Math 132 - Week 8
Textbook sections: 5.5-5.7
Topics covered:

- Zeroes and poles
- The point at infinity

Overview

- In the last two weeks we've shown that analytic functions can be written as a convergent power series. If there are singularities, then sometimes we can't write these functions as convergent power series, but we can usually still write them as convergent Laurent series.
- In this set of notes we use these power series expansions to analyze the behaviour of analytic functions near zeroes and near singularities. We'll be able to classify singularities into different categories, namely non-isolated singularities, essential singularities, poles, and removable singularities.

Zeroes of functions

- Before we classify singularities of functions, we'll first classify zeroes of functions, which are a bit easier. Later we'll see that singularities and zeroes are closely related; basically, $1/f$ has a singularity whenever f has a zero.
- A point z_0 is called a *zero* of a function f if $f(z_0) = 0$. However, this is not the end of the story; some zeroes are "stronger" than others.
- For instance, consider the polynomials $f(z) = (z-1)(z+2)$ and $g(z) = (z-1)^2(z+2)$. Both f and g have zeroes at 1, but the g has a "double zero" at 1 whereas f only has a "single" zero. Another way of saying this is that both $f(z)$ and $g(z)$ converge to zero as $z \to 1$, but g converges "twice as quickly" as f. For instance, $f(1.01) \approx 0.02$, but $g(1.01) \approx 0.0002$, and so forth.

- More precisely, if $f(z)$ is a polynomial, we say that f has a *zero of order* k at z_0 if you can divide out k factors of $(z - z_0)$ and still be continuous at z_0, but you can't factor out $k + 1$ factors of $(z - z_0)$ without introducing a discontinuity at z_0.

- In other words, if

$$\lim_{z \to z_0} \frac{f(z)}{(z - z_0)^k} \text{ exists, but}$$

$$\lim_{z \to z_0} \frac{f(z)}{(z - z_0)^{k+1}} \text{ does not exist}$$

  we say that f has a zero of order k at z_0.

- Thus, $(z - 1)(z + 2)$ has a zero of order 1 at 1, and $(z - 1)^2(z + 2)$ has a zero of order 2 at 1. Both functions have zeroes of order 1 at -2.

- Zeroes of order 1 are also called *simple zeroes* or *single zeroes*; zeroes of order 2 are also called *double zeroes*; zeroes of order 3 are called *triple zeroes*, and so forth. Thus $(z - 1)^2(z + 2)$ has a double zero at 1 and a simple zero at -2.

- Somewhat confusingly, if $f(z_0) \neq 0$ then by the above definition, f will have a zero of order 0; e.g. $(z - 1)^2(z + 2)$ has a zero of order 0 at 3. So a zero of order 0 isn't actually a zero!

- The above definition works for all analytic functions, and not just polynomials. For instance, consider the function $f(z) = \sin(z)$ at $z_0 = 0$. This function has a zero at 0; to work out what order of zero it has, we use the Taylor expansion of $f(z)$ around 0:

$$f(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$$

  We can divide out one factor of $(z - 0) = z$ and still be continuous at $z = 0$:

$$\frac{f(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots$$

  Even though $\frac{f(0)}{0}$ is undefined, we clearly see that $\frac{f(z)}{z}$ converges to 1 as $z \to 0$.

- However, if one divides out by two powers of z then the function develops a singularity at 0:

$$\frac{f(z)}{z^2} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \ldots$$

Because of this, we see that $\sin(z)$ has a simple zero at $z_0 = 0$.

- More generally, if f is analytic at z_0, the order of the zero at z_0 is equal to the smallest number k for which the $(z - z_0)^k$ co-efficient does not vanish. For instance, the function

$$f(z) = 2(z+1)^3 + 3(z+1)^4 + 4(z+1)^5 + \ldots$$

has a triple zero at -1.

- Let's return to the example $f(z) = (z-1)^2(z+2)$. The Taylor expansion around 1 is

$$f(z) = 3(z-1)^2 + (z-1)^3$$

while the Taylor expansion around -2 is

$$f(z) = 9(z+2) - 6(z+2)^2 + (z+2)^3.$$

Thus we have a double zero at 1 and a simple zero at -2.

- From Taylor's formula, the $(z-z_0)^k$ co-efficient of the Taylor expansion of f at z_0 is equal to $f^{(k)}(z_0)/k!$. Thus, the order of the zero of f at z_0 is equal to the first k such that $f^{(k)}(z_0) \neq 0$.

- For instance, take $f(z) = \sin(z)$. We have $f(0) = 0$, but $f'(0) = 1 \neq 0$, so f has a simple zero at 0.

- Or, take $f(z) = (z-1)^2(z+2)$. One can calculate $f'(z) = 2(z-1)(z+2) + (z-1)^2$ and $f''(z) = 2(z+2) + 2(z-1) + 2(z-1)$. Thus $f(1) = f'(1) = 0$ but $f''(1) \neq 0$, so f has a double zero at 1.

- Or, take $f(z) = 1 - \cos(z)$. We have $f(0) = 0$, $f'(0) = \sin(0) = 0$, but $f''(0) = \cos(0) = 1 \neq 0$, so f has a double zero at 0.

- If $f(z)$ has a zero of order k at z_0, and $g(z)$ has a zero of order l at z_0, then $f(z)g(z)$ has a zero of order $k+l$ at z_0. This is because when you multiply a power series of the form

$$a_k(z-z_0)^k + \text{ higher order terms}$$

with

$$b_l(z-z_0)^l + \text{ higher order terms}$$

you get

$$a_k b_l(z-z_0)^{k+l} + \text{ higher order terms.}$$

- For instance, we already know that z and $\sin(z)$ both have simple zeroes at 0. Thus $z\sin(z)$ has a double zero at 0, $z\sin^2(z)$ has a triple zero at 0, etc. Similarly, $(1-\cos(z))^{10}$ has a zero of order 20 at 0.

- Note that the order of a zero is always an integer; there's no such thing as a zero of order 1/2. (One might think that, e.g. $p.v.z^{1/2}$ ought to have a zero of order 1/2 at 0, but this function is not analytic at 0, because of the branch cut!)

- By convention, the zero function $f(z) = 0$ has a zero of infinite order at every point (one can factor out as many powers of $(z - z_0)$ as one pleases!) Later on, we shall see that this is the only function which can have zeroes of infinite order.

- If $f(z)$ has a zero of order k at z_0, then we can factor out exactly k copies of $(z-z_0)$, and obtain a factorization

$$f(z) = (z-z_0)^k g(z)$$

where $g(z)$ is analytic and non-zero at z_0. For instance, since $\sin(z)$ has a simple zero at 0, we may factor

$$\sin(z) = zg(z)$$

where g is the function

$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots.$$

Since $\sin(z)$ has an infinite radius of convergence, g also has an infinite radius of convergence, and thus g is entire. Also, $g(0) = 1 \neq 0$, so we cannot factor out any further powers of z.

L'hopital's rule

- One can use the theory of zeroes to prove
- **L'hopital's rule.** Let $f(z)$ and $g(z)$ be functions which are analytic and zero at z_0. Then

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f'(z)}{g'(z)}.$$

- Let k be the order of the zero of f at z_0, and l be the order of the zero of g at z_0, so

$$f(z) = a_k(z - z_0)^k + \text{ higher order terms}$$

$$g(z) = b_l(z - z_0)^l + \text{ higher order terms}$$

  where a_k, b_l are non-zero numbers.

- If $k > l$, then the left-hand limit is 0; if $k < l$, the left-hand limit is infinite. If $k = l$, then the left-hand limit is a_k/b_l.
- Differentiating the above power series, we get

$$f'(z) = ka_k(z - z_0)^{k-1} + \text{ higher order terms}$$

$$g'(z) = lb_l(z - z_0)^{l-1} + \text{ higher order terms}$$

- If $k > l$, then the right-hand limit is 0, if $k < l$, then the right-hand limit is infinite. If $k = l$, then the right-hand limit is $ka_k/lb_l = a_k/b_l$. Thus in all cases the limits are equal.

Isolated and non-isolated singularities

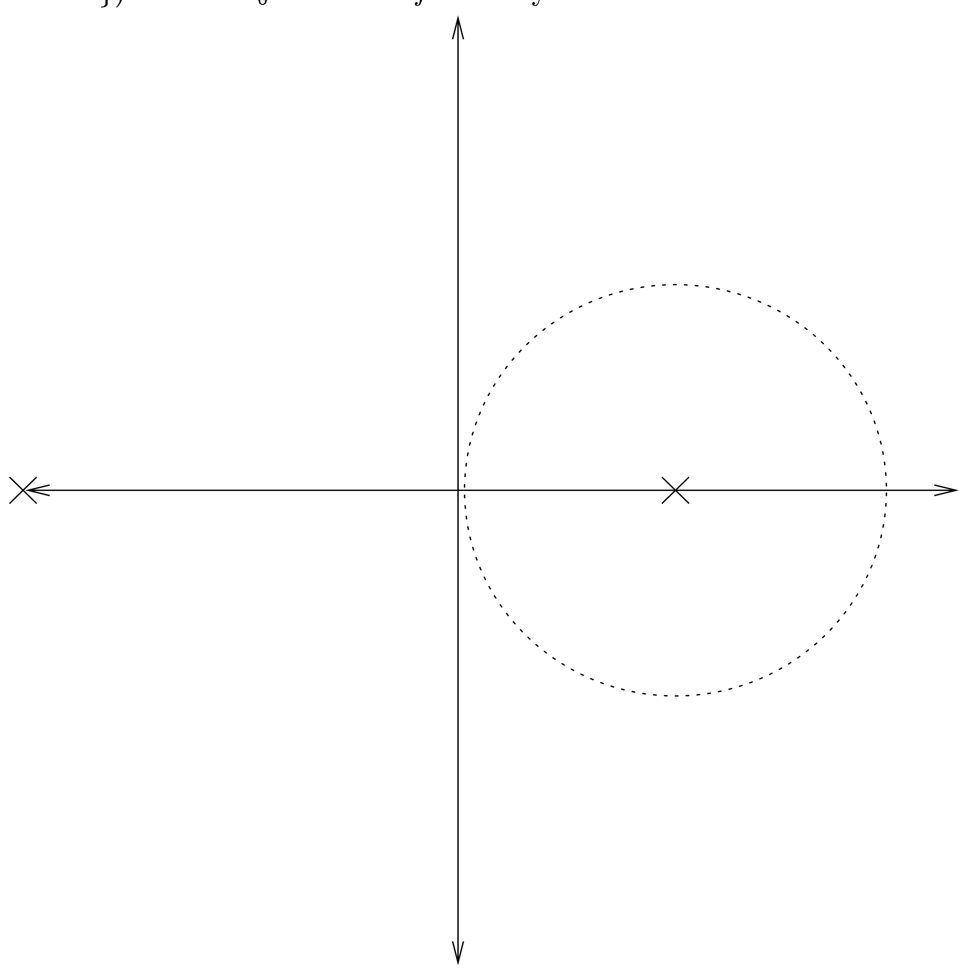
- Now that we've studied zeroes of functions, let's study the singularities of functions.
- **Definition** We call a point z_0 a *singularity* of a function f if f is not analytic at z_0.

- The first major distinction we have to make is between an isolated and non-isolated singularity.

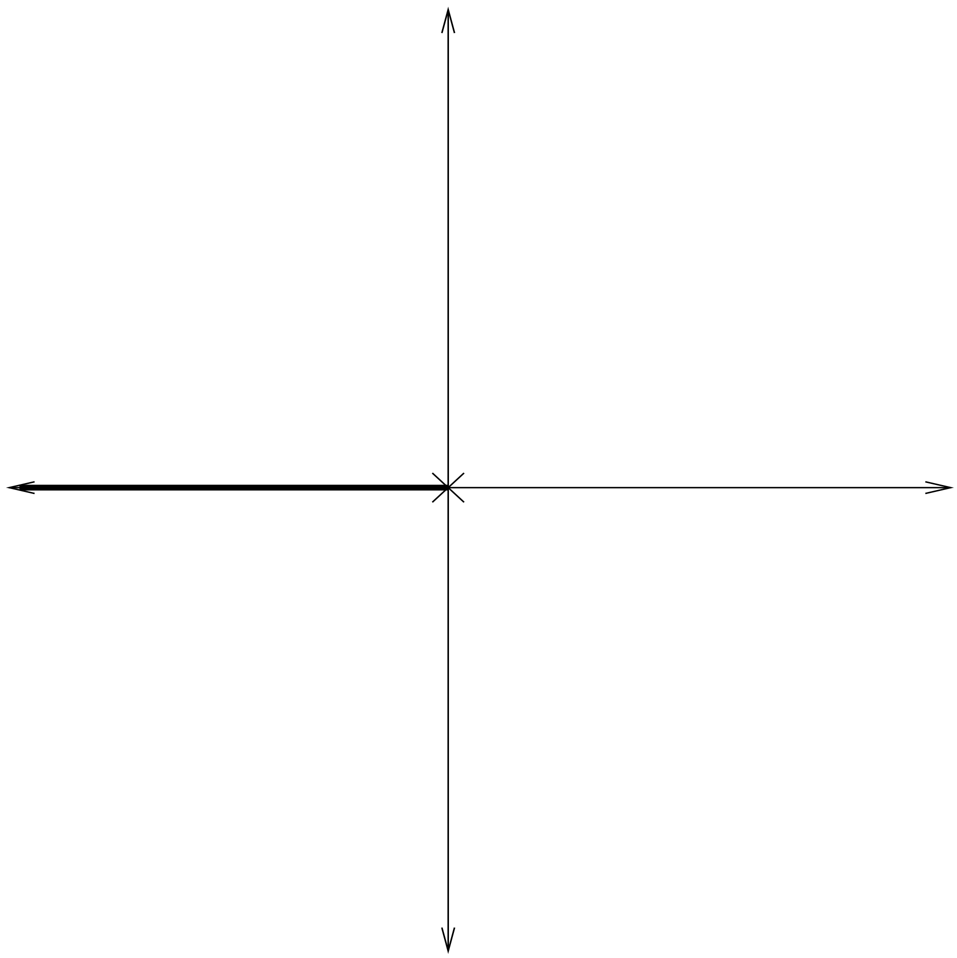
- **Definition.** A singularity z_0 of a function f is said to be *analytic* if we can find a radius $r > 0$ such that f is analytic on the punctured disk $\{z : 0 < |z - z_0| < r\}$.

- Another way of saying this is that a singularity is isolated if it is separated from all other singularities of f by a non-zero distance $r > 0$.

- For instance, the function $f(z) = \frac{1}{(z-1)^2(z+2)}$ has an isolated singularity at $z_0 = 1$, because we can find a punctured disk (e.g. $\{z : 0 < |z - 1| < 1\}$) around z_0 on which f is analytic.



- On the other hand, the function $f(z) = \text{Log}(z)$ has a non-isolated singularity at 0, because no matter how small one chooses the radius r, the punctured disk $\{z : 0 < |z| < r\}$ must contain singularities other than 0.



- If $f(z)$ has an isolated singularity at z_0, then f is analytic on an annulus $\{z : 0 < |z - z_0| < r\}$, and therefore we have a convergent Laurent series around z_0 in this region. We shall abbreviate notation and call this series *the* Laurent series of f around z_0, although strictly speaking there may be other annuli around z_0 which have Laurent series.

Removable singularities

- Isolated singularities can be divided into three classes: removable singularities, poles, and essential singularities. We first study the concept of a removable singularity.

- You should already have encountered the concept of a *removable discontinuity* in real analysis. This occurs when a real-variable function $f(x)$ is not continuous at x_0, but can be made continuous by re-defining $f(x)$ at x_0. An example is the function $f(x) = \frac{\sin(x)}{x}$. This function is not defined at $x = 0$ and so has a discontinuity there, however if one redefines f at 0 as

$$f = \begin{cases} \sin(x)/x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

  then f becomes continuous at 0. Since this discontinuity can be so easily removed, we call it a removable discontinuity.

- Not all discontinuities are removable this way; for instance the function $1/x$ has a discontinuity at 0 no matter how one redefines it at 0.

- Similarly, in complex analysis we say that a function f has a *removable singularity at* z_0 if f has an isolated singularity at z_0, but f can be made analytic by redefining f at z_0.

- For instance, consider the function $f(z) = \frac{1-\cos(z)}{z^2}$. This function is undefined at $z = 0$, but is otherwise analytic, so we have an isolated singularity at 0. To figure out whether it is removable, we compute the Laurent expansion of $f(z)$ around 0. Starting with

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$$

  we have

$$1 - \cos(z) = \frac{z^2}{2!} - \frac{z^4}{4!} + \ldots$$

  and thus

$$f(z) = \frac{1 - \cos(z)}{z^2} = \frac{1}{2!} - \frac{z^2}{4!} + \ldots$$

  for $|z| > 0$.

- This Laurent series contains no negative powers of z, and is thus a power series, and hence analytic on its disk of convergence (which in this case is the entire complex plane). Since the Laurent series matches $f(z)$ for all z except the origin, we have thus demonstrated that f has a removable singularity at 0.

- More generally, if $f(z)$ has an isolated singularity at z_0 and the Laurent expansion of $f(z)$ around z_0 has no negative power terms, then f has a removable singularity. If the Laurent expansion contains at least one negative power term, then the singularity cannot be removed, because otherwise f would be given by a Taylor series with no negative power terms, a contradiction since Laurent series are unique.

- If a function $f(z)$ has a removable singularity at z_0, then it is possible that $f(z)$ has a zero of some order after the singularity is removed. For instance, consider the function $f(z) = \frac{1-\cos(z)}{z}$. This function has an isolated singularity at 0; its Laurent expansion around 0 is

$$f(z) = \frac{z}{2!} - \frac{z^3}{4!} + \dots$$

The power series on the right-hand side has a simple zero at 0. Thus $f(z)$ has a simple zero at 0 once the singularity is removed.

Poles

- Suppose $f(z)$ is a function with an isolated singularity at z_0. Even if this singularity is not removable, it may be possible to remove it after multiplying f by enough powers of $(z - z_0)$. (The function $(z - z_0)$ is zero at z_0, and so multiplying by $(z - z_0)$ should make the singularity "better").

- For example, consider the function $f(z) = \frac{1-\cos(z)}{z^6}$. This function has a singularity at 0, with Laurent expansion

$$f(z) = \frac{1}{2!z^4} - \frac{1}{4!z^2} + \frac{1}{6!} - \dots$$

Because the Laurent expansion has negative powers, this singularity is not removable. However, if we multiply f by four powers of z, then it the resulting function does have a removable singularity at 0:

$$z^4 f(z) = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \ldots$$

If we multiply by any fewer powers of z, we cannot completely remove the singularity. Because of this, we say that this function has a *pole of order 4*, or a *quadruple pole.*

- More generally, if the Laurent expansion of a function f around an isolated singularity z_0 has the form

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \text{ higher order terms}$$

where $a_{-k} \neq 0$, then we say that f has a *pole of order k*. For instance, the Laurent series

$$\frac{3}{(z+i)^3} + \frac{2}{(z+i)^2} + \frac{1}{z+i} + 0 + (-1)(z+i) + \ldots$$

has a triple pole at $-i$. More generally, the order of a pole is equal to the largest k which has a non-zero $(z - z_0)^{-k}$ term in the Laurent expansion.

- Removable singularities are also poles of order 0, although they are generally not considered poles (the same way zeroes of order 0 are not considered zeroes).

- If f has a pole of order k at z_0, then we can write $f(z) = \frac{g(z)}{(z-z_0)^k}$, where

$$g(z) = a_{-k} + \text{ higher order terms}$$

is a convergent power series near z_0, and is hence an analytic function at z_0, with $g(z_0) = a_{-k} \neq 0$. In other words, poles of order k arise by dividing a non-zero analytic function by $(z - z_0)^k$. (Compare this to zeroes of order k, which arise by *multiplying* a non-zero analytic function by $(z - z_0)^k$).

- It is easy to see what happens when multiplying or dividing poles and zeroes together. We already know that a zero of order k multiplied by a zero of order l is a zero of order $k+l$. Similarly, a pole of order k multiplies by a pole of order l is a pole of order $k+l$. If one wants to multiply a pole of order k by a zero of order l, there will be some cancellation, but the end result will depend on whether k is larger than, equal to, or less than l.

- For instance, suppose $f(z)$ has a triple pole at 2 and $g(z)$ has a double zero at 2. This means that f is a non-zero analytic function divided by $(z-2)^3$, while $g(z)$ is a non-zero analytic function multiplied by $(z-2)^2$. Multiplying the two together we see that $f(z)g(z)$ is a non-zero analytic function divided by $(z-2)$, which gives a simple pole at 2.

- Take the same example, and now consider $f(z)g(z)^2$. When $z \neq 2$ this is a non-zero analytic function multiplied by $(z-2)$; when $z=2$ this is undefined because $f(2)$ is undefined. This means that $f(z)g(z)^2$ has a removable singularity at 2, and once the singularity is removed we get a simple zero.

- By similar arguments, one can divide one function by another and determine whether one gets a pole or zero as a result. The reciprocal of a zero of order k is a pole of order k, and conversely

- Once one gets the hang of this, computing the order of a pole or zero can become quite quick. For instance, to figure out the nature of the singularity of
$$\frac{e^z(1-\cos(z))^2}{\sin(z)^3 z^3}$$
at zero, we note that e^z is non-zero at 0 (hence a zero of order 0), $(1-\cos(z))$ has a double zero, while $\sin(z)$ and z both have simple zeroes. This gives a total of four zeroes in the numerator and six zeroes in the denominator, so the function as a whole has a double pole at 0. (Alternatively, one could try to compute the Laurent series of the above function around 0, and find the most negative order term, but this will take a long time).

Essential singularities

- There are some singularities which are neither removable or poles. For instance, consider the function $f(z) = e^{1/z}$. This function has a singularity at zero, and its Laurent expansion around zero is

$$f(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots$$

- Because of the presence of negative powers in the Laurent expansion, this function does not have a removable singularity at 0. The singularity is not a pole either, because no matter how many times one multiplies by z, one cannot eliminate all the negative powers of z and make the singularity removable. For instance, if one multiplies by z^3, one still gets negative power terms:

$$z^3 f(z) = z + z^2 + \frac{z}{2!} + \frac{1}{3!} + \frac{1}{4!z} + \ldots$$

  Because of this, we say that $e^{1/z}$ has an *essential singularity* at 0.

- More generally, whenever the Laurent expansion of $f(z)$ around z_0 contains an infinite number of negative power terms, we say that f has an essential singularity. (If it has a finite number of negative terms, then the singularity is a pole, and if it has no negative terms, the singularity is removable). Thus every isolated singularity is of exactly one type: removable, pole, or essential.

- Suppose $f(z)$ had an essential singularity at 0, and $g(z)$ had (for instance) a double zero at 0. Then $f(z)g(z)$ cannot have a removable singularity, since by dividing by $g(z)$ this would mean that $f(z)$ would be a double pole or better, a contradiction. $f(z)g(z)$ cannot be a pole of order k, since by dividing by g this would mean that f would be a pole of order $k+2$, a contradiction. By elimination, we conclude that $f(z)g(z)$ has an essential singularity at 0.

- More generally, if you multiply or divide an essential singularity by a pole, removable singularity, or zero, you still get an essential singularity (kind of like how if you add or subtract any finite number from infinity, you still get infinity). Also, the reciprocal of an essential singularity is again an essential singularity (because it can't be anything else).

- However, when you multiply or divide two essential singularities together, anything can happen (it's somewhat like subtracting infinity from infinity). For instance, $e^{1/z}$ and $z^2 e^{-1/z}$ both have essential singularities at 0, but when you multiply them together you get a double zero at 0 (after the singularity is removed).

Limiting behaviour at singularities

- Suppose $f(z)$ has an isolated singularity at z_0. We ask the question of how $f(z)$ behaves as $z \to z_0$. The answer depends on what kind of singularity one has at z_0.

- If z_0 is a removable singularity, then f can be redefined to be analytic, hence continuous at z_0. Thus $\lim_{z \to z_0} f(z)$ exists and is finite.

- If z_0 is a pole of order k, then $f(z) = g(z)/(z - z_0)^k$ for some non-zero analytic function $g(z)$. As $z \to z_0$, the numerator of $g(z)/(z - z_0)^k$ tends to some non-zero number, whereas the denominator tends to zero. Thus we have $f(z) \to \infty$ as $z \to z_0$. Roughly speaking, the higher the order of the pole, the faster the rate at which f goes to infinity.

- If z_0 is an essential singularity, then the behaviour becomes very strange. For instance, consider the essential singularity of $f(z) = e^{1/z}$ at $z_0 = 0$. If we approach 0 from the right, f goes to ∞ very quickly:

$$\lim_{x \to 0^+} e^{1/x} = \infty.$$

  On the other hand, if we approach 0 from the left, f goes to 0 very quickly:

$$\lim_{x \to 0^-} e^{-1/x} = -\infty.$$

  If we approach 0 from above, f spins round the unit circle infinitely often

$$\lim_{y \to 0^+} e^{1/(iy)} = \lim_{y \to 0^+} \cos(\frac{1}{y}) - i \sin(\frac{1}{y} = \text{ does not exist}$$

  and so forth. In fact, by choosing the approach path appropriately, one can make $e^{1/z}$ exhibit just about any kind of behaviour one wants, although one can never make it equal zero.

- More generally, there is a very difficult theorem known as the Great Picard Theorem, which states that near an essential singularity, a function takes on every value in the complex plane, with at most one exception. Thus, for instance, near zero, $e^{1/z}$ can equal any complex number other than zero. We won't prove the Great Picard Theorem here, as the proof is extremely complicated.

Using singularity theory to compute Laurent expansions

- To find the Laurent series of a function near a singularity, it sometimes helps to first work out the nature of the singularity. For instance, suppose we want to work out the Laurent expansion of

$$f(z) = \frac{1}{e^z - 1}$$

around $z = 0$. We expand the denominator as

$$e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

- Thus $e^z - 1$ has a simple zero at 0, so f has a simple pole at 0:

$$f(z) = \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \ldots$$

- To work out the co-efficients, we multiply the above two equations together to get

$$1 = (\frac{a_{-1}}{z} + a_0 + a_1 z + \ldots)(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots)$$

- Multiplying the series together and comparing co-efficients we get

$$a_{-1} = 1$$

$$\frac{a_{-1}}{2!} + a_0 = 0$$

$$\frac{a_{-1}}{3!} + \frac{a_0}{2!} + a_1 = 0$$

etc., and one can solve for a_{-1}, then a_0, then a_1, etc. recursively.

- This function has singularities at integer multiples of $2\pi i$, so is analytic in the annulus $\{0 < |z| < 2\pi\}$ and not analytic in any larger annulus. Thus this annulus is the largest region on which the above Laurent series converges.

The point at infinity

- This section will be rather informal.

- We've looked at many limits of the form

$$\lim_{z \to z_0} f(z)$$

where z_0 is a complex number. This measures how f behaves as z approaches a finite number z_0. However, it's also useful to find out how f behaves near infinity, and to try to compute a limit such as

$$\lim_{z \to \infty} f(z)$$

One can convert limits at infinity to limits at finite points (such as 0), e.g. by the identity

$$\lim_{z \to \infty} f(z) = \lim_{z \to 0} f(\frac{1}{z}).$$

- Because of things like this, it is sometimes convenient to treat ∞ on equal footing with other complex numbers.

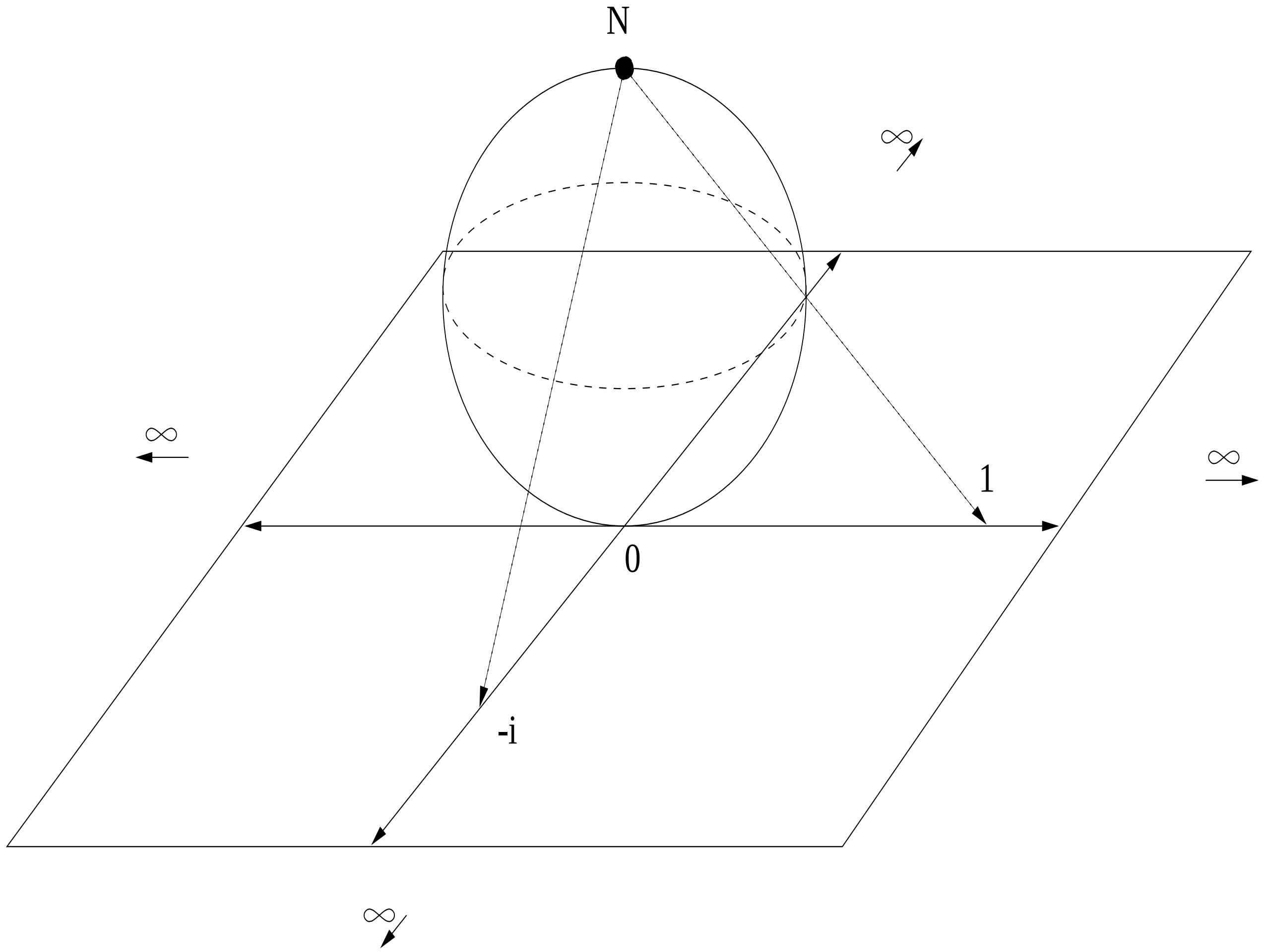
- **Definition.** The *extended complex plane* is the complex plane $\mathbf{C}$, union with the single point $\{\infty\}$, which is referred to as *the point at infinity.*

- Of course, we can't depict the extended complex plane accurately in the usual Cartesian plane, since ∞ is infinitely far away. To visualize the extended complex plane we use a different model based on the *stereographic projection.*

- Think of the complex plane as embedded in three-dimensional space $\mathbf{R}^3$, so that each complex number $x + yi$ becomes the point $(x, y, 0)$. Now consider the sphere S of radius $\frac{1}{2}$ centered at $(0, 0, \frac{1}{2})$:

$$S = \{(x, y, z) : x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}\}.$$

We call $(0, 0, 1)$ the *north pole* of S.



- We can map the complex plane **C** to S by the following procedure. For every point z in **C**, join the line segment from the north pole to z, and find out where this intersects S. This maps points in **C** to points in S. We can reverse this process: if x is any point in S other than the north pole, we can join the ray from the north pole passing through x, and find the point for which this ray hits **C**. Thus we can make a one-to-one map between the complex plane **C** and the sphere S with the north pole removed.

- We can fill in the hole in the north pole, and can make a one-to-one map between the extended complex plane to the sphere S by mapping the point at infinity to the north pole. So the extended complex plane can be visualized as a sphere.

- (The exact map is given by the formula

$$x + yi \mapsto (\frac{x}{x^2+y^2+1}, \frac{y}{x^2+y^2+1}, \frac{x^2+y^2}{x^2+y^2+1}),$$

  but this formula isn't really worth remembering).

- There are several nice properties of this projection. For instance, if we have a circle or line on the complex plane, and use the stereographic projection to map this to the sphere, then we always get a circle. (Strictly speaking, we have to adopt the convention that lines include the point at infinity, otherwise the projected circle will be missing one point at the north pole). Conversely, if we take any circle on the sphere, and project it back to the plane, we either get a circle or line. (Depending on whether the original circle went through the north pole or not). It is sometimes helpful to think of a line as a circle of infinite radius (and center infinitely far away from the origin).

- The map is *conformal*, which means it preserves angles. If two curves in $\mathbf{C}$ intersect at angle θ, their stereographic projections to the sphere also intersect at angle θ.

- Also, some maps on the complex plane become simpler using the stereographic projection. Back in week 1 we studied the inversion map $z \mapsto 1/z$, which took a point with magnitude r and phase θ, and mapped it to a point with magnitude $1/r$ and phase $-\theta$. On the sphere, the inversion map becomes a 180 degree rotation around the east-west axis

$$\{(t, 0, 1/2) : t \in \mathbf{R}\}.$$

- Note that we don't make any distinction between $+\infty$ or $-\infty$ as one does with the real line; there is just a single point at infinity, and all directions in the complex plane eventually lead to this point.

Singularities at infinity

- We know how to classify singularities at any point z_0 on the complex plane into removable singularities, poles, and essential singularities. But it is also convenient sometimes to classify the behaviour of a function at infinity - whether one has a removable singularity, pole, or essential singularity at ∞.

- The rule is simple: the classification of the singularity of $f(z)$ at ∞ is the same as that of the singularity of $f(1/z)$ at 0.

- For instance, consider the function

$$f(z) = \frac{z^2 + 1}{z}.$$

To find out what type of singularity f has at ∞, we look at $f(1/z)$:

$$f(1/z) = \frac{(1/z)^2 + 1}{1/z} = \frac{1 + z^2}{z}.$$

The numerator is non-zero at 0, and the denominator has a simple zero at 0, so $f(1/z)$ has a simple pole at 0, thus $f(z)$ has a simple pole at infinity.

- Or, take the function

$$f(z) = \frac{z}{z^2 + 1}.$$

The function $f(1/z)$ is

$$f(1/z) = \frac{(1/z)^2 + 1}{1/z} = \frac{z}{z^2 + 1}$$

for $z \neq 0$. Of course, $f(1/z)$ is not defined at $z = 0$, but the function $\frac{z}{z^2+1}$ has a simple zero at 0. Thus $f(z)$ has a removable singularity at ∞, with a simple zero once the singularity is removed.

- As a last example, consider the function $f(z) = e^z$. Even though this function has no singularities in the complex plane (it is entire), it has quite a nasty singularity at ∞. The function $f(1/z)$ is just

$$f(1/z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots,$$

which has an essential singularity at 0. Thus f has an essential singularity at ∞.

- As one can expect from the definition, singularities at ∞ are much like singularities at other points, except that the roles of $z - z_0$ is now played by $1/z$.

- Just as the type of singularity at a finite point z_0 influences the behaviour of $f(z)$ as $z \to z_0$, the type of singularity at ∞ influences the behaviour of $f(z)$ as $z \to \infty$. For instance, if $f(z)$ has a removable singularity at ∞ then
$$\lim_{z\to\infty} f(z)$$
exists. If f has a zero of order m at ∞, this means that
$$\lim_{z\to\infty} z^m f(z)$$
exists, but that
$$\lim_{z\to\infty} z^{m+1} f(z)$$
doesn't; another way of saying this is that $f(z)$ decays like $1/z^m$ as $z \to \infty$.

- If f has a pole of order m at ∞, this means that
$$\lim_{z\to\infty} f(z)/z^m$$
exists, but
$$\lim_{z\to\infty} f(z)/z^{m-1}$$
does not. Another way of saying this is that $f(z)$ grows like z^m as $z \to \infty$.

- Finally, if f has an essential singularity at ∞ then the behaviour as $z \to \infty$ is quite wild, and depends on exactly what path one takes to go to ∞. Along one path it might converge to a limit; along another path it might go off to infinity; and on a third path it might just oscillate infinitely often.

- Without the point at infinity, the number of zeroes and poles of a function do not have to agree. However, adding ∞ happens to balance these out:

- **Theorem** Let f be a function which has a finite number of poles and zeroes in the extended complex plane, but no other singularities. Then the total number of poles of f is equal to the total number of zeroes of f.

- (Proof omitted).