!} + \frac{-iy^3}{3!} + \frac{y^4}{4!} + \ldots$$

Separating into real and imaginary parts he got

$$e^y = (1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots) + i(y - \frac{y^3}{3!} + \ldots)$$

which he then recognized as the series for cosine and sine.

Complex trigonometric functions

- If x is a real number, we can define $\sin(x)$ and $\cos(x)$ geometrically, using right-angled triangles. This doesn't work for complex numbers, because there's no such thing as a complex angle. So how can we define sines and cosines of complex numbers?

- Start with the identities

$$e^{ix} = \cos x + i \sin x, \quad e^{-ix} = \cos x - i \sin x$$

and solve for cosine and sine to get

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Based on these equations, it is now natural to define the complex cosine and sine as

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

- Examples:

$$\cos \pi = \frac{e^{i\pi} + e^{-i\pi}}{2} = \frac{-1 - 1}{2} = -1$$

$$\sin i = \frac{e^{-1} - e^{1}}{2i} = \frac{i}{2}(e - 1/e).$$

- Both these functions are clearly entire, and have derivatives

$$\frac{d}{dz}\cos z = -\sin z, \quad \frac{d}{dz}\sin z = \cos z.$$

- All the trig identities that hold for the real sine and cosine, also hold for the complex sine and cosine. For instance:

$$\cos(-z) = \cos(z)$$
$$\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$$
$$\sin(\pi/2 - z) = \cos(z)$$
$$\cos^2(z) + \sin^2(z) = 1$$

  etc., etc. There's a reason for this which has to do with analytic continuation, which we'll see later in the course.

- Both $\sin(z)$ and $\cos(z)$ are periodic with period 2π.

- Caution: $\sin(z)$ and $\cos(z)$ aren't always real numbers, and don't always lie between -1 and 1. That's only when z is real. For complex z it is not necessarily true that $\mathrm{Re}(e^{iz}) = \cos(z)$, for instance.

- Example: find all solutions to $\cos(z) = 2$. We rewrite this as

$$\frac{e^{iz} + e^{-iz}}{2} = 2$$

$$e^{iz} - 4 + e^{-iz} = 0.$$

Writing $w = e^{iz}$, this becomes

$$w - 4 + 1/w = 0$$

$$w^2 - 4w + 1 = 0$$

$$(w-2)^2 = 3$$

$$w = 2 \pm \sqrt{3}$$

$$e^{iz} = e^{\ln(2\pm\sqrt{3})}$$

$$iz = \ln(2 \pm \sqrt{3}) + 2k\pi i$$

$$z = -i\ln(2 \pm \sqrt{3}) + 2k\pi.$$

- One can also define other complex trig functions by the usual formulae:

$$\tan(z) = \sin(z)/\cos(z), \quad \sec(z) = 1/\cos(z), \text{ etc.}$$

- These functions satisfy the usual identities (e.g. $\frac{d}{dz}\tan(z) = \sec^2(z)$ if $z \neq (k+\frac{1}{2})\pi$).

Hyperbolic trig functions

- You may remember the hyperbolic trig functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

from real analysis. Their complex counterparts are

$$\cosh(z) = \frac{e^z + e^{-z}}{2}, \quad \sinh(z) = \frac{e^z - e^{-z}}{2}.$$

Thus, for instance,

$$\cosh(\pi i) = \frac{e^{\pi i} + e^{-\pi i}}{2} = \frac{-1 - 1}{2} = -1.$$

- These functions are clearly related to the trig functions

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Indeed, we have

$$\cosh(z) = \cos(iz), \sinh(z) = -i\sin(iz),$$

$$\cos(z) = \cosh(iz), \sin(z) = -i\sinh(iz).$$

- This explains why every trig identity is paired up with a hyperbolic trig identity which is almost the same except for some sign changes. For instance, we have

$$\cosh^2(z) - \sinh^2(z) = \cos^2(iz) - (-i\sin(iz))^2$$
$$= \cos^2(iz) + \sin^2(iz).$$

Since we have the trig identity $\cos^2(w)+\sin^2(w) = 1$, we thus have the hyperbolic trig identity $\cosh^2(z) - \sinh^2(z) = 1$. And so forth.

- $\sinh(z)$ and $\cosh(z)$ are entire, with derivatives

$$\frac{d}{dz}\sinh(z) = \cosh(z), \quad \frac{d}{dz}\cosh(z) = \sinh(z).$$

- One can also define $\tanh(z)$, $\mathrm{sech}(z)$, etc. as usual.

The complex logarithm - introduction

- The natural logarithm $\ln(x)$ is very useful in real analysis (for instance, when one wants to integrate $1/x$), and is defined for positive x by the relationship

$$y = \ln(x) \iff x = e^y$$

In other words, the natural logarithm is the inverse of exponentiation.

- In analogy to this, one can define the complex logarithm $\log(z)$ in the same way:

$$w = \log(z) \iff z = e^w.$$

(We always use ln() for the real logarithm, and log() for the complex logarithm). This definition is quite reasonable, but has one major defect - a single complex number z will have infinitely many logarithms w, because the complex exponential is periodic. For instance $\log(1)$ can equal 0, or $\pm 2\pi i$, or $\pm 4\pi i$, etc.

- This problem occurs also in real analysis, whenever one tries to invert a periodic function. For instance, if one were to naively define the inverse sine function by

$$y = \sin^{-1}(x) \iff x = \sin(y)$$

then a single value of x could have infinitely many inverse sines. For instance, $\sin^{-1}(0)$ could be 0, $\pm\pi$, $\pm 2\pi$, etc.

- To get around this, one usually restricts the value of an inverse function to a small set in order to only get one possible inverse. For instance, the inverse sine function is traditionally restricted to the interval $[-\pi/2, \pi/2]$ - so $\sin^{-1}(0) = 0$, for instance. This is not the only possible choice - for instance, one could restrict $\sin^{-1}$ to $[\pi/2, 3\pi/2]$ - but it is the convention which is most frequently used in practice (and on your calculators).

- A similar situation occurs with the complex logarithm. In order to get a single value for $w = \log(z)$, we will need to make a restriction on w; for instance, we can restrict the imaginary part of w to $(-\pi, \pi]$. Such a restriction is known as a "branch" of the log function. Unfortunately, any such restriction will create a discontinuity in the function, known as a "branch cut". Different branches have cuts in different places, and sometimes it is better to use one branch over another.

The multi-valued logarithm $\log(z)$.

- Let's try to find all possible values of $\log(z)$. In other words, we choose a z and then try to solve the equation

$$z = e^w$$

for w.

- If $z = 0$, then there is no solution to this equation, because e^w is never zero. Thus $\log(0)$ is undefined.

- Otherwise, we can write $w = x + iy$ to get

$$e^x e^{iy} = z.$$

This means that e^x is the magnitude of z, and y is one of the phases of z:

$$e^x = |z|, \quad y = \arg(z).$$

So we can solve for $w = x + iy$ as

$$w = \ln|z| + i\arg(z).$$

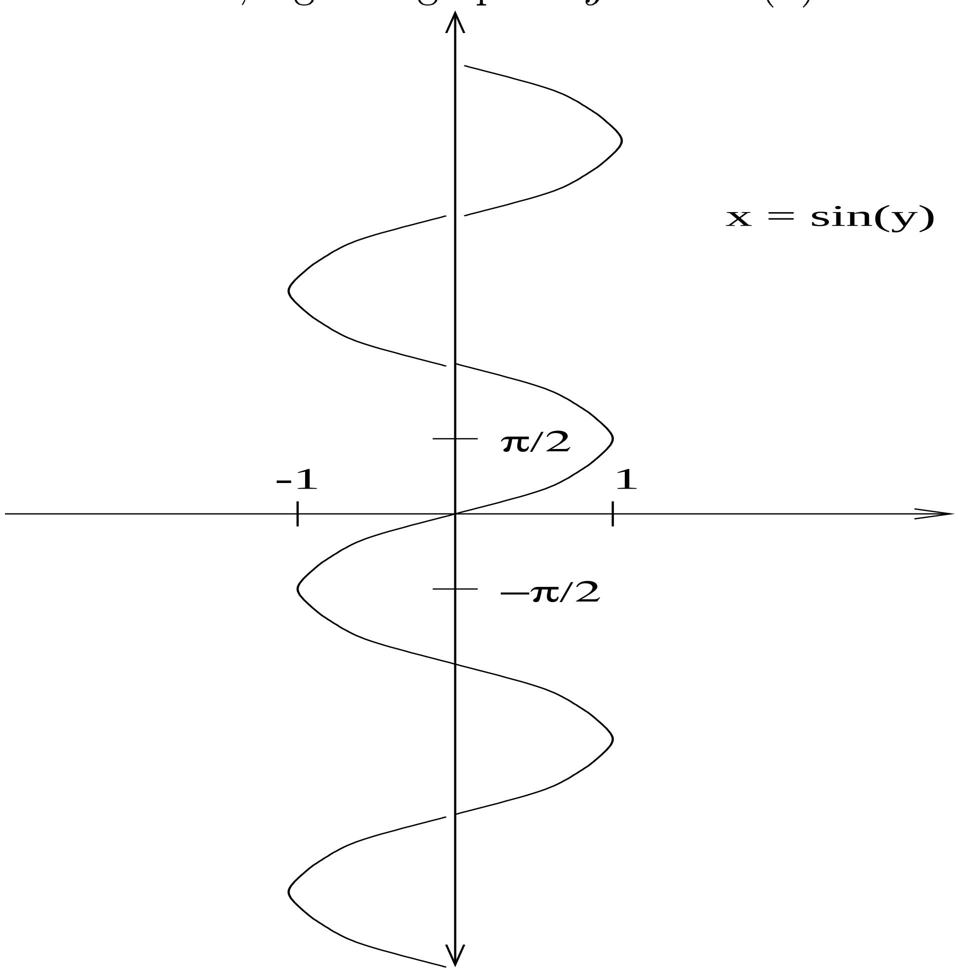
Recall that $\arg(z)$ has multiple values (it is only determined up to a multiple of 2π), so w is also multiple-valued (it is only determined up to a multiple of $2\pi i$).

- Example:

$$\log(1+i) = \ln|1+i| + i\arg(1+i)$$
$$= \ln\sqrt{2} + i(\frac{\pi}{4} + 2k\pi).$$

Branches

- If $y = f(x)$ is a multi-valued function, its graph does not always intersect each vertical line segment in a single point (which is what a single-valued function). Usually the graph looks like a tree with many branches, e.g. the graph of $y = \sin^{-1}(x)$:



- When we restrict the range of y to $[-\pi/2, \pi/2]$, however, the we prune all but one branch off of the tree, and the function becomes single-valued. The restriction of $\sin^{-1}(x)$ to $[-\pi/2, \pi/2]$ is thus called a *branch* of the multi-valued inverse sine function.

- We use the same terminology for complex functions, even though the graphs of these functions are much harder to visualize.

Branches of the argument

- The multi-valued complex logarithm is given by

$$\log(z) = \ln|z| + i\arg(z).$$

In order to find branches of this function, we need to find a branch for the multi-valued arg function.

- The standard (or principal) branch Arg of the argument, which takes values in $(-\pi, \pi]$, is one example of a branch of arg. This branch is discontinuous on the negative real axis; this axis is said to be a *branch cut* for Arg.

- However, there are other possible branches of arg. In fact, for every half-open interval $(\alpha, \alpha + 2\pi]$ we can create a branch of arg which takes values in that interval. In the textbook this branch is denoted

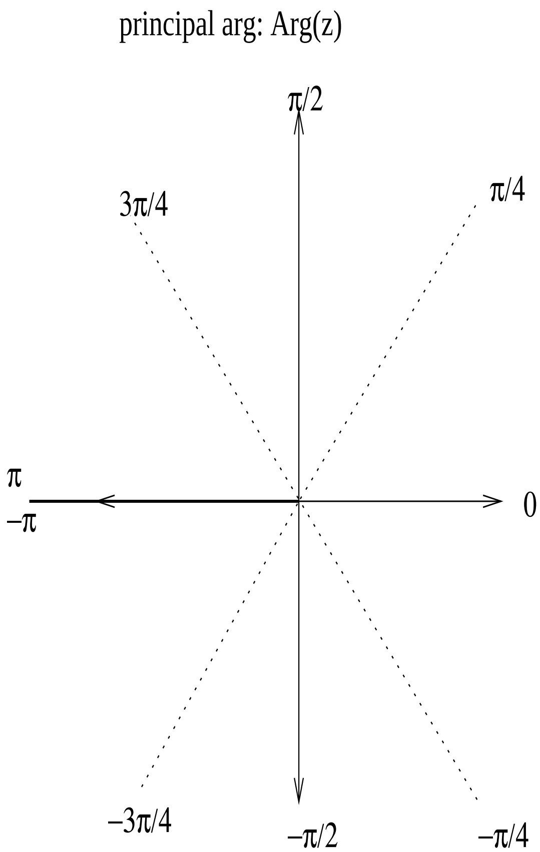
$$w = \arg(z) \qquad \alpha < \arg(z) \leq \alpha + 2\pi;$$

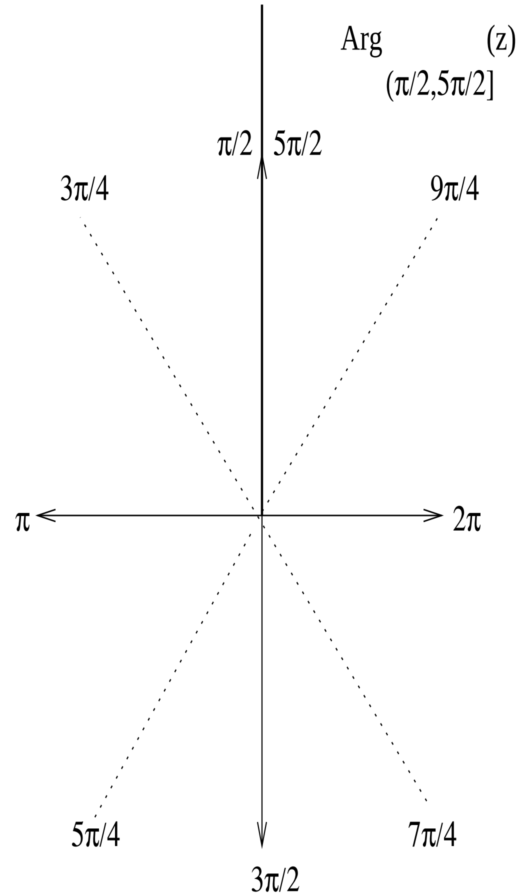
in other books it is denoted

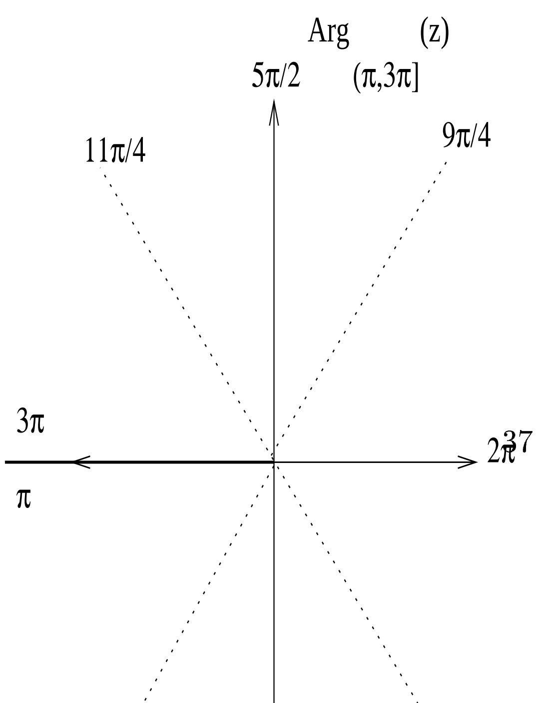
$$w = \text{Arg}_{(\alpha,\alpha+2\pi]}(z).$$

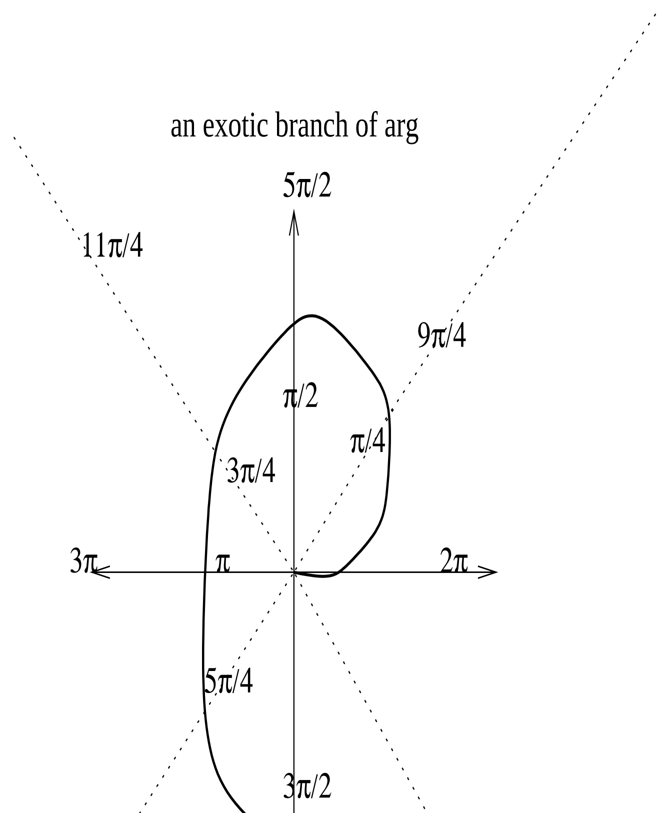
Thus, $\text{Arg}_{(-\pi,\pi]}$ is the standard branch of the argument.

- The branch $\mathrm{Arg}_{(\alpha,\alpha+2\pi]}$ has a branch cut on the ray $\{z : \alpha = \arg(z)\}$. Thus different branches usually have different branch cuts.
- There are more exotic branches with curved branch cuts, but they are almost never needed in practice.


principal arg: Arg(z)
π/2
3π/4
π/4
π
−π
0
−3π/4
−π/2
−π/4


Arg (z)
(π/2,5π/2]
π/2
5π/2
3π/4
9π/4
π
2π
5π/4
7π/4
3π/2


Arg (z)
(π,3π]
5π/2
11π/4
9π/4
3π
π
2π


an exotic branch of arg
5π/2
11π/4
9π/4
π/2
π/4
3π/4
3π
π
2π
5π/4
3π/2

Branches of the logarithm

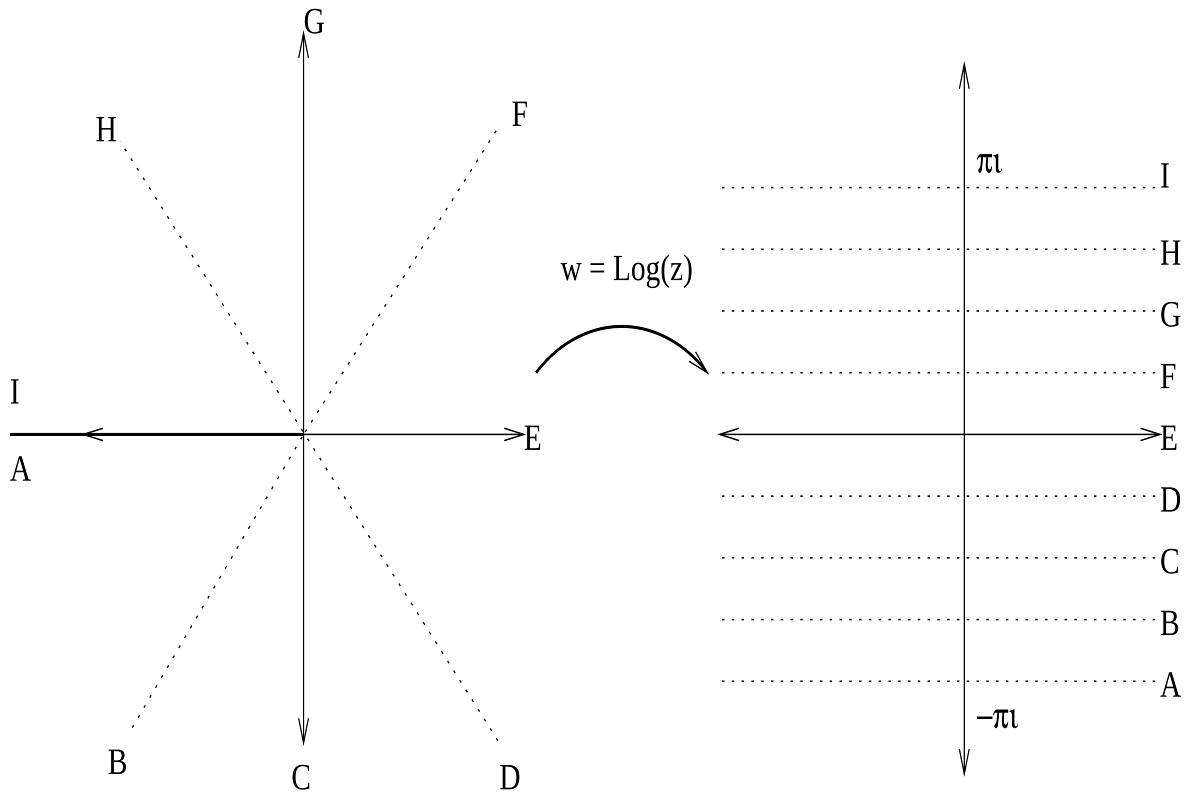
- Every branch of the argument determines a branch of the logarithm. The principal branch Arg of the argument determines the principal branch Log of the logarithm:
$$\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z).$$
Thus, for instance, $\mathrm{Log}(1+i) = \ln\sqrt{2} + i\pi/4$.

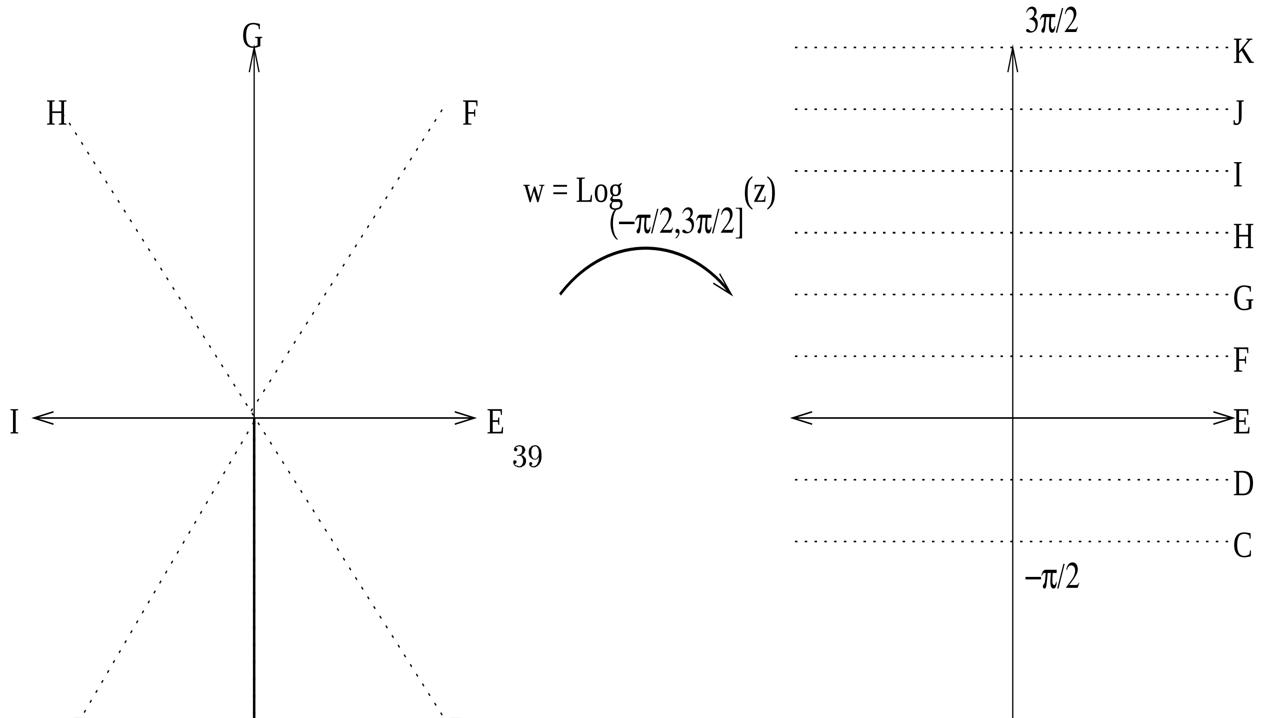
- More generally, every branch $\mathrm{Arg}_{(\alpha,\alpha+2\pi]}$ gives rise to a branch $\mathrm{Log}_{(\alpha,\alpha+2\pi]}(z)$ of the logarithm:
$$\mathrm{Log}_{(\alpha,\alpha+2\pi]}(z) = \ln|z| + i\mathrm{Arg}_{(\alpha,\alpha+2\pi]}(z).$$
This branch is denoted by
$$w = \log(z), \quad \alpha < \mathrm{Arg}(z) \le \alpha + 2\pi$$
in the textbook.

- Thus, for instance, $\mathrm{Log}_{(\pi/2,5\pi/2]}(1+i) = \ln\sqrt{2} + i9\pi/4$.

- The branch $\mathrm{Log}_{(\alpha,\alpha_2\pi]}(z)$ sends the complex plane to the half-open strip $\{w \in \mathbf{C} : \alpha < \mathrm{Im}(w) \le \alpha + 2\pi\}$. This strip is called a *fundamental domain* for the exponential function, because the exponential function is invertible on this set.

- More exotic branches of the logarithm also exist.


G
H
F
I
A
E
B
C
D
w = Log(z)
πι
I
H
G
F
E
D
C
B
A
−πι


G
H
F
I
E
w = Log
(−π/2,3π/2]
(z)
3π/2
K
J
I
H
G
F
E
D
C
−π/2

Properties of the logarithm

- The exponential always inverts the logarithm, but the logarithm does not always invert the exponential!

$$e^{\log(z)} = z, \text{ but } \log(e^z) = z + 2k\pi i!$$

- The multi-valued log converts products and quotients to sums and differences:

$$\log(zw) = \log(z) + \log(w), \quad \log(z/w) = \log(z) - \log(w);$$

- However, this is not quite true for branches of the logarithm, as one can be off by a factor of $2\pi i$. For instance:

$$0 = \text{Log}(-1 \cdot -1) \neq \textit{Log}(-1) + \text{Log}(-1) = i\pi + i\pi.$$

- In polar co-ordinates, the branches of the logarithm can be written

$$\text{Log}_{(\alpha,\alpha+2\pi]}(re^{i\theta}) = \ln r + i\theta \qquad \alpha < \theta \leq \alpha + 2\pi.$$

One can verify the polar co-ordinate Cauchy-Riemann equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

for all θ *strictly* between α and $\alpha + 2\pi$. Thus the $\text{Log}_{(\alpha,\alpha_2\pi]}$ branch is complex differentiable everywhere except at the branch cut $\{z : \arg(z) = \alpha\}$, and at the origin, where it is undefined.

- To differentiate a branch of the logarithm - let's say the principal branch $\mathrm{Log}(z)$ - it's easiest to start with the identity
$$e^{\mathrm{Log}(z)} = z$$
and differentiate this using the chain rule:
$$e^{\mathrm{Log}(z)} \frac{d}{dz}\mathrm{Log}(z) = 1$$
$$z\frac{d}{dz}\mathrm{Log}(z) = 1$$
$$\frac{d}{dz}\mathrm{Log}(z) = \frac{1}{z}.$$
Thus the principal branch of the logarithm has derivative $\frac{1}{z}$ away from the branch cut. (As we said before, on the branch cut Log isn't even continuous, let alone differentiable).

- The same is true for every other branch of the logarithm - they are differentiable away from the branch cut, with derivative $1/z$. It may seem odd that so many functions have the same derivative, but bear in mind that the different branches only differ from each other by a multiple of 2π.