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Boundedness by 3 of the Whitney Interpolation Constant

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TO BL. SENDOV ON THE OCCASION OF HIS SEVENTIETH BIRTHDAY

Let the function $f \in C[0,1]$ satisfy $f(\frac{j}{k-1}) = 0$, $j = 0, \ldots, k-1$. We prove the estimate

$$\sup_{x \in [0,1]} |f(x)| \leqslant 3 \sup_{x, x+kh \in [0,1]} \left| \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+jh) \right|.$$

Key Words: Whitney constant; interpolation; modulus of smoothness.

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1. INTRODUCTION AND FORMULATION OF MAIN RESULT

Let C be the space of continuous functions f on $I := [0, 1]$ equipped with the uniform norm

$$||f|| := \max_{x \in I} |f(x)|.$$

Everywhere below $f \in C$ and $k \in N, \ k > 1$. For a function f, we denote the kth difference with a step h by

$$\Delta_h^k f(x) := \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh),$$

and the kth modulus of continuity at the point $1/k$ by

$$\omega_k(f, 1/k) := \sup_{x, x+kh \in I} |\Delta_h^k f(x)|.$$

Let $L_{k-1}(f, x)$ be the Lagrange polynomial of degree $\leqslant k - 1$, which interpolates f at the equidistant points $x_m := m/(k-1)$, i.e.,

$$f(x_m) = L_{k-1}(f, x_m), \qquad m = 0, \ldots, k-1.$$

Whitney interpolation constants are defined by

$$W'(k) := \sup_{f \in C} \frac{||f - L_{k-1}(f, \cdot)||}{\omega_k(f, 1/k)},$$

where the supremum is taken over all functions $f \in C$ which are not algebraic polynomials of degree less than k.

First results that concern $W'(k)$ were given by Burkill [2] and Whitney [11]. Burkill noticed that $W'(2) = 1$ and conjectured that $W'(k)$ are finite numbers for all k. Whitney proved this conjecture and estimated $W'(k)$ for $k \leqslant 5$. His attempts to obtain a good estimate for $W'(k)$ led to inequalities $\frac{16}{15} \leqslant W'(3) \leqslant \frac{14}{9}$, $W'(k) \geqslant 1$, and to a conclusion that the problem of finding $W'(k)$ is probably extremely difficult.

Sendov [6] conjectured that the constants $W'(k)$ are bounded by two. For $k = 4$ this conjecture has been confirmed by Danilenko [3], and for $k = 5, 6, 7$ by Zhelnov [12].

We have the following history of $W'(k)$ estimates for all k. Sendov and Popov [8, Chap. 2, Theorem 25] deduced the estimate $W'(k) = O(\ln k)$ from the Sendov [7] integral representation. Takev [10] applied this representation to prove the inequality $W'(k) < 36$. Kryakin and Takev [5] used a new, so called “interpolation in the average” method, and a modified integral representation [4] to prove the estimate $W'(k)<5$. In a different way Bojanov [1] obtained the inequality $W'(k)<6$. Shevchuk [9] announced that $W'(k)\leqslant\pi$. Lemma 1.1 (see below) is an essential part of his unpublished proof.

The main result of this paper is the following.

THEOREM 1.1. *For $k>1$ we have*

$$W'(k)\leqslant 3.$$

Since the inequality

$$|f(x)-L_{k-1}(f,x)|\leqslant\omega_k(f,1/k),\qquad x\in[1/k,1-1/k],$$

is well-known (see, for example, estimates in [8]), we only have to prove that

$$|f(x)-L_{k-1}(f,x)|\leqslant 3\omega_k(f,1/k),\qquad x\in[0,1/k). \tag{1.1}$$

To obtain (1.1) we shall use the method that was proposed in [5]. This method is connected with the intermediate approximation by polynomials Q_{k-1} such that

$$\int_0^{i/k}(f(t)-Q_{k-1}(f,t))\,dt=0,\qquad i=1,\ldots,k.$$

By using the notation $g(x):=f(x)-Q_{k-1}(f,x)$ we get

$$\begin{aligned}|f(x)-L_{k-1}(f;x)|&\leqslant|f(x)-Q_{k-1}(f,x)-L_{k-1}(f,x)+Q_{k-1}(f,x)|\\&\leqslant|f(x)-Q_{k-1}(f,x)|+|L_{k-1}(f-Q_{k-1},x)|\\&\leqslant|g(x)|+\left|\sum_{m=0}^{k-1}g(x_m)l_m(x)\right|,\end{aligned}$$

where

$$l_m(x)=\prod_{\substack{j=0\\j\neq m}}^{k-1}((k-1)x-j)/(m-j),\qquad m=0,1,\ldots,k-1.$$

Thus our problem is to estimate the value of $|g(x)|$ for $x\in[0,1/k)$, and in points $x_m,\ m=0,\ldots,k-1$. For this purpose we will use the following.

LEMMA 1.1. *If* $\omega_k(f, 1/k) \leqslant 1$ *and* $m < k/2$, $x \in [m/k, (m+1)/k]$, $\delta = (1-x)/(k-m)$, *then*

$$\binom{k}{m}|g(x)| \leqslant 1 + (k\delta)^k - (-1)^{k-m}\binom{k}{m}A_k'(x) + \frac{2}{\delta}\sum_{j=0}^{m-1}\binom{k}{j}\frac{1}{m-j}|A_k(x+\delta(j-m))|,$$

where

$$A_k(x) := \frac{k^k}{k!}x\left(x-\frac{1}{k}\right)\left(x-\frac{2}{k}\right)\cdots\left(x-\frac{k}{k}\right) = x(-1)^k\prod_{j=1}^{k}\left(1-\frac{kx}{j}\right).$$

The paper is organized as follows. In Section 2, we give the proof of Lemma 1.1. Section 3 is devoted to the proof of Theorem 1.1. Interpolation in the mean and estimates for classical Whitney constants $W(k)$ are considered in Section 4. One can read Section 4 directly after Section 2.

2. PROOF OF LEMMA 1.1

We need the next two well-known lemmas from [4, 13]. For reader's convenience we give also the proofs. To this end we put $F(x) := \int_0^x f(u)\,du$ and apply the identity

$$\int_0^1 f(x_1 + (x_2 - x_1)t)\,dt = \frac{F(x_2) - F(x_1)}{x_2 - x_1}, \qquad x_1, x_2 \in I, \quad x_1 \neq x_2. \tag{2.1}$$

LEMMA 2.1. *If* $m \in \{0, 1, \ldots, k\}$, $x \in I$ *and* $\delta > 0$ *are such that* $[x - m\delta, x + (k-m)\delta] \subset I$, *then*

$$(-1)^{k-m}\binom{k}{m}f(x) = \int_0^1 \Delta_{t\delta}^k f(x - m\delta t)\,dt - (-1)^{k-m}\frac{1}{\delta}\binom{k}{m}(\sigma_{k-m} - \sigma_m)F(x) - \frac{1}{\delta}\sum_{j=0, j\neq m}^{k}(-1)^{k-j}\binom{k}{j}\frac{1}{j-m}F(x+(j-m)\delta), \tag{2.2}$$

where

$$\sigma_0 := 0, \qquad \sigma_m := \sum_{j=1}^{m} \frac{1}{j}, \quad m = 1, 2, \ldots \ .$$

Proof of Lemma 2.1. The definition of kth difference and (2.1) give

$$\int_0^1 \Delta_{t\delta}^k f(x - m\delta t)\,dt - (-1)^{k-m}\binom{k}{m} f(x)$$

$$= \sum_{j=0, j\neq m}^{k} (-1)^{k-j}\binom{k}{j}\int_0^1 f(x + (j-m)\delta t)\,dt$$

$$= \frac{1}{\delta}\sum_{j=0, j\neq m}^{k} (-1)^{k-j}\binom{k}{j}\frac{F(x+(j-m)\delta) - F(x)}{j-m}.$$

Now (2.2) follows from the identity

$$\sum_{j=0, j\neq m}^{k} (-1)^{k-j}\binom{k}{j}\frac{1}{j-m} = \sigma_m - \sigma_{k-m}. \quad \blacksquare$$

Lemma 2.2. *If* $F(i/k) = 0$, $i = 1, \ldots, k$, *then*

$$F(x) = A_k(x)\int_0^1 \Delta_{t/k}^k f(x(1-t))\,dt, \qquad x \in I. \tag{2.3}$$

Proof of Lemma 2.2. For $x \neq i/k$ the identities

$$A_k(x)\sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j}\frac{1}{x - j/k} = L_k(1, x) = 1$$

and

$$\frac{F(x)}{x - j/k} = \frac{F(x) - F(j/k)}{x - j/k} = \int_0^1 f(x + (j/k - x)t)\,dt,$$

give

$$F(x) = A_k(x)\sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j}\int_0^1 f(x + (j/k - x)t)\,dt. \quad \blacksquare$$

Proof of Lemma 1.1. Lemmas 2.1, 2.2, and the obvious estimates

$$|\Delta_{t\delta}^k g(x-m\delta t)|\leqslant \omega_k(f,1/k)\leqslant 1, \qquad 0\leqslant t\leqslant 1,$$

$$|\Delta_{t/k}^k g(x(1-t))|\leqslant \omega_k(f,1/k)\leqslant 1, \qquad 0\leqslant t\leqslant 1,$$

imply

$$\binom{k}{m}|f(x)|\leqslant 1+\frac{1}{\delta}\binom{k}{m}(\sigma_{k-m}-\sigma_m)|A_k(x)| + \frac{1}{\delta}\sum_{j=0,j\neq m}^{k}\binom{k}{j}\frac{|A_k(x+(j-m)\delta)|}{|j-m|}.$$

Taking into account that $|A_k(x)|=(-1)^{k-m}A_k(x)$ and

$$\int_0^1 \Delta_{t\delta}^k A_k'(x-m\delta t)\,dt=(k+1)(k\delta)^k\int_0^1 t^k\,dt=(k\delta)^k,$$

we again apply Lemma 2.1, for A_k instead of F, and obtain

$$\begin{aligned}\frac{1}{\delta}\binom{k}{m}&(\sigma_{k-m}-\sigma_m)|A_k(x)|\\ &= (-1)^{k-m}\frac{1}{\delta}\binom{k}{m}(\sigma_{k-m}-\sigma_m)A_k(x)\\ &= (k\delta)^k-(-1)^{k-m}\binom{k}{m}A_k'(x)\\ &\quad -\frac{1}{\delta}\sum_{j=0,j\neq m}^{k}(-1)^{k-j}\binom{k}{j}\frac{A_k(x+(j-m)\delta)}{j-m}.\end{aligned}$$

That is,

$$\binom{k}{m}|f(x)|\leqslant 1+(k\delta)^k-(-1)^{k-m}\binom{k}{m}A_k'(x)+\frac{1}{\delta}\sum_{j=0,j\neq m}^{k}\binom{k}{j} \times\left(\frac{|A_k(x+(j-m)\delta)|}{|m-j|}+(-1)^{k-j}\frac{A_k(x+(j-m)\delta)}{m-j}\right).$$

To end the proof we show that in the last sum each term with $j > m$ vanishes. Indeed, since $j > m$ and $m/k \leqslant x \leqslant (m+1)/k$, then

$$x + (j-m)\delta - \frac{j}{k} = x + (j-m)\frac{1-x}{k-m} - \frac{j}{k} \geqslant \frac{m}{k} + (j-m)\frac{1-m/k}{k-m} - \frac{j}{k} = 0,$$

$$x + (j-m)\delta - \frac{j+1}{k} \leqslant \frac{m+1}{k} + (j-m)\frac{1-(m+1)/k}{k-m} - \frac{j+1}{k}$$
$$= \frac{m-j}{(k-m)k} < 0,$$

and therefore $x + (j-m)\delta \in [j/k, (j+1)/k]$. Hence

$$|A_k(x + (j-m)\delta)| = (-1)^{k-j} A_k(x + (j-m)\delta), \qquad j > m. \quad \blacksquare$$

3. PROOF OF THEOREM 1.1

It is clear that we may assume that $\omega_k(f, 1/k) \leqslant 1$. To make the presentation more transparent we split the proof into several lemmas. Lemma 3.1 is a consequence of Lemma 1.1.

LEMMA 3.1. *Let* $x \in [0, 1/k)$. *Then*

$$|f(x) - Q_{k-1}(x)| = |g(x)| \leqslant 1 + (1-x)^k - (-1)^k A_k'(x).$$

In order to estimate the quality

$$|L_{k-1}(g, x)| = \left| \sum_{m=0}^{k-1} g(x_m) l_m(x) \right|, \qquad x_m = \frac{m}{k-1}$$

we need Lemma 3.2.

LEMMA 3.2. *For each* $m = 0, \ldots, k-1$ *we have*

$$|g(x_m)| \leqslant \binom{k-1}{m}^{-1} + 2(k-1)\sigma_{k-1}|A_k(x_m)|.$$

The proof of Lemma 3.2 is the most technical part of this paper. We will use Lemma 3.2 to deduce Lemma 3.3.

LEMMA 3.3. *For* $x \in [0, 1/k)$ *we have*

$$|L_{k-1}(g,x)| \leqslant \left(\frac{1}{x} + C_{k-1}(x)\right)|A_{k-1}(x)| + 2(k-1)\sigma_{k-1}(k/(k-1))^{k-1} \times (|A_{k-1}(x)|(\tfrac{1}{2} - x) - |A_k(x)|),$$

where

$$C_{k-1}(x) := \frac{k-1}{1-(k-1)x} + \cdots + \frac{k-1}{k-1-(k-1)x} = -\sum_{m=1}^{k-1} \frac{1}{x - x_m}.$$

Lemma 3.4 follows from Lemmas 3.1–3.3.

LEMMA 3.4. *Let* $x \in [0, 1/k),\ k > 6$. *Then*

$$|f(x) - L_{k-1}(f,x)| \leqslant 2 + e(k-1)\sigma_{k-1}|A_{k-1}(x)|.$$

LEMMA 3.5. *For* $x \in [0, 1/k)$ *we have*

$$e(k-1)\sigma_{k-1}|A_{k-1}(x)| \leqslant 1.$$

Lemma 3.5 is a direct consequence of inequalities

$$1 - t \leqslant \exp(-t), \qquad t\exp(-t) \leqslant 1, \quad t \geqslant 0.$$

Now Theorem 1.1 follows easily from Lemmas 3.4 and 3.5.

In the remaining part of this section we will prove Lemmas 3.2–3.4.

Proof of Lemma 3.2. Let us introduce first some new notations:

$$B_k(y) := kA_k(y/k) = \frac{1}{k!}(y-1)\cdots(y-k),$$

$$y_m := m + x_m = kx_m, \qquad z_m := \frac{k - y_m}{k - m},$$

$$s_m := \frac{1}{z_m}\sum_{j=0}^{m-1}\binom{k}{j}\frac{1}{m-j}|B_k(y_m + (j-m)z_m)|,$$

$$b_m(x) := \left(1 - \frac{x}{k-m}\right)\cdots\left(1 - \frac{x}{1}\right)\left(1 + \frac{x}{1}\right)\cdots\left(1 + \frac{x}{m}\right),$$

$$c_m(x) := \frac{1}{k-m-x} + \cdots + \frac{1}{1-x} - \frac{1}{1+x} - \cdots - \frac{1}{m+x}.$$

Without loss of generality we assume that $x_m \leqslant \frac{1}{2}$. An application of Lemma 1.1 reduces Lemma 3.2 to the inequality

$$1 + z_m^k - (-1)^{k-m} \binom{k}{m} B_k'(y_m) + 2s_m \leqslant \frac{k}{k-m} + 2\sigma_{k-1} \frac{k-1}{k} \binom{k}{m} |B_k(y_m)|. \tag{3.1}$$

The relation

$$-(-1)^{k-m} \binom{k}{m} B'(y) = -b_m(x) + x b_m(x) c_m(x), \qquad y = m + x,$$

allows us to rewrite (3.1) in the form

$$z_m^k + b_m(0) - b_m(x_m) + x_m b_m(x_m) c_m(x_m) + 2s_m \leqslant \frac{k}{k-m} + 2\sigma_{k-1} \frac{k-1}{k} x_m b_m(x_m).$$

We will use the inequalities

$$z_m^k \leqslant \exp \frac{x_m}{x_m - 1} + \frac{4.8 x_m}{k} - \frac{x_m}{1 - x_m} - 1 + \frac{k}{k-m}, \tag{3.2}$$

$$s_m \leqslant \frac{k-1}{k} \sigma_m x_m b_m(x_m) + \frac{2\sigma_m}{k} x_m b(x_m). \tag{3.3}$$

In view of (3.2), (3.3) we conclude that to prove (3.1) it is sufficient to establish the inequality

$$\frac{4\sigma_m b_m(x_m) + 4.8}{k} - \frac{1}{1 - x_m} + \frac{\left(\exp \frac{x_m}{x_m - 1} - 1\right)}{x_m} + \frac{b_m(0) - b_m(x_m)}{x_m} + b_m(x_m) c(x_m) \leqslant 2(\sigma_{k-1} - \sigma_m) \frac{k-1}{k} b_m(x_m),$$

which may be rewritten as

$$\frac{4\sigma_{k-1}+4.8}{k}-\frac{1}{1-x_m}+\frac{1}{x_m}\left(\exp\frac{x_m}{x_m-1}-1\right)$$
$$+\left(\frac{b_m(0)-b_m(x_m)}{x_m}-(\sigma_{k-1}-\sigma_m)b_m(x_m)\right)$$
$$+\ b_m(x_m)(c_m(x_m)-\sigma_{k-1}+\sigma_m)\leqslant 0. \tag{3.4}$$

Therefore, our next task is to prove inequalities (3.2)–(3.4).

3.1. Proof of (3.2)

Estimate (3.2) follows by combining the inequalities

$$\begin{aligned} z_m^k-\exp\frac{x_m}{x_m-1} &= \left(1-\frac{x_m}{k-m}\right)^k-\exp\frac{x_m}{x_m-1}\\ &\leqslant \exp\frac{kx_m}{m-k}-\exp\frac{x_m}{x_m-1}\\ &\leqslant \left(\frac{kx_m}{m-k}-\frac{x_m}{x_m-1}\right)\exp\frac{kx_m}{m-k}=\frac{x_m}{1-x_m}\frac{x_m}{k-m}\exp\frac{kx_m}{m-k}\\ &\leqslant \frac{x_m}{1-x_m}\frac{1}{ke}\leqslant\frac{2}{ek}x_m<\frac{0.8}{k}x_m \end{aligned}$$

and

$$\frac{x_m}{1-x_m}-\frac{k}{k-m}+1<\frac{4}{k}x_m.$$

3.2. Proof of (3.3)

We shall deduce (3.3) from the estimate

$$\binom{k}{j}|B_k(y_m+(j-m)z_m)|\leqslant\binom{k}{j+1}|B_k(y_m+(j+1-m)z_m)|,$$
$$j=0,\dots,m-1. \tag{3.5}$$

To prove (3.5), set $u_j := y_m + (j-m)z_m - j = \frac{(k-j)m}{(k-1)(k-m)}$ and note that $u_j > u_{j+1}$. Therefore

$$\frac{\binom{k}{j}|B_k(j+u_j)|}{\binom{k}{j+1}|B_k(j+1+u_{j+1})|}$$

$$= \frac{(1+\frac{u_j}{j})\cdots(1+\frac{u_j}{1})u_j(1-\frac{u_j}{1})\cdots(1-\frac{u_j}{k-j})}{(1+\frac{u_{j+1}}{j+1})\cdots(1+\frac{u_{j+1}}{1})u_{j+1}(1-\frac{u_{j+1}}{1})\cdots(1-\frac{u_{j+1}}{k-j-1})}$$

$$< \frac{u_j(1-u_j/(k-j))}{(1+u_{j+1}/(j+1))u_{j+1}} = \frac{k-j}{k-j-1}\frac{k-m-1}{k-m-1+m/(j+1)} < 1,$$

and this yields (3.5). It implies

$$s_m \leqslant \frac{1}{z_m}\sigma_m x_m b_m(x_m),$$

and (3.3) follows from the estimate

$$\frac{1}{z_m} = \frac{k-m}{k-m-1}\frac{k-1}{k} \leqslant \frac{k-1}{k} + \frac{2}{k}.$$

3.3. Proof of (3.4)

We have divided this proof into three parts.

3.3.1.

Here we shall use the notations $\sigma := \sigma_{k-1} - \sigma_m$, $d(x) := \frac{23}{10}x - \frac{27}{20}x^3$. We begin with the proof of the inequality

$$(c_m(x) - \sigma)b_m(x) \leqslant \frac{1.2}{k-1} + d(x), \qquad 0 \leqslant x \leqslant x_m. \tag{3.6}$$

First, suppose that $m > 2$. Since

$$\sigma_{k-m} - \sigma_m - \sigma \leqslant -x_m + \frac{1}{k} \leqslant -x + \frac{1}{k},$$

then

$$c_m(x) - \sigma \leqslant \frac{1}{k} + \left(\frac{2x}{1-x^2} - x\right) + \frac{2x}{2^2-x^2} + \cdots + \frac{2x}{m^2-x^2}$$
$$+ \frac{x}{(m+1)(m+1-x)} + \cdots + \frac{x}{(k-m)(k-m-x)}.$$

Put $a := \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} - \frac{5}{4}$. By using the inequality $(1-t)^k \leqslant 1 - kt + (kt)^2/2$, $t \geqslant 0$, we obtain

$$\left(1-\frac{x^2}{3^2}\right)\cdots\left(1-\frac{x^2}{m^2}\right) \leqslant \left(1 - \frac{x^2}{m-2}\left(\frac{1}{3^2} + \cdots + \frac{1}{m^2}\right)\right)^{m-2}$$
$$\leqslant 1 - ax^2 + \frac{a^2}{2}x^4 + x^2\left(\frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \cdots\right)$$
$$\leqslant 1 - ax^2 + \frac{a^2}{2}x^4 + \frac{xx_m}{m} = \frac{x}{k-1} + 1 - ax^2 + \frac{a^2}{2}x^4.$$

Therefore, we have

$$\left(\frac{1}{k} + \frac{2x}{1-x^2} - x + \frac{2x}{2^2-x^2}\right)b_m(x)$$
$$\leqslant \frac{1.2}{k-1} + \left(\frac{2x}{1-x^2} - x + \frac{2x}{2^2-x^2}\right)(1-x^2)\left(1-\frac{x^2}{2^2}\right)\left(1 - ax^2 + \frac{a^2}{2}x^4\right).$$

Now we apply the estimate $b_m(x) \leqslant (1-x^2)(1-x^2/4)(j^2-x^2)/j^2$ for $j = 3, \ldots, m$, and the estimate $b_m(x) \leqslant (1-x^2)(1-x^2/4)(j-x)/j$, for $j = m+1, \ldots, k-m$, to obtain

$$\left(\frac{2x}{3^2-x^2} + \cdots + \frac{x}{(k-m)(k-m-x)}\right)b_m(x) \leqslant 2ax(1-x^2)\left(1-\frac{x^2}{2^2}\right).$$

Finally, we add the last two inequalities and have

$$b_m(x)(c_m(x) - \sigma)$$
$$\leqslant \frac{1.2}{k-1} + \left(2a + \frac{3}{2}\right)x + \left(\frac{1}{4} - 4a\right)x^3 + \left(\frac{a}{4} + \frac{3a^2}{4} - \frac{1}{4}\right)x^5 + \left(\frac{a}{4} + \frac{a^2}{8}\right)x^7$$
$$\leqslant \frac{1.2}{k-1} + \left(2a + \frac{3}{2}\right)x + \left(\frac{1}{4} - 4a\right)x^3 \leqslant \frac{1.2}{k-1} + d(x),$$

where, in the last line, we used the fact that $a < \frac{2}{5}$.

For $m = 1$ we may combine the inequalities $b_m(x) \leqslant 1 - x^2$, $b_m(x) \leqslant (j - x)/j$, $j = 2, \ldots, k - 1$, and get

$$b_m(x)(c_m(x) - \sigma) = \left(\frac{2x}{1 - x^2} + \sum_{i=2}^{k-1} \frac{x}{i(i - x)}\right) b_m(x) \leqslant \left(\frac{9}{4} + a\right) x.$$

Since $0 \leqslant x \leqslant 1/(k - 1)$, we obtain (3.6). The proof for $m = 2$ follows the same pattern.

3.3.2.

Here we prove the estimate

$$\frac{b_m(0) - b_m(x_m)}{x_m} - \sigma b_m(x_m)$$

$$\leqslant \frac{1.6}{k - 1} + d(x_m)(1 - x_m \ln x_m) + x_m \ln^2 x_m. \tag{3.7}$$

In order to do this, note that

$$\frac{b_m(0) - b_m(x_m)}{x_m}$$

$$= \ -b'(\theta) = b_m(\theta)(c_m(\theta) - \sigma) + b_m(\theta)\sigma$$

$$\leqslant \frac{1.2}{k - 1} + d(\theta) + \sigma$$

$$\leqslant \frac{1.2}{k - 1} + d(x_m) + \sigma = \frac{1.2}{k - 1} + d(x_m) + \sigma b_m(x_m) + \sigma(b_m(0) - b_m(x_m)).$$

Hence

$$\frac{b_m(0) - b_m(x_m)}{x_m} - \sigma b_m(x_m)$$

$$\leqslant \frac{1.2}{k - 1}(1 + x_m \sigma) + d(x_m)(1 + x_m \sigma) + x_m \sigma^2.$$

Now (3.7) follows from the evident estimate $\sigma < \ln \frac{k-1}{m} = -\ln x_m$.

3.3.3.

Combining (3.7) with (3.6) yields that the left-hand side of (3.4) does not exceed the quantity

$$\begin{aligned}
&d(x_m)(2 - x_m \ln x_m) + x_m \ln^2 x_m - \frac{1}{1 - x_m} \\
&\quad + \frac{\left(\exp \frac{x_m}{x_m - 1} - 1\right)}{x_m} + \frac{4\sigma_{k-1} + 4.8}{k} + \frac{2.8}{k-1} \\
&\leqslant \frac{4\sigma_{k-1} + 4.8}{k} + \frac{2.8}{k-1} + \max_{x \in (0,1/2]} \Bigg(d(x)(2 - x \ln x) \\
&\quad + x \ln^2 x - \frac{1}{1-x} + \frac{(\exp \frac{x}{x-1} - 1)}{x} \Bigg) \\
&= \frac{4\sigma_{k-1} + 4.8}{k} + \frac{2.8}{k-1} - 0.56873 \cdots,
\end{aligned}$$

which implies (3.4) for $k > 72$. Direct calculations provide the validity of (3.1) for $k \leqslant 72$. ∎

Proof of Lemma 3.3. It is clear that

$$l_m(x) := (-1)^{k-1-m} \binom{k-1}{m} \frac{1}{x - x_m} A_{k-1}(x).$$

This implies

$$\sum_{m=0}^{k-1} \binom{k-1}{m}^{-1} |\, l_m(x)| = \left(\frac{1}{x} + C_{k-1}(x)\right) |A_{k-1}(x)|. \tag{3.8}$$

Thus our aim is to prove that

$$\begin{aligned}
&\sum_{m=0}^{k-1} |A_k(m/(k-1))| \, |l_m(x)| \\
&\quad = (k/(k-1))^{k-1} (|A_{k-1}(x)|(1/2 - x) - |A_k(x)|).
\end{aligned} \tag{3.9}$$

Proof of (3.9). Put $a_k(x) := \frac{k!}{k^k} A_k(x) = x(x - \frac{1}{k}) \cdots (x - \frac{k}{k})$. Then

$$\tilde{a}_k(x) := x a_{k-1}(x) = x^{k+1} - \frac{k}{2} x^k + \cdots$$

and

$$a_k(x) = x^{k+1} - \frac{(k+1)}{2} x^k + \cdots .$$

Therefore

$$\begin{aligned} a_k(x) - \tilde{a}_k(x) - L_{k-1}(a_k, x) &= a_k(x) - \tilde{a}_k(x) - L_{k-1}(a_k - \tilde{a}_k, x) \\ &= -\tfrac{1}{2} a_{k-1}(x). \end{aligned}$$

Since $a_k(0) = a_k(1) = 0$ and, for all $m = 1, \ldots, k-3$,

$$\operatorname{sign} l_m(x) a_k(m/(k-1)) = \operatorname{sign} l_{m+1}(x) a_k((m+1)/(k-1)),$$

then

$$\begin{aligned} \sum_{m=0}^{k-1} |l_m(x)\, a_k(m/(k-1))| &= |L_{k-1}(a_k, x)| \\ &= |(\tfrac{1}{2} - x)\, a_{k-1}(x) + a_k(x)| \\ &= |a_{k-1}(x)|(\tfrac{1}{2} - x) - |a_k(x)|, \end{aligned} \tag{3.10}$$

where, in the last line, we have used the relations

$$a_k(x) a_{k-1}(x) \leqslant 0 \quad \text{and} \quad |a_{k-1}(1/k)| > 0 = a_k(1/k).$$

Now we multiply both sides of (3.10) by $k^k/k!$ and get (3.9). ∎

Proof of Lemma 3.4. Lemmas 3.1, 3.3, and the identity

$$-(-1)^k A_k'(x) = C_k(x)|A_k(x)| - \frac{1}{x}|A_k(x)|,$$

reduce Lemma 3.4 to the estimate

$$\begin{aligned} h(x) \coloneqq (1-x)^k &+ C_k(x)|A_k(x)| - \frac{1}{x}|A_k(x)| \\ &+ \frac{1}{x}|A_{k-1}(x)| + C_{k-1}(x)|A_{k-1}(x)| \\ &- 2(k-1)e\sigma_{k-1} x|A_{k-1}(x)| - 2(k-1)\sigma_{k-1}|A_k(x)| \leqslant 1. \end{aligned} \tag{3.11}$$

For $6 < k \leqslant 31$ we check (3.11) by direct calculations. Everywhere below we assume that $k > 31$.

LEMMA 3.6. *If* $x \in [0, 1/k)$, *then*

$$|A_{k-1}(x)| - |A_k(x)| \leqslant (1+\sigma_{k-2})x|A_{k-1}(x)| + \frac{x}{1-kx}|A_k(x)|.$$

Proof of Lemma 3.6. Taking into account the inequalities $1+t \leqslant e^t$, $(1-t)e^t \leqslant 1,\ t \geqslant 0$, we get

$$\begin{aligned}(1-x)\frac{|A_{k-1}(x)|}{|A_k(x)|} &= \left(1+\frac{x}{1-kx}\right)\left(1+\frac{x}{2-kx}\right)\cdots\left(1+\frac{x}{(k-1)-kx}\right)\\ &\leqslant \left(1+\frac{x}{1-kx}\right)\left(1+\frac{x}{1}\right)\cdots\left(1+\frac{x}{k-2}\right)\\ &\leqslant \left(1+\frac{x}{1-kx}\right)\frac{1}{1-x\sigma_{k-2}}. \quad \blacksquare\end{aligned}$$

Lemma 3.6 implies

$$\begin{aligned}h(x) \leqslant{}& \tfrac{1}{2}(1-x)^k + (C_k(x) - k\sigma_k)|A_k(x)| + \frac{1+\sigma_{k-1}}{1-kx}|A_k(x)| + |A_k(x)|\\ &+ \tfrac{1}{2}(1-x)^k + (C_{k-1}(x) - (k-1)\sigma_{k-1})|A_{k-1}(x)|\\ &+ (1+\sigma_{k-2})|A_{k-1}(x)| + x(k-1)\sigma_{k-1}(1+\sigma_{k-2}-2e)|A_{k-1}(x)| \quad (3.12)\end{aligned}$$

Since $1+t < \frac{1}{1-t},\ 0 < t < 1$, the last line in (3.12) is less than

$$\frac{1+\sigma_{k-2}}{1-(k-1)x}|A_{k-1}(x)| + x(k-1)\sigma_{k-1}(\sigma_{k-2}-2e)|A_{k-1}(x)|.$$

Using the notation

$$g_k(y) := \left(\frac{y}{(1-y)1} + \cdots + \frac{y}{(k-y)k}\right)|B_k(y)| + \frac{1+\sigma_{k-1}}{k(1-y)}|B_k(y)|,$$

we see that

$$\begin{aligned}h(x) \leqslant{}& \frac{1}{2}e^{-u} + g_k(u) + \frac{1}{k}|B_k(u)| + \frac{1}{2}e^{-v} + g_{k-1}(v)\\ &+ \frac{\sigma_{k-1}}{k-1}(\sigma_{k-2}-2e)v|B_{k-1}(v)|,\end{aligned}$$

where $0 < u := kx < 1$, and $0 < v =: (k-1)x < 1$.

LEMMA 3.7. *For* $y \in (0,1)$ *we have*

$$g_k(y) + \frac{1}{k}|B_k(y)| \leqslant \frac{1}{2} y e^{-y}, \tag{3.13}$$

$$g_{k-1}(y) + \frac{\sigma_{k-1}}{k-1}(\sigma_{k-2} - 2e) y |B_{k-1}(y)| \leqslant \frac{1}{2} y e^{-y}. \tag{3.14}$$

Lemma 3.4 follows from Lemma 3.7, since

$$h(x) \leqslant \tfrac{1}{2}(1+u)e^{-u} + \tfrac{1}{2}(1+v)e^{-v} \leqslant 1.$$

Proof of Lemma 3.7. We start by proving (3.13). To estimate the first term in g_k we use the inequalities

$$\frac{y}{(1-y)1} + \cdots + \frac{y}{(k-y)k} = \frac{y}{1-y}\left(\frac{1-y}{(1-y)1} + \cdots + \frac{1-y}{(k-y)k}\right)$$
$$\leqslant \frac{y}{1-y}\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots\right) = \frac{y}{1-y}\frac{\pi^2}{6},$$

$$\frac{y}{1-y}|B_k(y)| \leqslant \frac{1}{e(\sigma_k - 2)} y e^{-y},$$

and get

$$\left(\frac{y}{(1-y)1} + \cdots + \frac{y}{(k-y)k}\right)|B_k(y)| \leqslant \frac{\pi^2}{6e(\sigma_k - 2)} y e^{-y} \leqslant \frac{\pi^2}{12e} y e^{-y}.$$

To estimate the second term in g_k we use the inequality $\frac{1}{1-y}|B_k(y)| \leqslant y e^{-(\sigma_k - 1)y} \leqslant y e^{-y}$ and obtain

$$\frac{1+\sigma_{k-1}}{k(1-y)}|B_k(y)| \leqslant \frac{1+\sigma_{k-1}}{k} y e^{-y} \leqslant \frac{5}{31} y e^{-y}.$$

Thus $g_k(y) \leqslant (\frac{\pi^2}{12e} + \frac{5}{31}) y e^{-y}$, and we may write

$$g_k(y) + \frac{1}{k}|B_k(y)| \leqslant g_k(y) + \frac{1}{k} y e^{-y} \leqslant \left(\frac{\pi^2}{12e} + \frac{5}{31} + \frac{1}{31}\right) y e^{-y} \leqslant \frac{1}{2} y e^{-y},$$

so (3.13) holds. Next we prove (3.14). Clearly,

$$g_{k-1}(y) + \frac{\sigma_{k-1}}{k-1}(\sigma_{k-2} - 2e) y |B_{k-1}(y)|$$
$$\leqslant \left(\frac{\pi^2}{12e} + \frac{5}{31}\right) y e^{-y} + \frac{\sigma_{k-1}}{k-1}(\sigma_{k-2} - 2e) y |B_{k-1}(y)|.$$

Therefore (3.14) holds if $\sigma_{k-2} - 2e < 0$. Otherwise, $k - 2 > 62$, hence $\frac{\sigma_{k-1}}{k-1} \leqslant \frac{2e}{62}$, and

$$\frac{\sigma_{k-1}}{k-1}(\sigma_{k-2} - 2e)y|B_{k-1}(y)| \leqslant \frac{\sigma_{k-1}}{k-1}\frac{\sigma_{k-2} - 2e}{e(\sigma_{k-1} - 1)} ye^{-y}$$

$$\leqslant \frac{\sigma_{k-1}}{k-1}\frac{1}{e} ye^{-y} \leqslant \frac{1}{31} ye^{-y}. \quad \blacksquare$$

Remark 3.1. For $k \leqslant 6$, the inequality in Lemma 3.4 follows from the estimate $W'(k) \leqslant 2$ (see Section 1). Note that for $1 < k \leqslant 6$, we can obtain Theorem 1 by straightforward computation from Lemmas 3.1 and 3.3.

4. ON WHITNEY CONSTANTS $\tilde{W}(k)$ AND $W(k)$

THEOREM 4.1. *Let the polynomials* Q_{k-1} *be defined by* (1.2). *Then*

$$||f - Q_{k-1}|| \leqslant \tilde{W}(k)\omega_k(f, 1/k),$$

with

$$\tilde{W}(k) \leqslant \begin{cases} 2, & k \leqslant 82{,}000, \\ 2 + \exp(-2), & k > 82{,}000. \end{cases}$$

Theorem 4.1. corrects an arithmetical mistake in [4], where it was claimed that $\tilde{W}(k) < 2$ for all k. Theorem 4.1 follows from Lemma 3.1 and Lemma 4.1.

LEMMA 4.1. *For* $x \in [0, 1/k)$ *we have*

$$(1 - x)^k - (-1)^k A_k'(x) \leqslant 1 + \frac{1}{e^2} \tag{4.1}$$

and

$$(1 - x)^k - (-1)^k A_k'(x) \leqslant 1, \qquad k \leqslant 82{,}000. \tag{4.2}$$

Proof of Lemma 4.1. We check (4.2) by direct calculations. Let us prove (4.1). After the change of variable $u = kx$, we get the inequality $(B_k(u) := kA(u/k))$

$$w_k(u) := \left(1 - \frac{u}{k}\right)^k - (-1)^k B_k'(u) \leqslant 1 + \frac{1}{e^2}, \qquad 0 < u < 1,$$

which is equivalent to (4.1). Evidently, $w_2(u) = \frac{5}{4}u(\frac{8}{5} - u) \leqslant 0.8$; similarly $w_3(u) < 0.8$. So we may suppose that $k \geqslant 4$ and $\sigma_k - 1 > 1$. By using the equality

$$-(-1)^k B_k'(u) = |B_k(u)| \left(c(u) - \frac{1}{u} \right),$$

where

$$c(u) := \frac{1}{1-u} + \cdots + \frac{1}{k-u} < \sigma_k + \frac{\pi^2}{6} \frac{u}{1-u} < \sigma_k + \frac{5}{3} \frac{u}{1-u},$$

we conclude that

$$\begin{aligned} -(-1)^k B_k'(u) &\leqslant \left(\sigma_k - \frac{1}{u} \right) |B_k(u)| + \frac{5}{3} \frac{u}{1-u} |B_k(u)| \\ &\leqslant \max\{0, e^{-\sigma_k u}(\sigma_k u - 1)\} + \frac{5}{3} u^2 e^{-(\sigma_k - 1)u} \\ &\leqslant \frac{1}{e^2} + \tfrac{5}{3} u^2 e^{-u}. \end{aligned}$$

Therefore

$$w_k(u) \leqslant \frac{1}{e^2} + e^{-u} + \frac{5}{3} u^2 e^{-u} \leqslant \frac{1}{e^2} + 1. \quad \blacksquare$$

We end the paper with Theorem 4.2 about Whitney constant $W(k)$. Let $E_{k-1}(f) := \inf_p ||f - p||$ be the error of the best uniform approximation of f by algebraic polynomials p of degree $\leqslant k - 1$. Whitney constants are defined by

$$W(k) := \sup_{f \in C} \frac{E_{k-1}(f)}{\omega_k(f, 1/k)}.$$

Evidently, $W(k) \leqslant \tilde{W}(k)$ and Theorem 4.1 implies

THEOREM 4.2. *We have*

$$W(k) \leqslant \begin{cases} 2, & k \leqslant 82{,}000, \\ 2 + \exp(-2), & k > 82{,}000. \end{cases}$$

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