Math 132 - Week 5
Textbook sections: 4.1-4.6
Topics covered:

- Curves and contours
- Contour integration
- Fundamental theorem of Calculus
- The Cauchy-Goursat theorem
- Cauchy integral formula
- Mean value principle
- Louville's theorem; Fundamental theorem of Algebra

Contour integration

- In real analysis of one variable, we integrate functions $y = f(x)$ along intervals $[a, b]$ to obtain a number, called the definite integral

$$\int_{[a,b]} f(x)\, dx = \int_a^b f(x)\, dx.$$

- In complex analysis, the analogue of this is called the contour integral, in which a complex function $w = f(z)$ is integrated along a path or *contour* γ from z_1 to z_2:

$$\int_\gamma f(z)\, dz = \int_{z_1}^{z_2} f(z)\, dz.$$

  This is analogous to the line integral, which occurs in calculus of several variables.

- (The contour integral is distinct from the area integral $\int\int f(x, y)\, dxdy$, which you may have encountered in calculus of several variables. There is a connection between the two, having to do with Stokes' theorem - but we'll come to that later).

- The definite integral in real analysis has an interpretation as the area under a graph. In complex analysis the contour integral does not have an area interpretation; it is more like a cumulative measure of work or flux.

- In real analysis, there is really only one way to integrate from a to b - namely, by going through the interval from a to b. However, in complex analysis there are infinitely many contours that go from z_1 to z_2, and these could potentially give different answers. However, in many cases the value of the integral from z_1 to z_2 does not depend on the path; we'll see why shortly.

- This week we'll define contours and contour integrals; we'll show how to compute these integrals directly, and state the Fundamental Theorem of Calculus for contour integrals. This is all pretty familiar-looking stuff, but then we'll do some more surprising things with contour integrals, including new ways to compute integrals which you wouldn't have seen in real analysis courses.

What is a contour?

- A contour is defined as a sequence of curves. To make this more precise, we have to first define what a curve is.

- **Definition.** A curve is a function $\gamma : [a, b] \to \mathbf{C}$ from an interval to the complex plane which is differentiable everywhere, and whose derivative is never zero: $\gamma'(t) \neq 0$ for all $a \leq t \leq b$.

- In other words, a curve is the trajectory of a particle which moves smoothly without ever stopping. Some examples:

- $\gamma(t) = e^{it}, 0 \leq t \leq 2\pi$ describes a curve that traverses the unit circle once anti-clockwise.

- $\gamma(t) = e^{it}, 0 \leq t \leq \pi$ describes a curve that traverses the upper unit semi-circle once anti-clockwise.

- $\gamma(t) = e^{it}, 0 \leq t \leq 4\pi$ describes a curve that traverses the unit circle twice anti-clockwise.

- $\gamma(t) = e^{-it}, 0 \leq t \leq 2\pi$ describes a curve that traverses the unit circle once clockwise.

- $\gamma(t) = e^{2it}, 0 \leq t \leq \pi$ describes a curve that traverses the unit circle once anti-clockwise. If we make the change of variables $s = 2t$ then this becomes the first example in this series. The two curves are said to be *reparameterizations* of each other. Often, we say that these two curves are two different parameterizations of the same curve.

- $\gamma(t) = 1 + it, 0 \leq t \leq 1$ describes a curve that traverses the line segment from 1 to $1 + i$.

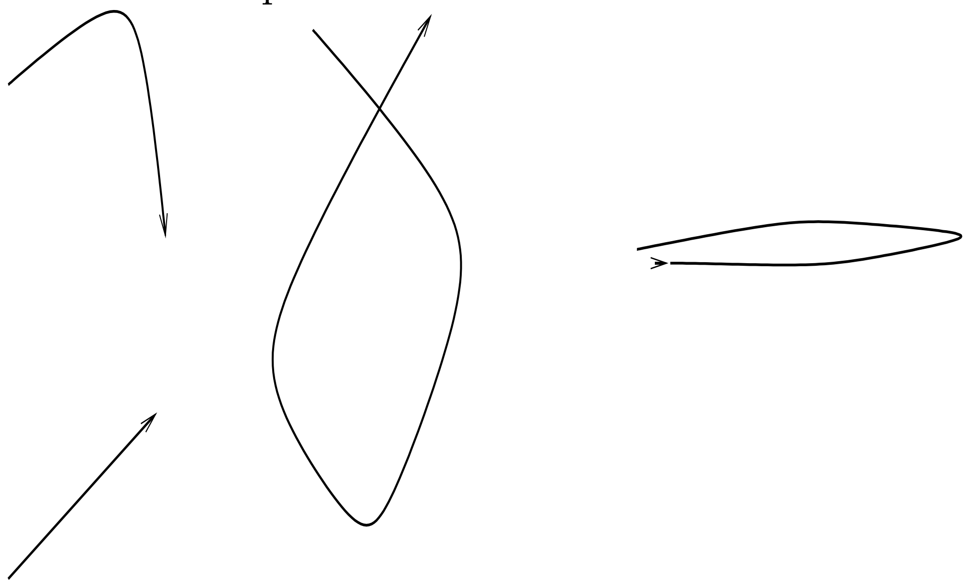
- $\gamma(a)$ is called the *initial point* of γ, and $\gamma(b)$ is called the *final point* of γ. For instance, $\gamma(t) = e^{it}, 0 \leq t \leq 2\pi$ has initial and final point both equal to 1.

- The *length* of a curve $\gamma : [a, b] \to \mathbf{C}$ is given by the formula
$$|\gamma| = \int_a^b |\gamma'(t)| \, dt.$$
For instance, the length of $\gamma(t) = e^{it}, 0 \leq t \leq 2\pi$ is
$$\int_0^{2\pi} |ie^{it}| \, dt = \int_0^{2\pi} dt = 2\pi.$$
Note that the length of $\gamma(t) = e^{it}, 0 \leq t \leq 4\pi$ is 4π even though it only lives on a circle of circumference 2π.

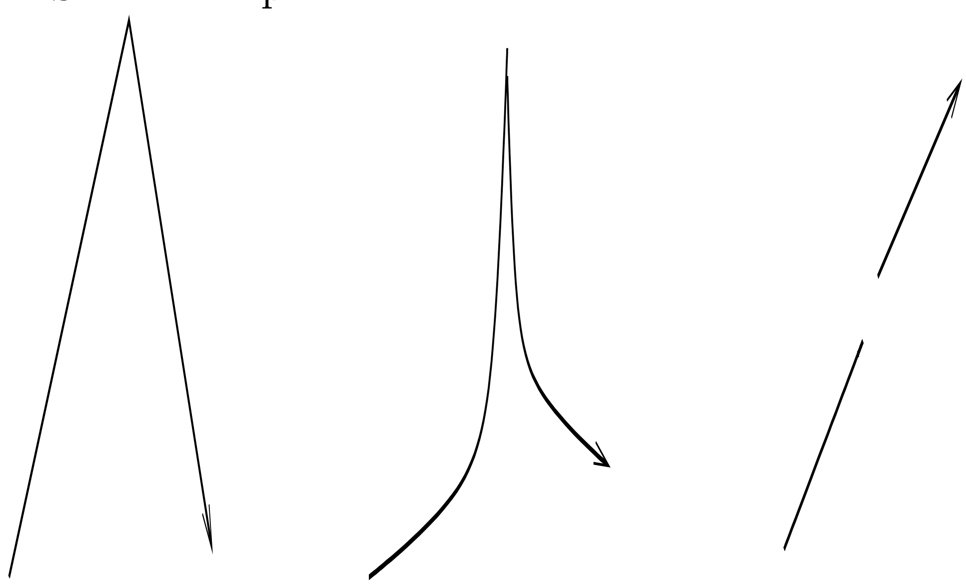
- If you reparameterize a curve, its length does not change. (This can be proven using the change of variables formula).

- If $\gamma : [a, b] \to \mathbf{C}$ is a curve, the negative $-\gamma : [-b, -a] \to \mathbf{C}$ of γ is defined by $-\gamma(t) = \gamma(-t)$. This curve traverses the same region of the complex plane as γ, but in the reverse direction. For instance, the negative of $\gamma(t) = e^{it}, 0 \leq t \leq 2\pi$ is $-\gamma(t) = e^{-it}, -2\pi \leq t \leq 0$.

Some examples of curves:



Some examples of non-curves:



Contours

- A contour is a finite sequence $\gamma_1 + \gamma_2 + \ldots + \gamma_n$ of curves, such that the final point of each curve matches up with the initial point of the next one. An example is $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, where

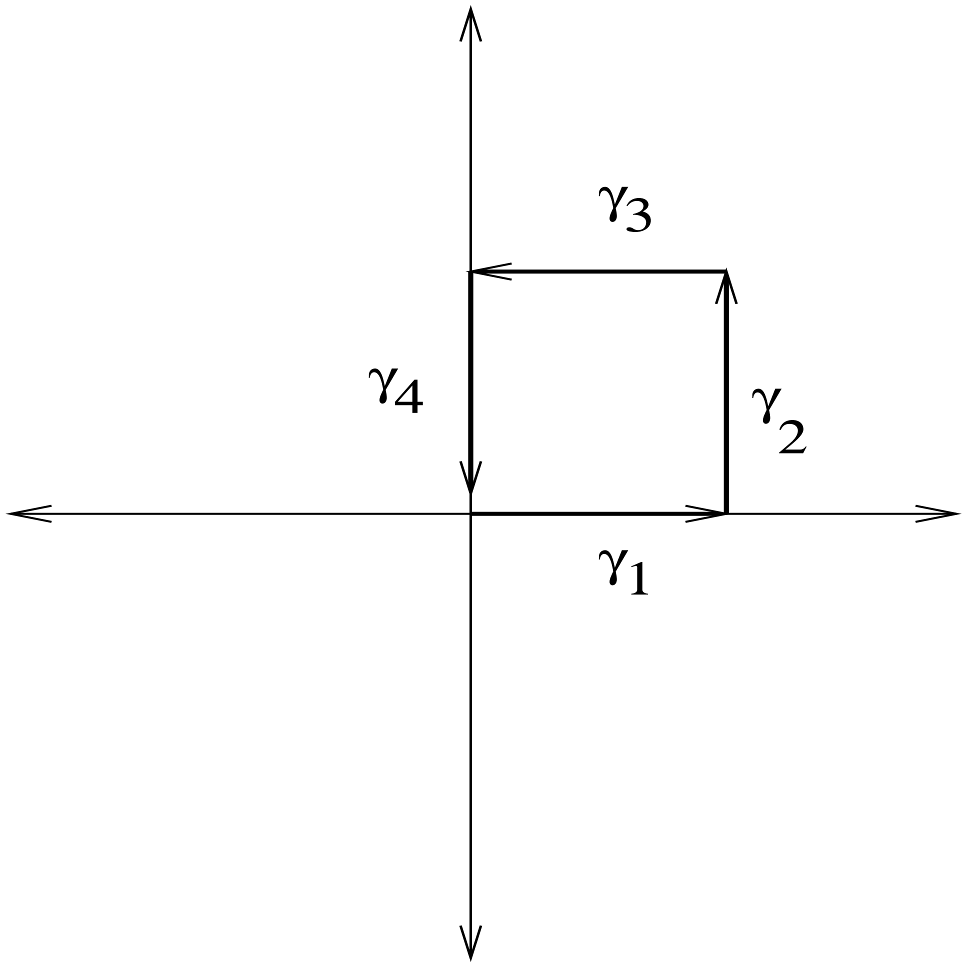
$$\gamma_1(t) = t, \quad 0 \leq t \leq 1$$

$$\gamma_2(t) = 1 + it, \quad 0 \leq t \leq 1$$

$$\gamma_3(t) = 1 + i - t, \quad 0 \leq t \leq 1$$

$$\gamma_4(t) = i - it, \quad 0 \leq t \leq 1$$

  This contour describes a square traversed once anti-clockwise.



- Two contours Γ and Γ' can be added together to form a longer contour $\Gamma + \Gamma'$ if the final point of Γ equals the initial point of Γ'. (Otherwise, we do not define addition of contours).

- The length of a contour is the sum of the lengths of the component curves. Thus the length of the square above is 4.

- A contour can be reversed; the negation of $\gamma_1 + \gamma_2 + \ldots + \gamma_n$ is $(-\gamma_n) + \ldots + (-\gamma_2) + (-\gamma_1)$.
- A contour is said to be *closed* if its final point is equal to its initial point, otherwise it is open. (This is not related to the notions of open and closed sets mentioned in first week).
- Generally, curves are denoted γ and contours are denoted Γ.

Integration along contours

- Recall that the definite integral in real analysis is defined as

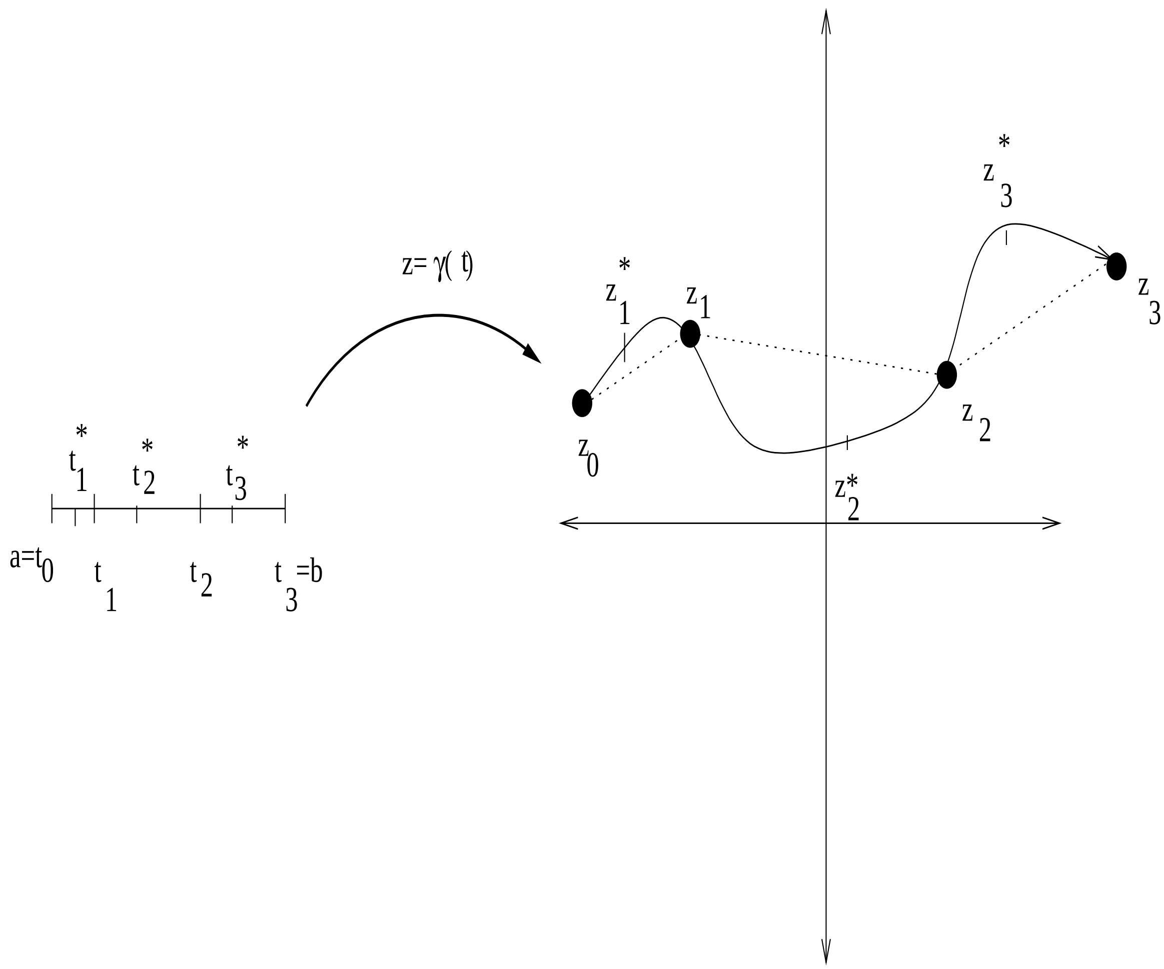
$$\int_a^b f(x)\ dx = \lim \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

where $a = x_0 < x_1 < \ldots < x_n = b$ is a partition of $[a, b]$, and each x_i^* is between x_{i-1} and x_i, and the limit is as the separation between adjacent x_i's go to zero. This limit exists as long as f is continuous on $[a, b]$.

- The integral along a curve γ is defined similarly as

$$\int_\gamma f(z)\ dz = \lim \sum_{i=1}^{n} f(z_i^*)(z_i - z_{i-1})$$

where $z_i = \gamma(t_i)$, $a = t_0 < t_1 < \ldots < t_n = b$, $z_i^* = \gamma(t_i^*)$, each t_i^* is between t_{i-1} and t_i and the limit is as the separation between adjacent t_i's go to zero. It turns out that this limit exists if f is continuous on some domain containing the range of γ, but we won't prove that in this course, as it is rather dull.



- This definition is quite cumbersome, but it can be quickly simplified. If we accept the approximation
$$\gamma'(t_i^*) \approx \gamma'(t_i) \approx \frac{z_i - z_{i-1}}{t_i - t_{i-1}}$$
then we can rewrite the right-hand side as
$$\lim \sum_{i=1}^{n} f(\gamma(t_i^*))\gamma'(t_i^*)(t_i - t_{i-1}).$$

But this is just

$$\int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

So we can convert a contour integral into an ordinary definite integral by a change of variables $z = \gamma(t)$.

- To integrate on a contour $\gamma_1 + \ldots + \gamma_n$, we simply integrate on each curve separately and add up:

$$\int_{\gamma_1+\ldots+\gamma_n} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \ldots + \int_{\gamma_n} f(z)\, dz.$$

Examples

- If γ_1 is the line segment from 1 to $1+i$, so $\gamma_1(t) = 1+it, 0 \le t \le 1$, then

$$\int_{\gamma_1} z\ dz = \int_0^1 (1+it)i\ dt = \int_0^1 i - t\ dt$$

$$= (it - \frac{t^2}{2})|_0^1 = i - \frac{1}{2}.$$

- We could parameterize γ_1 differently, e.g. $\gamma_1(s) = 1+2is, 0 \le s \le 1/2$, but the integral you get is the same (thanks to the change of variables formula). In other words, the choice of parameterization does not affect the value of the integral.

- If γ_2 is the line segment from $1+i$ to i, so $\gamma_2(t) = 1+i-t, 0 \le t \le 1$, then

$$\int_{\gamma_2} z\ dz = \int_0^1 (1+i-t)(-1)\ dt = \int_0^1 t-1-i\ dt$$

$$= (\frac{t^2}{2} - t - it)|_0^1 = -\frac{1}{2} - i.$$

- Combining the two, we have

$$\int_{\gamma_1+\gamma_2} z\ dz = \int_{\gamma_1} z\ dz + \int_{\gamma_2} z\ dz = -1.$$

Properties of contour integration

- Integration is linear:

$$\int_\Gamma f(z) + g(z)\ dz = \int_\Gamma f(z) + \int_\Gamma g(z)\ dz$$

$$\int_\Gamma cf(z)\ dz = c\int_\Gamma f(z)\ dz.$$

- Reversing a contour negates the integral:

$$\int_{-\Gamma} f(z)\ dz = -\int_\Gamma f(z)\ dz$$

- Adding two contours adds integrals together:

$$\int_{\Gamma_1+\Gamma_2} f(z)\ dz = \int_{\Gamma_1} f(z)\ dz + \int_{\Gamma_2} f(z)\ dz.$$

- In particular, integration along a doubled-back contour $\Gamma + -\Gamma$ is always zero:

$$\int_{\Gamma+-\Gamma} f(z)\ dz = 0.$$

Estimating integrals

- Sometimes, an integral is too hard to compute exactly. In that case we often just settle for finding an upper bound for an integral.
- **Lemma (triangle inequality for integrals).** We have
$$|\int_a^b f(t)\ dt| \leq \int_a^b |f(t)|\ dt.$$
- **Proof.** Write $\int_a^b f(t)\ dt$ in polar co-ordinates as
$$\int_a^b f(t)\ dt = re^{i\theta}.$$
Then
$$|\int_a^b f(t)\ dt| = r = \mathrm{Re}(r) = \mathrm{Re}(e^{-i\theta}\int_a^b f(t)\ dt)$$
$$= \int_a^b \mathrm{Re}(f(t)e^{-i\theta})\ dt$$
$$\leq \int_a^b |f(t)|\ dt.$$
- **Corollary.** If γ is a curve and $|f(z)| < M$ for all z in γ, then
$$|\int_\gamma f(z)\ dz| \leq M|\gamma|.$$

- **Proof.** We have

$$|\int_\gamma f(z)\ dz| = |\int_a^b f(\gamma(t))\gamma'(t)\ dt|$$

$$\leq \int_a^b |f(\gamma(t))||\gamma'(t)|\ dt$$

$$\leq M \int_a^b |\gamma'(t)|\ dt$$

$$= M|\gamma|.$$

- Example: suppose we want to estimate

$$\int_\gamma \frac{dz}{z^4+1}$$

where γ is the circle $\gamma(t) = 10e^{it}$, $0 \leq t \leq 2\pi$. For all z in γ, $|z| = 10$, so $|z^4| = 1000$, so $|z^4+1| \geq 999$, so $|\frac{1}{z^4+1}| \leq \frac{1}{999}$. Thus we have

$$|\int_\gamma \frac{dz}{z^4+1}| \leq \frac{1}{999} 2\pi \cdot 10.$$

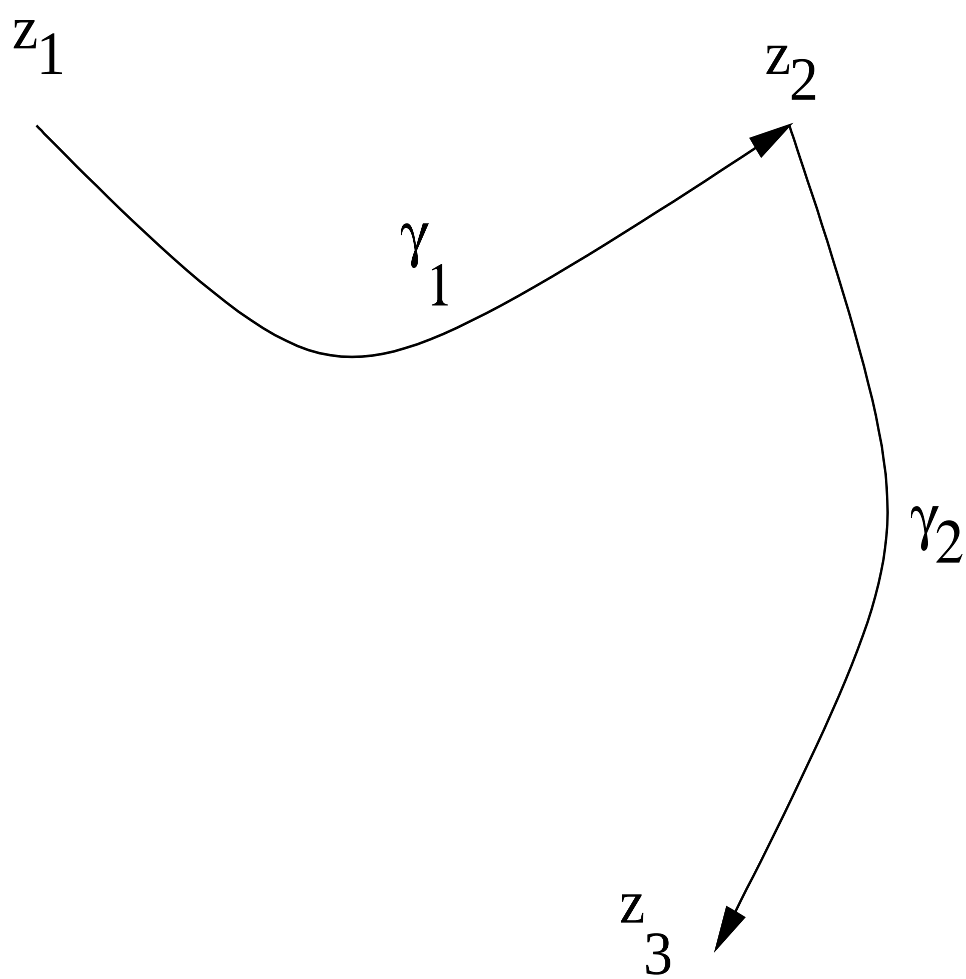
The Fundamental theorem of calculus

- The easiest way to compute a definite integral is to find an anti-derivative and use the Fundamental theorem of calculus. One can do the same in complex analysis (although we shall soon see that there are some more powerful tools than the FToC available!)

- **Fundamental Theorem of Calculus I.** Suppose that $f(z)$, $F(z)$ are functions such that F is compl ex differentiable on a domain D, and $f(z) = F'(z)$ for all z in D. Then for any z_1, z_2 in D and any contour Γ from z_1 to z_2, we have

$$\int_\Gamma f(z)\, dz = F(z)|_{z_1}^{z_2} = F(z_2) - F(z_1).$$

- **Proof** We first prove this theorem when Γ is a curve; once we prove it for curves, the statement for contours follows (see figure)



If $\int_{\gamma_1} f(z)\ dz = F(z)|_{z_1}^{z_2}$, and $\int_{\gamma_2} f(z)\ dz = F(z)|_{z_2}^{z_3}$, then $\int_{\gamma_1+\gamma_2} f(z)\ dz = F(z)|_{z_1}^{z_3}$.

- If $\Gamma : [a, b] \to \mathbf{C}$ is a curve, then

$$\int_\Gamma f(z)\ dz = \int_a^b f(\Gamma(t))\Gamma'(t)\ dt$$

$$
\begin{aligned}
&= \int_a^b F'(\Gamma(t))\Gamma'(t)\ dt \\
&= \int_a^b \frac{d}{dt} F(\Gamma(t))\ dt \\
&= F(\Gamma(b)) - F(\Gamma(a)) \\
&= F(z_2) - F(z_1).
\end{aligned}
$$

- **Corollary:** If f has an anti-derivative on a domain D, then $\int_\Gamma f(z)\ dz = 0$ for all *closed* contours Γ in D.

- It turns out this corollary can be reversed, to obtain

- **Fundamental Theorem of Calculus II:** If $\int_\Gamma f(z)\ dz = 0$ for all closed contours Γ in a domain D, then f has an anti-derivative F.

- **Proof** This will remind you of how the second Fundamental Theorem of Calculus was proven for real variables.

- Pick a z_0 in D, and for each $z_1 \in D$ we define

  $$F(z_1) = \int_\Gamma f(z)\ dz$$

  where Γ is any contour in D from z_0 to z_1. In theory this definition could depend on the choice of Γ, but our hypotheses will stop this from happening. In fact, if we use two different paths Γ, Γ' from z_0 to z_1, then $\Gamma + -\Gamma'$ is a closed contour, so by hypothesis we have

$$\int_{\Gamma+-\Gamma'} f(z)\, dz = 0$$

and therefore

$$\int_{\Gamma} f(z)\, dz = \int_{\Gamma'} f(z)\, dz.$$

Thus it doesn't matter which path you choose to perform the integral.

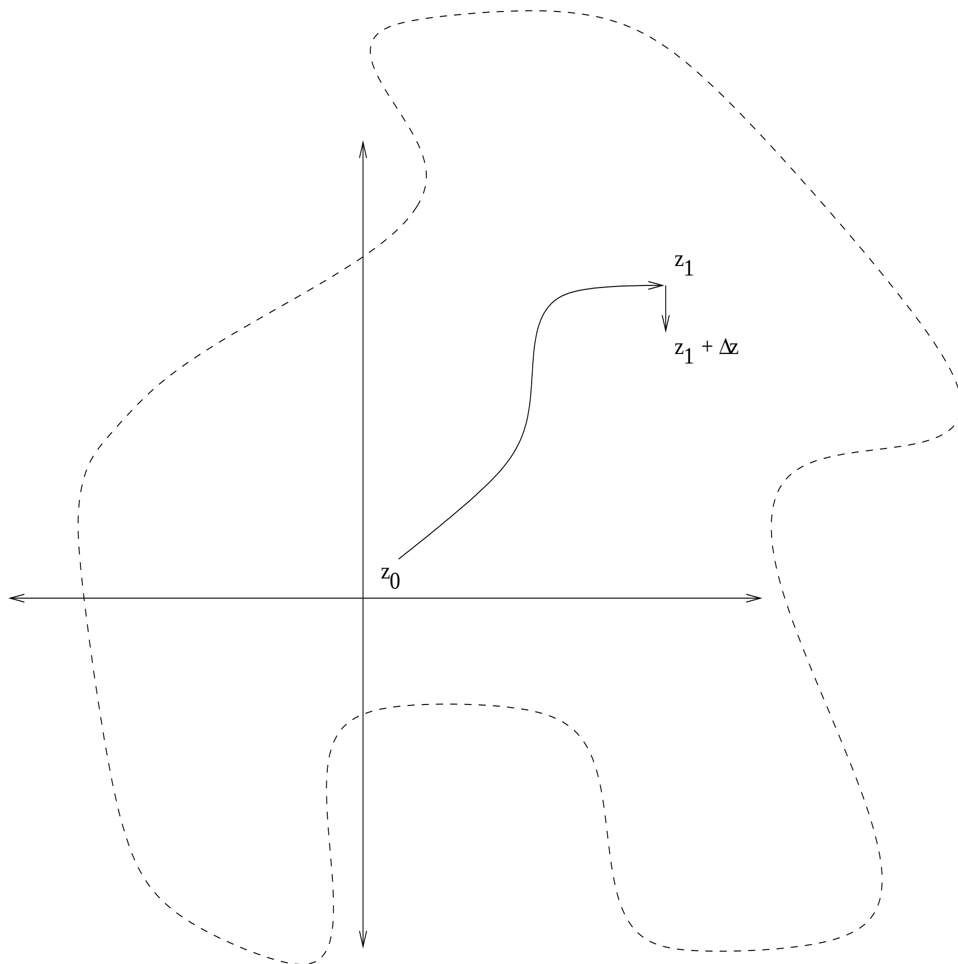
- We need to show that $F'(z_1) = f(z_1)$, or in other words that

$$\lim_{\Delta z \to 0} \frac{F(z_1 + \Delta z) - F(z_1)}{\Delta z} = f(z_1).$$

From definition, we have

$$F(z_1 + \Delta z) - F(z_1) = \int_{\gamma} f(z)\, dz,$$

where γ is the line segment from z_1 to $z_1 + \Delta z$.



Parameterizing γ as $\gamma(t) = z_1 + \Delta z t, 0 \leq t \leq 1$, we thus get

$$F(z_1 + \Delta z) - F(z_1) = \int_0^1 f(z_1 + \Delta z t)\Delta z \; dt,$$

so

$$\frac{F(z_1 + \Delta z) - F(z_1)}{\Delta z} = \int_0^1 f(z_1 + \Delta z t) \; dt.$$

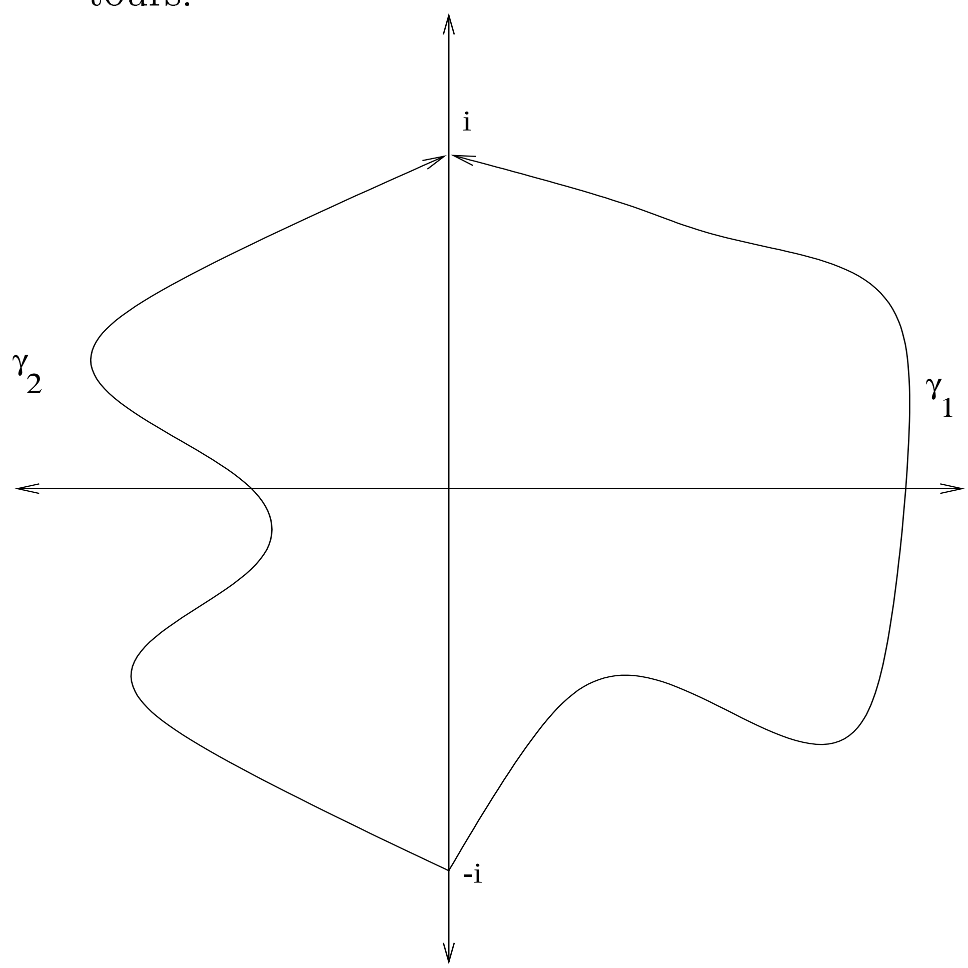
- If we let $\Delta z \to 0$ we thus get

  $$\lim_{\Delta z \to 0} \frac{F(z_1 + \Delta z) - F(z_1)}{\Delta z} = \int_0^1 f(z_1)\, dt = f(z_1)$$

  so $F' = f$. QED

Example: integration of $1/z$.

- Suppose we wish to integrate $1/z$ on these two contours:



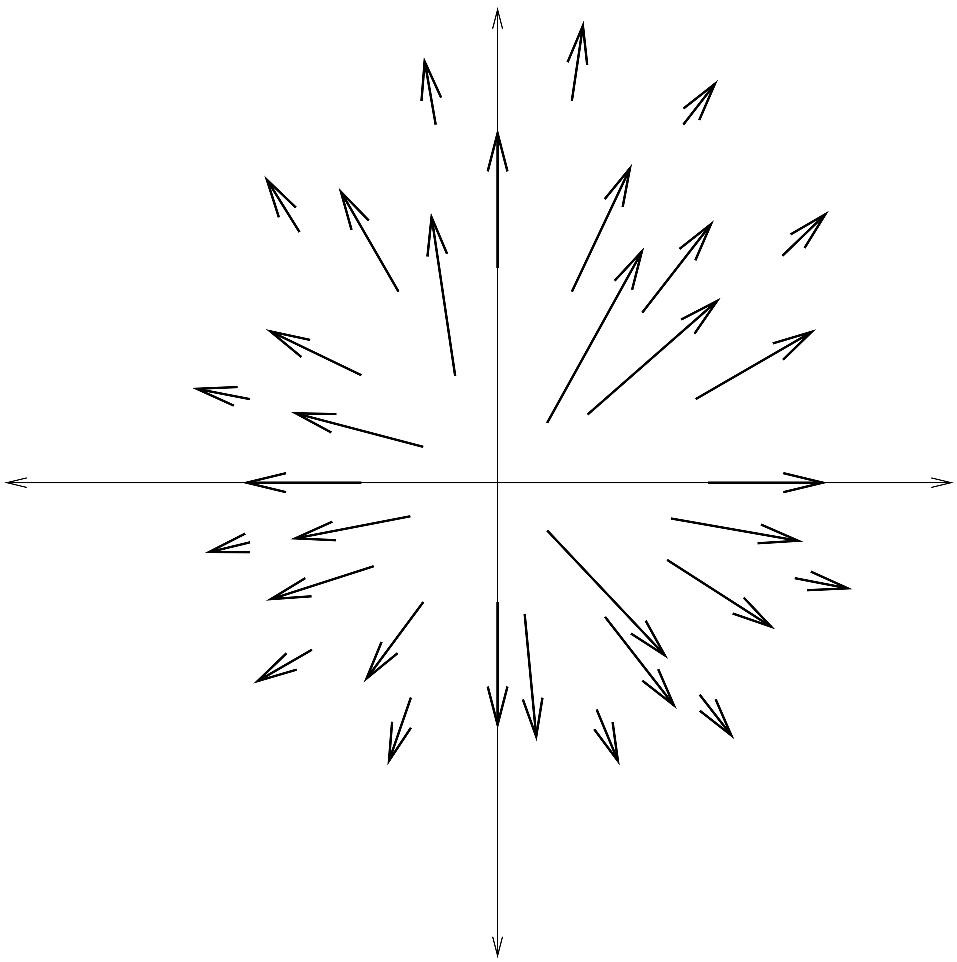
- To integrate $1/z$ on γ_1, we need an anti-derivative of $1/z$ which is differentiable on a domain containing γ_1. The principal logarithm $\mathrm{Log}(z)$ fits these requirements, and so we can calculate

$$\int_{\gamma_1} \frac{dz}{z} = \mathrm{Log}(z)|_{-i}^{i} = \pi i/2 - (-\pi i/2) = \pi i.$$

There are other branches which also work for this integral, and they all give the same answer of course. (It's kind of like how one can add $+C$ to an anti-derivative without affecting the value of a definite integral).

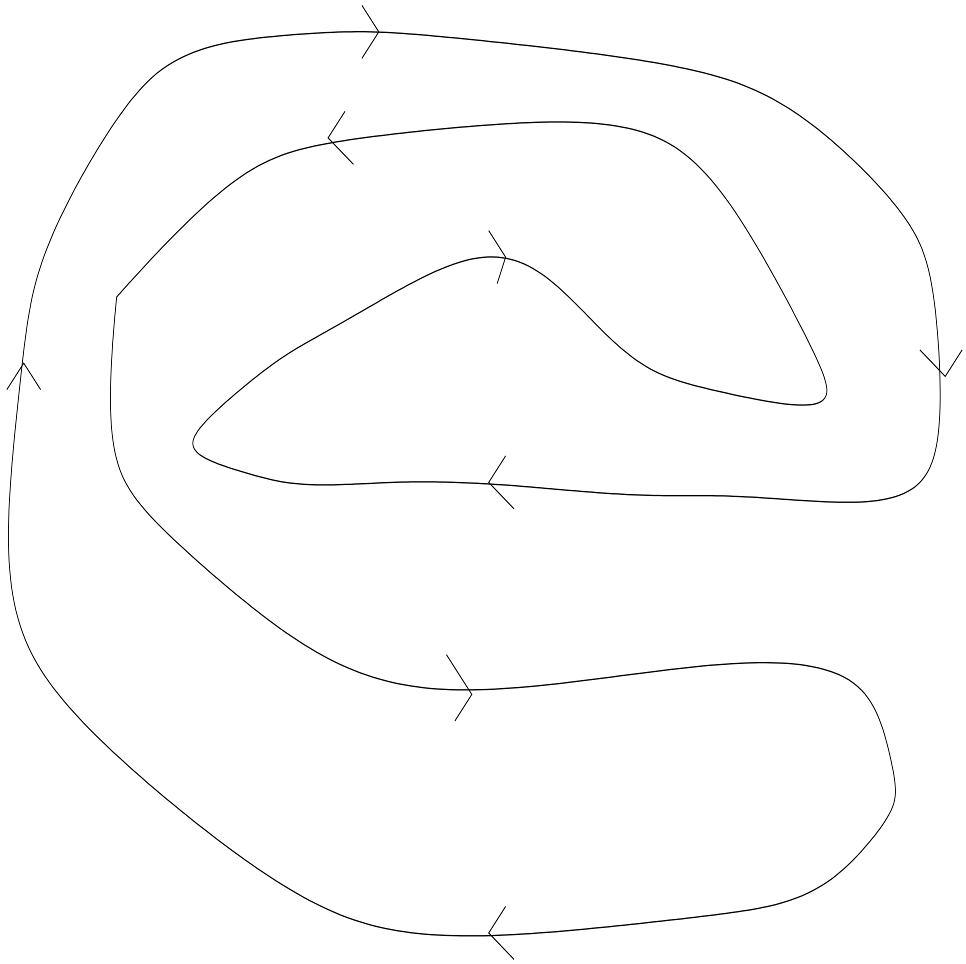
- To integrate $1/z$ on γ_2, the same calculation does not work, because $\mathrm{Log}(z)$ fails to be differentiable (and thus fails to be an anti-derivative of $1/z$) on its branch cut - the negative real axis. However , one can pick a different branch, such as $\mathrm{Log}_{(0,2\pi]}(z)$, whose branch cut does not intersect γ_2, and compute this integral as

$$\int_{\gamma_2} \frac{dz}{z} = \mathrm{Log}_{(0,2\pi]}(z)|_{-i}^{i} = \pi i/2 - (3\pi i/2) = -\pi i.$$



Simple contours

- A contour is said to be *simple* if it does not cross itself, that is if every point in space is covered at most once by the contour. (If the initial point and final point coincide, we do not consider this a crossing).
- Thus, for instance, the contour $\gamma(t) = e^{it} : 0 \leq t \leq 2\pi$ is a simple closed contour, while $\gamma(t) = e^{it} : 0 \leq t \leq 4\pi$ is a non-simple but closed contour.
- It is intuitively obvious that a simple closed contour must divide the complex plane into two regions. Formally, we have
- **Jordan curve theorem** If Γ is a simple closed contour, then the complement $\mathbf{C}\backslash\Gamma$ is the union of two disjoint domains, one of which is bounded.



- The bounded region is called the *interior* of Γ, and the unbounded region the *exterior* of Γ.
- This theorem is plausible, but remarkably difficult to prove, requiring some sophisticated topology. We won't prove it here, as it is beyond the scope of the course.

- We call a simple closed contour *anti-clockwise oriented* or *positively oriented* if the interior is always on the left. Otherwise we say that the contour is *clockwise-oriented* or *negatively oriented*.

- By convention, if an orientation of a closed contour is not specified, it is assumed to be anti-clockwise. Thus if we don't specify the orientation of the unit circle, it is understood to be parameterized by e.g. $\gamma(t) = e^{it}, 0 \leq t \leq 2\pi$ as opposed to $\gamma(t) = e^{-it}, 0 \leq t \leq 2\pi$.

Integration on closed contours

- On the real line, an integral along a closed contour is always equal to zero, regardless of the integrand:

$$\int_a^b f(x)\ dx + \int_b^a f(x)\ dx = 0$$

$$\int_a^b f(x)\ dx + \int_b^c f(x)\ dx + \int_c^a f(x)\ dx = 0, \text{ etc.}$$

- In the complex plane, this is not always the case; for instance, in the previous set of notes we found a closed contour $\gamma_1 + -\gamma_2$ such that

$$\int_{\gamma_1 + -\gamma_2} \frac{1}{z}\ dz = -2\pi i \neq 0.$$

- Another example: let γ be the unit circle $|z| = 1$ traversed once anti-clockwise, and consider the contour integral

$$\int_\gamma \overline{z}\ dz.$$

We may parameterize γ as $\gamma(t) = e^{it}, 0 \leq t \leq 2\pi$. Since $z = e^{it}$, $dz = ie^{it}dt$, and we have

$$\int_\gamma \overline{z}\ dz = \int_0^{2\pi} \overline{e^{it}} ie^{it}\ dt$$

$$= \int_0^{2\pi} i \, dt = 2\pi i \neq 0$$

- In the previous set of notes we showed that a function $f(z)$ had integral zero on every closed contour in a domain D if and only if it had an anti-derivative on D. Thus, many functions do not have an anti-derivative.

- For instance, we now know that $1/z$ cannot have an anti-derivative on any domain containing $\gamma_1 + -\gamma_2$. (This explains why we need branch cuts in order to form an anti-derivative of $1/z$). Similarly, $\overline{z}$ cannot have an anti-derivative on any domain that contains the unit circle. (In fact, $\overline{z}$ does not have an anti-derivative anywhere).

- However, many functions do have integral zero on every closed contour. For instance, we know that $\oint_\Gamma z^2 \, dz = 0$ for every closed contour Γ, because z^2 has an anti-derivative, namely $z^3/3$.

- (The notation $\oint$ is often used instead of $\int$ when integrating over a closed contour, but otherwise the symbols have the same meaning.).

- However, it does not seem easy to check whether a function has an anti-derivative or not. For instance, we can't assert yet that $\oint_\Gamma e^{z^2}\,dz = 0$ for every closed contour Γ, because we don't know an anti-derivative of e^{z^2} exists.

- Fortunately, there is a very simple and important theorem which covers these cases:

- **Cauchy-Goursat theorem.** If Γ is a simple closed contour, and f is analytic at every point on Γ and in the interior of Γ, then $\oint_\Gamma f(z)\,dz = 0$.

- This theorem is perhaps the most important result in complex analysis, and most of the course will use this theorem as a foundation.

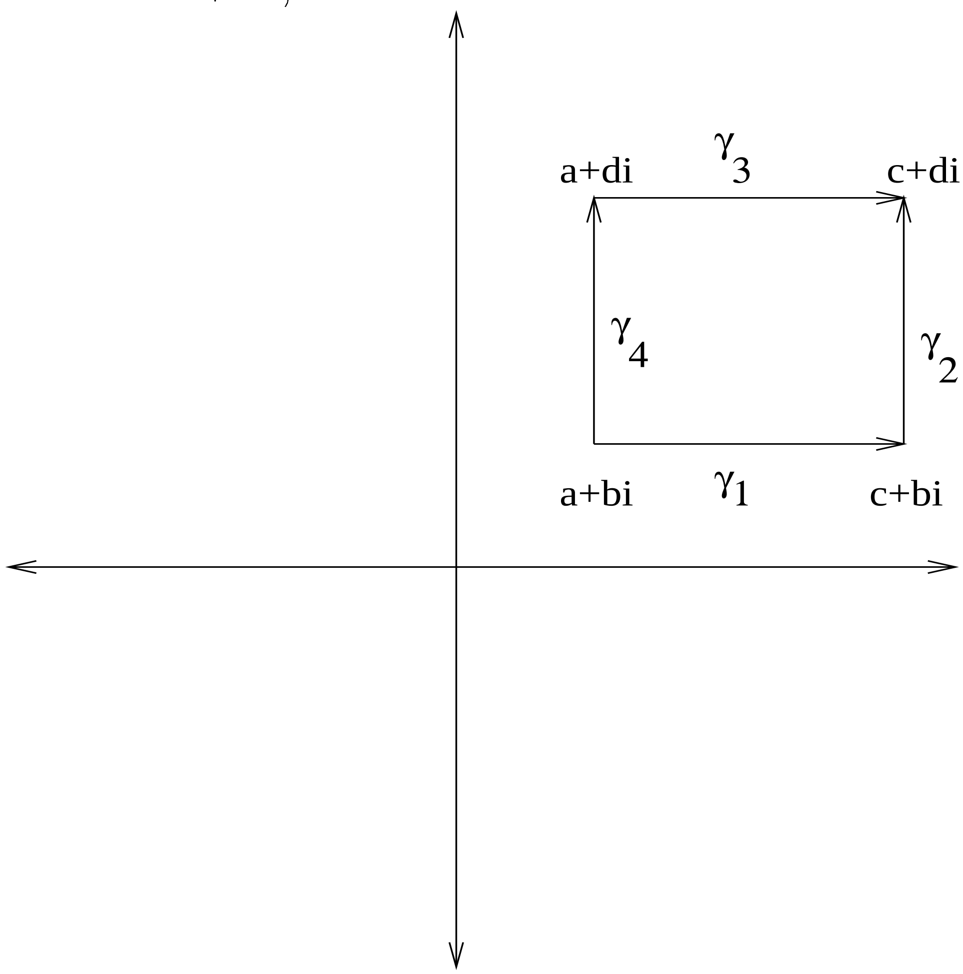
- This theorem is often referred to as "Cauchy's theorem". Augustus Cauchy proved the theorem (c. 1830) assuming that the derivative of f was continuous, and Eduard Goursat removed this restriction much later (c. 1880). I'll only prove Cauchy's version of the theorem, as the proof is a bit simpler.

- Interestingly, the theorem requires f to be analytic on the *interior* of the contour, even though the integration is only performed on the contour itself. For instance, when integrating $1/z$ on $\gamma_1 + -\gamma_2$, the

Cauchy-Goursat theorem does not apply because $1/z$ is not analytic at 0, which is inside $\gamma_1 + \gamma_2$.

The case of a rectangle

- Let us first prove Cauchy's theorem when Γ is a rectangle connecting four points $a + bi$, $c + bi$, $c + di$, and $a + di$, as shown.



- In this case we have $\Gamma = \gamma_1 + \gamma_2 + -\gamma_3 + -\gamma_4$, where

$$\gamma_1(x) = x + bi, \quad a \leq x \leq c;$$

$$\begin{aligned}\gamma_2(y) &= c + yi, \quad b \le y \le d;\\ \gamma_3(x) &= x + di, \quad a \le x \le c;\\ \gamma_4(y) &= a + yi, \quad b \le y \le d.\end{aligned}$$

- By changing variables, we thus have

$$\oint_\Gamma f(z)\ dz = \int_a^c f(x+bi)\ dx + \int_b^d f(c+yi)\ idy$$
$$- \int_a^c f(x+di)\ dx - \int_b^d f(a+yi)\ idy$$

  which we can simplify as

$$-\int_a^c (f(x+di)-f(x+bi))\ dx + i\int_b^d (f(c+yi)-f(a+yi))\ dy.$$

- However, from the (real variable) Fundamental Theorem of Calculus, we have

$$f(x+di) - f(x+bi) = \int_b^d \frac{\partial f}{\partial y}(x+yi)\ dy$$

  and

$$f(c+yi) - f(a+yi) = \int_a^c \frac{\partial f}{\partial x}(x+yi)\ dx.$$

- Substituting these equations into the first equation, we get

$$\oint_\Gamma f(z)\,dz = -\int_a^c \int_b^d \frac{\partial f}{\partial y}(x+yi)\,dydx$$
$$+\int_b^d \int_a^c i\frac{\partial f}{\partial x}(x+yi)\,dxdy.$$

If we use Fubini's theorem to interchange the integrations, this becomes

$$\oint_\Gamma f(z)\,dz = \int_a^c \int_b^d (i\frac{\partial f}{\partial x}(x+yi)-\frac{\partial f}{\partial y}(x+yi))\,dydx,$$

which is zero by the Cauchy-Riemann equations, since f is analytic everywhere on and inside the rectangle.

The general case

- To prove the general case, we need to use Stokes' theorem, which states that

$$\oint_\Gamma f(x,y)dx + g(x,y)dy = \int\int_D (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})\ dxdy$$

  for any continuously differentiable functions f, g on a domain D, where Γ is the boundary of D.

- Since $z = x + iy$, we have $dz = dx + idy$. Thus

$$\oint_\Gamma f(z)\ dz = \oint_\Gamma f(x+iy)\ dx + if(x+iy)\ dy.$$

  By Stokes' theorem, we thus have

$$\oint_\Gamma f(z)\ dz = \int\int_D i\frac{\partial f}{\partial x}(x+iy) - \frac{\partial f}{\partial y}(x+iy)\ dxdy.$$

  where D is the interior of Γ.

- Since f is analytic on D, it satisfies the Cauchy-Riemann equations on D, so

$$i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = 0,$$

  and Cauchy's theorem is proved.

Example

- Consider the function e^z/z. This function is analytic everywhere except at 0. Thus, Cauchy's theorem says that $\oint_{|z-3|=1} \frac{e^z}{z}\, dz = 0$. However, this theorem says nothing about $\oint_{|z|=1} \frac{e^z}{z}\, dz$.

Cauchy's integral formula

- The Cauchy-Goursat theorem says that if $f(z)$ is analytic everywhere on and inside a simple closed contour Γ, then $\int_\Gamma f(z)\, dz = 0$.

- However, when $f(z)$ contains a singularity in the interior of Γ, then Cauchy's theorem does not seem to apply directly. For instance, we cannot currently compute
$$\oint_{|z|=1} \frac{e^z}{z}\, dz$$
where the integral is over the unit circle traversed once anti-clockwise.

- However, one can modify the Cauchy-Goursat theorem to deal with singularities. The first result in this direction is the Cauchy integral formula:

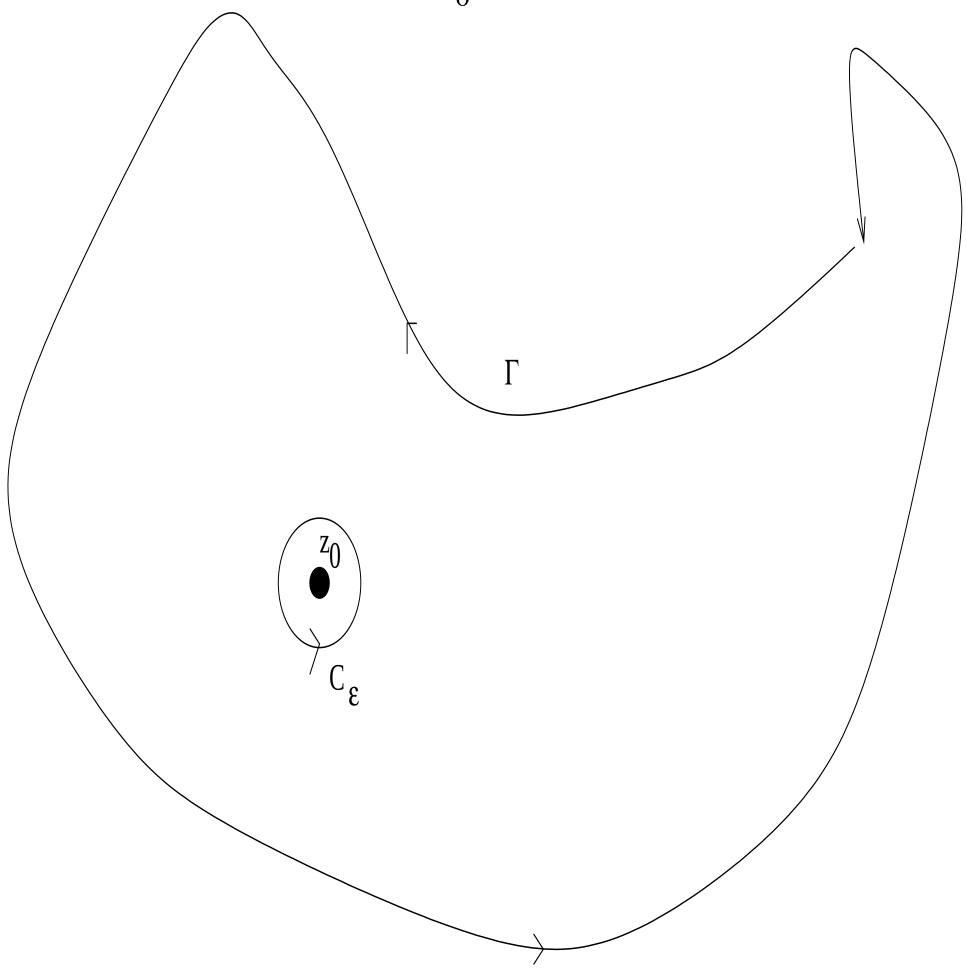
- **Cauchy integral formula.** Let Γ be a simple closed anticlockwise-oriented contour, and let z_0 be a point in the interior of Γ. If $f(z)$ is analytic on and inside Γ, then
$$\int_\Gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0).$$

- For instance, applying this theorem with $f(z) = e^z$ and $z_0 = 0$ we can now compute

$$\oint_{|z|=1} \frac{e^z}{z}\, dz = 2\pi i e^0 = 2\pi i.$$

Proof of Cauchy's integral formula

- Let $\varepsilon > 0$ be a small number, and let C_ε be the circle of radius ε around z_0 traversed once anti-clockwise.



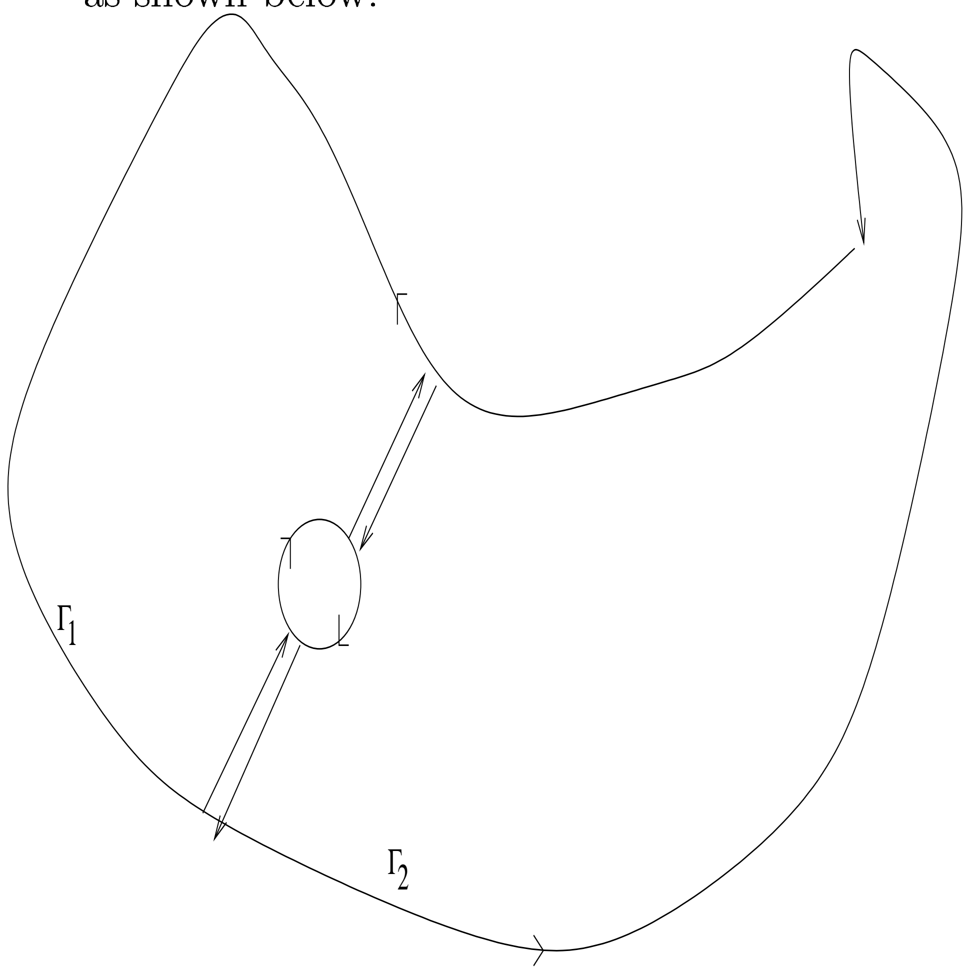
- The integral

$$\oint_\Gamma \frac{f(z)}{z - z_0}\, dz + \oint_{-C_\varepsilon} \frac{f(z)}{z - z_0}\, dz$$

can be rewritten as

$$\oint_{\Gamma'} \frac{f(z)}{z - z_0}\, dz + \oint_{\Gamma''} \frac{f(z)}{z - z_0}\, dz$$

as shown below.



- Since $\frac{f(z)}{z-z_0}$ is analytic on and inside the simple closed contour Γ', and on and inside the simple closed contour Γ', both integrals vanish by Cauchy-Goursat.

Hence

$$\oint_{\Gamma} \frac{f(z)}{(z-z_0)}\, dz = \oint_{C_\varepsilon} \frac{f(z)}{z-z_0}\, dz.$$

- The circle C_ε can be parameterized by $\gamma(t) = z_0 + \varepsilon e^{it}$, for $0 \leq t \leq 2\pi$. So $dz = \varepsilon i e^{it}$, and we have

$$\oint_{C_\varepsilon} \frac{f(z)}{z-z_0}\, dz = \int_0^{2\pi} \frac{f(z_0+\varepsilon e^{it})}{z_0+\varepsilon e^{it}-z_0}\, \varepsilon i e^{it}\, dt.$$

- Simplifying this, we get

$$\oint_{\Gamma} \frac{f(z)}{(z-z_0)}\, dz = i\int_0^{2\pi} f(z_0+\varepsilon e^{it})\, dt.$$

Taking limits as $\varepsilon \to 0$, we obtain

$$\oint_{\Gamma} \frac{f(z)}{(z-z_0)}\, dz = i\int_0^{2\pi} f(z_0)\, dt = 2\pi i f(z_0)$$

which is Cauchy's integral formula.

Computing integrals

- The Cauchy integral formula can be used to compute many contour integrals which are extremely difficult to compute by more traditional techniques.
- For instance, consider the integral $\oint_{|z|=1} \frac{e^z}{z}\, dz$ where the unit circle is traversed once anti-clockwise. The Cauchy integral formula states that this integral is $2\pi i$. If we were to compute this by parameterizing γ instead, we would obtain
$$\int_0^{2\pi} \frac{\exp(e^{it})}{e^{it}}\, ie^{it}\, dt,$$
which would simplify to the impossible-seeming integral
$$i \int_0^{2\pi} e^{\cos t}(\cos(\sin(t)) + i\sin(\sin(t)))\, dt.$$
- The theorem is phrased for anti-clockwise or positively oriented contours, but can be extended to clockwise contours simply by reversing the contour. For instance, if $-\gamma$ is the unit circle traversed once clockwise, we have
$$\oint_{-\gamma} \frac{e^z}{z}\, dz = -2\pi i.$$

Or if 2γ denotes the unit circle traversed twice anti-clockwise, we have

$$\oint_{2\gamma} \frac{e^z}{z}\, dz = 4\pi i.$$

And so forth.

- More sophisticated integrals can also be computed. For instance, to compute

$$\oint_{|z|=1} \frac{e^z}{z(z-2)}\, dz,$$

we move the $z-2$ factor onto the numerator as

$$\oint_{|z|=1} \frac{e^z/(z-2)}{z}\, dz,$$

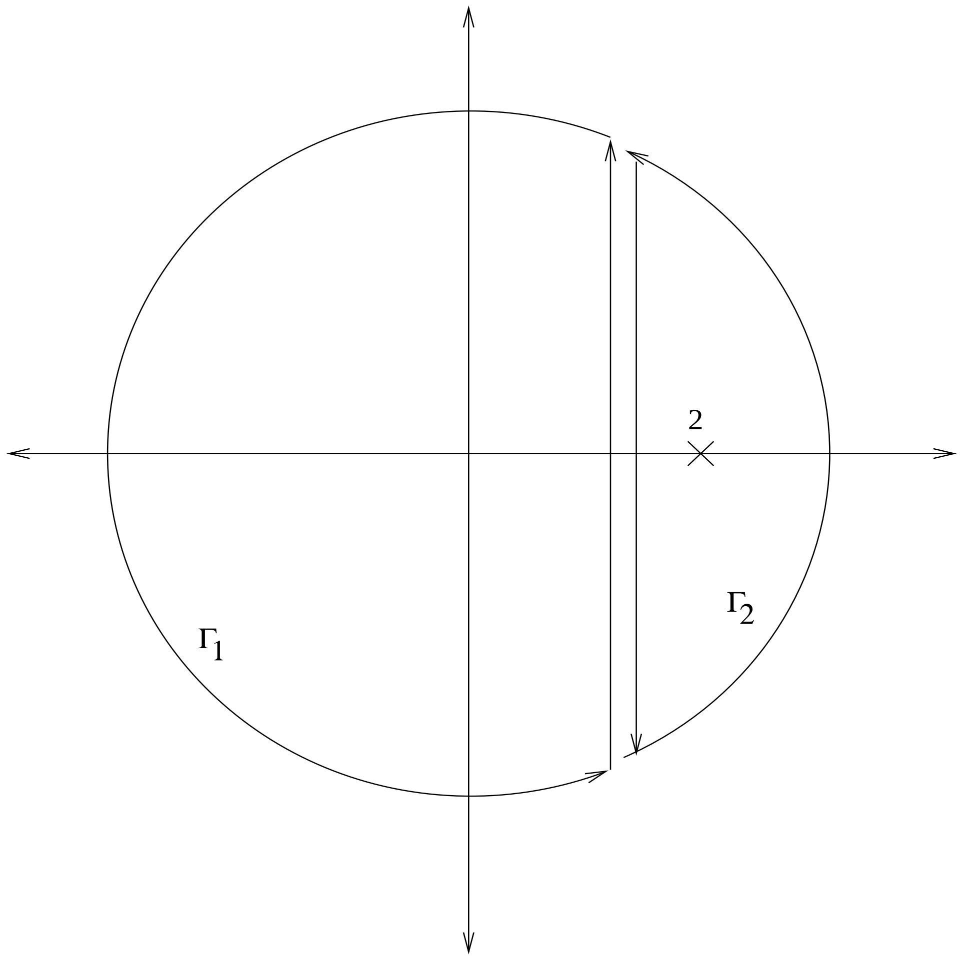
The function $e^z/(z-2)$ has a singularity at 2, but is analytic on and inside the contour $|z| = 1$. Thus the Cauchy integral formula gives this integral as

$$2\pi i e^0/(0-2) = -\pi i.$$

Now consider the integral

$$\oint_{|z|=3} \frac{e^z}{z(z-2)}\, dz.$$

The previous trick does not quite work. To resolve this integral we may use partial fractions, or alternatively divide up the circle $|z| = 3$ into two closed contours $\Gamma_1 + \Gamma_2$, where Γ_1 goes around 0 once anti-clockwise and Γ_2 goes around 2 once anti-clockwise:



As before, we have

$$\oint_{\Gamma_1} \frac{e^z/(z-2)}{z}\, dz = -\pi i,$$

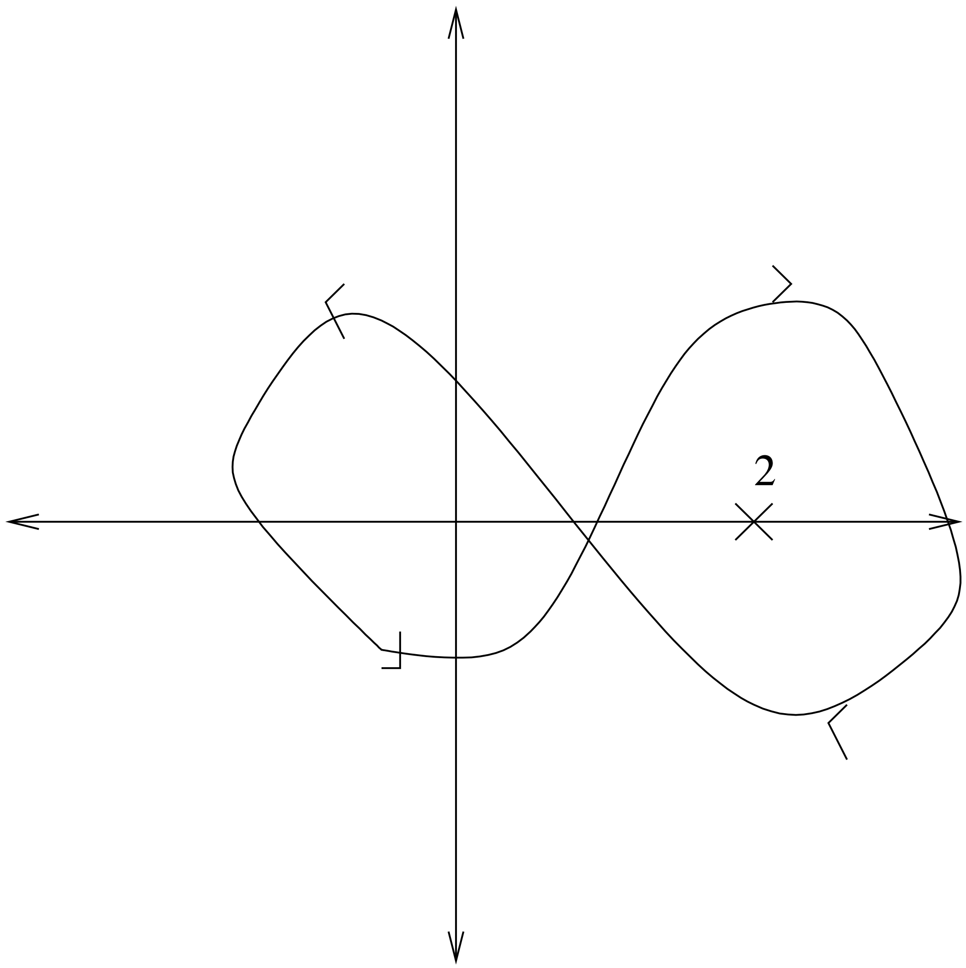
while

$$\oint_{\Gamma_2} \frac{e^z/z}{(z-2)}\, dz = 2\pi i e^2/2 = \pi i e^2.$$

Thus

$$\oint_{|z|=3} \frac{e^z}{z(z-2)}\, dz = \pi i(e^2 - 1).$$

- This trick of adding and subtracting a contour is a useful trick, and will be used often in this course.
- Now suppose we integrate the same function, but on a figure-eight contour:



The contribution of the left loop is $-\pi i$ as before, but the contribution of the right loop is now $-\pi i e^2$ instead of $\pi i e^2$ because that loop is clockwise instead of anti-clockwise. So the integral of $e^z/z(z-2)$ on this contour is $\pi i(-e^2-1)$ instead.

- As one can see, the shape of the contour is not particularly important when evaluating closed contour integrals. What is important is how the contour winds around each singularity.

Differentiating the Cauchy integral formula

- The Cauchy integral formula can compute many integrals, but there are some which are out of its reach. For instance, the integral

$$\oint_{|z|=1} \frac{\sin(z)}{z^2}\, dz$$

cannot be directly computed using the Cauchy integral formula, even if we place one of the z's in the numerator.

- To get around this, we use a trick known as "differentiating under the integral sign".

- From the Cauchy integral formula that

$$\oint_{|z|=1} \frac{\sin(z)}{z - z_0}\, dz = 2\pi i \sin(z_0)$$

for all z_0 inside the unit circle. If we differentiate this with respect to z_0 (!) we get

$$\oint_{|z|=1} \frac{e^z}{(z - z_0)^2}\, dz = 2\pi i \cos(z_0)$$

for all z_0.

- More generally, we have

- **Generalized Cauchy Integral Formula.** Let Γ be a closed positively oriented contour, let z_0 be a point in the interior of Γ, and let f be a function which is analytic on and inside Γ. Then

$$\oint_{\Gamma} \frac{f(z)}{(z-z_0)^2}\, dz = 2\pi i f'(z_0).$$

More generally, we have

$$\oint_{\Gamma} \frac{f(z)}{(z-z_0)^{m+1}}\, dz = \frac{1}{m!} 2\pi i f^{(m)}(z_0),$$

where $f^{(m)}$ denotes the m^{th} derivative of f.

- This result has an interesting consequence, though:

- **Corollary** If f is analytic at z_0, then $f^{(m)}(z_0)$ exists for all positive integers m. In other words, analytic functions are infinitely differentiable.

- Proof. Let γ be a small circle around z_0. By definition of analyticity, f is analytic on and inside γ if the radius is small enough. From the GCIF we have

$$f^{(m)}(z_0) = \frac{m!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-z_0)^{m+1}}\, dz.$$

- The integral on the right exists because $\frac{f(z)}{(z-z_0)^{m+1}}$ is continuous (in fact, it is analytic) on γ. So the left-hand side must exist as well.

- Thus, if a function is complex differentiable on a domain, it is automatically twice differentiable, three times differentiable, etc. This is very different from the situation in real analysis, and again shows how strong the property of complex differentiability is.

Example

- We can now compute integrals such as

$$\oint_{|z|=1} \frac{\sin(z)}{z^2(z-2)}\, dz.$$

  We can write this as

$$\oint_{|z|=1} \frac{f(z)}{z^2}\, dz$$

  where $f(z) = \sin(z)/(z-2)$. Since $f(z)$ is analytic on and inside $|z| = 1$, the generalized Cauchy integral formula applies, and the integral is equal to

$$2\pi i f'(0) = 2\pi i \frac{\cos(z)(z-2) - \sin(z)}{(z-2)^2}|_{z=0} = -\pi i.$$

- Integrals for more complex contours, such as $|z| = 3$, can be handled by the techniques of the previous section, or by using partial fractions. Of course, both methods give the same answer.

Morera's theorem

- Cauchy's theorem says that if f is analytic on a simply connected domain D, then the integral of f on any closed contour in D is zero. Morera (c. 1890) showed that the converse is also true:

- **Morera's theorem**. Let D be a domain, and suppose that f is a continuous function such that the integral of f on any closed contour in D is zero. Then f is analytic on D.

- Proof. If the integral of f on any closed contour is zero, then f must have an anti-derivative F on D, by the previous week's notes. Since F is analytic, it is differentiable infinitely often, by the GCIF. In particular, F' is differentiable at every point in D, hence analytic on D. But $F' = f$. Thus f is analytic on D.

- This shows that the properties of path independence, analyticity, and having an anti-derivative are virtually identical (at least on simply connected domains).

- So far, we've viewed Cauchy's integral formula mainly as a way to compute integrals. But it can also be used to deduce many surprising properties of analytic functions. We've already seen one already (that every analytic function is infinitely differentiable). Now we'll see some others.

- Perhaps the simplest consequence comes from rewriting Cauchy's formula in a different way:

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz.$$

One consequence of this formula is that the value of f at z_0 is completely determined by the value of f on γ. In other words, *the value of f on a closed contour determines the value of f inside that contour!* This shows that analytic functions are somehow very "rigid" - if one fixes the value on the boundary of a domain, this automatically fixes the values in the interior as well.

Louiville's theorem

- In real analysis there are many functions which are bounded; for instance, the function $f(x) = \sin(x)$ is bounded by 1 for all x, as is $f(x) = 1/(x^2+1)$. One might think that we could also find many bounded analytic functions in complex analysis, but there is a surprising result of Louiville, which says that there are very few such functions:

- **Louiville's theorem** If $f(z)$ is an entire function which is bounded, so that $|f(z)| \leq M$ for some M and all z, then f must be constant.

- **Proof.** Let z_0 be any point in the complex plane, and let $R > 0$ be any positive number. Since f is entire, f is analytic on the disk $\{z : |z - z_0| \leq R\}$ and is bounded by M on this disk.

- Now we use the generalized Cauchy integral formula

$$f'(z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2}\, dz.$$

If $|z - z_0| = R$, then

$$\left|\frac{f(z)}{(z-z_0)^2}\right| = \frac{|f(z)|}{R^2} \leq \frac{M}{R^2}.$$

Since the contour has length $2\pi R$, we therefore have

$$|\oint_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2}\, dz| \leq 2\pi R \frac{M}{R^2}.$$

Putting this back into the previous identity we obtain

$$|f'(z_0)| \leq \frac{M}{R}.$$

But R is arbitrary, and M is a constant. So by letting $R \to \infty$ we obtain $f'(z_0) = 0$. Since z_0 could have been any complex number, we thus see that f must be constant.

The fundamental theorem of algebra

- Consider the problem of factoring a polynomial $P(x)$ in the reals. Sometimes we are able to completely factor P into linear factors, for instance

$$x^2 - 4 = (x-2)(x+2).$$

  However, many polynomials cannot be factored over the reals, e.g. $x^2 + 4$ cannot be factored because it has no roots in the reals.

- In the complex numbers, though, the situation is much better:

- **Fundamental theorem of algebra.** Every polynomial $P(z)$ can be completely factored into linear factors in the complex numbers.

- **Proof** Let n be the degree of the polynomial P. We prove by induction on n. If $n = 1$ then clearly P is factorable into linear factors. Now suppose that $n > 1$, and every polynomial of degree $n - 1$ can already be factored into linear factors. If P has a root z_0, then we can factor $z - z_0$ from P and be left with a polynomial of degree $n-1$, which by induction can already be factored. So P can be completely factored if we can find a root.

- Now suppose that P does not have a root. We will get a contradiction from this. If $P(z)$ has no roots, then $P(z)$ is never zero, which means that $1/P(z)$ is analytic everywhere, i.e. it is entire. Now we argue that $1/P(z)$ is bounded.

- Write
$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0,$$
so that
$$1/P(z) = \frac{1}{z^n} \frac{1}{a_n + a_{n-1}/z + \ldots + a_0/z^n}.$$
As $z \to \infty$, the first factor converges to zero, and the second factor converges to $1/a_n$. So $1/P(z) \to 0$ as $z \to \infty$. This means that $|1/P(z)| \leq 1$ for sufficiently large z (e.g. $|z| > R$). For the remaining z (i.e. $|z| \leq R$), $1/P(z)$ is bounded by the extreme value theorem. (Every continuous function on a closed bounded set has a maximum and minimum, and is therefore bounded.) So $1/P(z)$ is bounded for all z. Since $1/P(z)$ is both bounded and entire, it is constant, so $P(z)$ is constant, which is a contradiction since $n > 1$.

- This is an example of a non-constructive argument - it uses a proof by contradiction to show that $P(z)$

must have a root somewhere, but doesn't give any indication as to where the root is or how to find it! In fact, for most polynomials P of degree 5 or greater, one can prove that there is no exact formula for the roots of P (in terms of familiar arithmetic operations such as addition, division, square roots, etc.)

- Later on we shall show another, more geometric proof of the Fundamental Theorem of Algebra.