New Series Vol. xx, 199x, Fasc. x

# Whitney's Constants and Sendov's Conjectures

*Yuri V. Kryakin*

*Dedicated to Professor Blagovest Sendov on the occasion of his seventieth birthday*

In this paper we review the history and the current state of the Whitney's constants problem.

## 1. Introduction

Let $C$ be a space of continuous functions $f$ on $I := [0,1]$ with a uniform norm

$$\|f\| := \max_{x \in I} |f(x)|.$$

For a function $f \in C$ denote the $k$–th difference with step $h$ by

$$\Delta_h^k f(x) := \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+jh)$$

and $k$-th modulus of continuity by

$$\omega_k(f) := \sup_{x, x+kh \in I} |\Delta_h^k f(x)|.$$

Let $\mathbf{P}_k$ be a space of algebraic polynomials of degree $\leq k$. Whitney constants are defined by

$$W_k := \sup_{f \in C \setminus \mathbf{P}_{k-1}} \inf_{p \in \mathbf{P}_{k-1}} \frac{\|f-p\|}{\omega_k(f)}.$$

Let $L_{k-1}(f, x)$ be the Lagrange polynomial of degree $\leq k-1$, which interpolates $f$ at equidistant points $x_m := m/(k-1)$:

$$f(x_m) = L_{k-1}(f, x_m), \quad m = 0, \ldots, k-1.$$

Whitney interpolation constants are defined by

$$W'_k := \sup_{f\in C\setminus \mathbf{P}_{k-1}} \frac{\|f-L_{k-1}(f,\cdot)\|}{\omega_k(f)} = \sup_{f\in C,\ f(x_m)=0} \frac{\|f\|}{\omega_k(f)}.$$

In this paper we are mainly interested in estimates of $W_k$ and $W'_k$. Namely, we intend to describe the current situation with the following Sendov's conjectures [17].

**Fist Sendov's conjecture** : $W_k \le 1$.

**Second Sendov's conjecture** : $W'_k \le 2$.

## 2. Historical remarks

**2.1. Results of Burkill, Whitney, Beurling and Brudnyi.** It is clear, that $W_1 = 1/2$ and $W'_1 = 1$. This is the case of approximation of continuous function by constant. The results $W_2 = 1/2$ and $W'_2 = 1$ are due to H.Burkill [4] and H. Whitney [27].

**Burkill's Lemma.** *If $f \in C$ and $f(0) = f(1) = 0$, then $\|f\| \le \omega_2(f)$.*

Proof. Suppose that $|f(a)| = \|f\|$ and $a \le 1/2$. Then $f(a) = -1/2(f(0) - 2f(a) + f(2a)) + 1/2f(2a)$ and $\|f\| \le |f(0) - 2f(a) + f(2a)| \le \omega_2(f)$. In the case $a > 1/2$ we have the same conclusion by symmetry. ■

H.Burkill conjectured, that $W_k \le W'_k < \infty$. This conjecture was proved in 1957 by H.Whitney [27]. The main ingredient of his proof is the following Lemma.

**Whitney's Lemma.** *Let $k,\nu \in N,\ k,\nu \ge 2,\ X = \{0, 1, \dots, \nu(k-1)\}$. Then for $f(s),\ s \in X$ exist numbers*

$$a_i = a_i(s,\nu,k), \quad i = 0,\dots,\nu(k-1)-k,$$

$$b_j = b_j(s,\nu,k),\ j = 0,\dots,k-1,$$

*such, that*

$$f(s) = \sum_{j=0}^{\nu(k-1)-k} a_j \Delta_1^k f(j) + \sum_{j=0}^{k-1} b_j f(j\nu).$$

*Moreover, for $s = 1$ and for arbitrary positive $\varepsilon > 0$ there exists $\nu$, such that*

$$\sum_{j=1}^{k-1} b_j(s,\nu,k) < \varepsilon.$$

This Lemma may be used in various generalizations of Whitney inequality $W'_k \leq \infty$. For example, it was used in proofs of analogs of Whitney's estimate for functions in $L^p$ [24], for functions on complex arcs [26, 15], for Chebyshevian approximations [28, 16]. But we can not obtain good estimates of $W'_k$ and $W_k$ on this way. We can not write any, may be bed estimate, for all $k$. By using the special identities Whitney proved the inequalities

$$\frac{8}{15} \leq W_3 \leq \frac{7}{10}, \quad \frac{16}{15} \leq W'_3 \leq \frac{14}{9}.$$

Whitney noted, that " the problem of finding the $W_k$, $W'_k$ is probably extremely difficult ...".

Another proof of Whitney theorem with estimates $W_k \leq Ck^{2k}$ was obtained by Yu.Brudnyi [2]. Modified Brudnyi's proof with estimate $W_k \leq (k+1)k^k$ one can find in Sendov's paper [19].

On the other hand, the situation is not difficult for integrable on $[0, +\infty)$ functions. We have the following Whitney–Beurling identity [27]: For $A > 0, \quad 0 \leq y < x$

$$1/A \left\{ \int_0^A \Delta_h^k f(x)\, dh - \int_0^A \Delta_h^k f(y) dh \right\}$$

$$= (-1)^k (f(x) - f(y)) + 1/A \left\{ \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \left( \int_{y+jA}^{x+jA} f - \int_y^x f \right) \right\}.$$

This identity implies (we may consider $A >> 1$) estimate

$$|f(x) - f(y)| \leq 2 \sup_{x,\, h>0} |\Delta_h^k f(x)|.$$

**2.2. Results of Ivanov, Takev, Binev and Sendov.** In papers [17,18,19] Sendov invited attention to the problem of Whitney's constants. In 1985, as the result of works [7, 1, 20] the following remarkable inequality was obtained [21, 22]:

$$W_k \leq \text{ Const} \leq 6. \tag{1}$$

Using of the Ivanov–Takev integral operators [7]

$$\psi_i(f, x) := \frac{(-1)^{k-i}}{h\binom{k}{i}} \int_0^h \Delta_y^k f(x - iy)\, dy,$$

$$h = (k+1)^{-1},\ t \in [0, h],\ x = ih + t,\ i = 0, 1, \ldots, k,$$

was essential for proving (1). This operators can be considered as the convenient tool for transplantation of Whitney–Beurling proof from $[0,\infty)$ to $I=[0,1]$. Sendov deduced (1) from the following Lemma.

**Sendov's Lemma.** *Let $f\in C$, $k\in N$. Then there exist $p\in \mathbf{P_{k-1}}$ such, that*

$$f(x)=p(x)+\psi_i(f,x)+\sum_{j=0}^{k}\frac{1}{h}\int_0^t \psi_j(f,jh+y)l'_{k,j}(\frac{x-y}{h})\,dy,$$

*where*

$$l_{k,j}(x):=\prod_{m=0,m\neq j}^{k}(x-m)/(j-m).$$

Analysis of Sendov's Lemma leads [9, 10] to the following

**Modified Sendov's Lemma.** *Let $f\in C$, $k\in N$ and $\int_0^{i/k} f(t)\,dt=0$, $i=1,\dots,k$. Then for $x\in(0,1/k]$ we have*

$$f(ix)=\varphi_i(f,x)+\int_{1/k}^{x}\sum_{j=1}^{k}\varphi_j(f,y)\cdot\frac{j}{i}\cdot\left[l_{k,j}(\frac{x}{y}i)\right]'_x\,dy,$$

*where*

$$\varphi_i(f,x):=\frac{(-1)^{k-i}}{\binom{k}{i}}\frac{1}{x}\int_0^x \Delta_y^k f(i(x-y))\,dy.$$

Modified Sendov's Lemma implies inequality

$$W_k\le 3. \tag{2}$$

Inequality was (2) independently announced by Yu.Brudnyi [3], B.Sendov [23] and author [8].

Estimate

$$W'_k<\quad\text{Const}\quad<36$$

was obtained by M.Takev [25]. Combination of Takev's method with the modified Sendov's Lemma gave the inequality [13]

$$W'_k<5.$$

## 3. Recent developments

**3.1. Estimates of $W_k$.** In modified Sendov's Lemma we can replace $f(x)$ on $f(x) - q(x)$ and remove the condition $\int_0^{i/k} f = 0$, by special choosing of $q \in \mathbf{P_{k-1}}$:

$$\int_0^{i/k} f(t) - q(t)\,dt = 0, \quad i = 1, \ldots, k.$$

It is natural to define the corresponding constants:

$$W_k^* := \sup_{f \in C \setminus \mathbf{P}_{k-1}} \frac{\|f - q\|}{\omega_k(f)}.$$

It is clear that $W_k \le W_k^*$.

**Theorem 1** [12, 5].

$$W_k^* \le 2 \qquad \textit{for} \qquad k \le 82000.$$
$$W_k^* \le 2 + \exp(-2) \qquad \textit{for} \qquad k > 82000.$$

To prove Theorem 1 we need an another modification of Whitney–Beurling idea. To this end put $F(x) := \int_0^x f(u)\,du$. The following identity one can check directly.

**Lemma 1** [12]. *If $m \in \{0, 1, \ldots, k\}$, $x \in I$ and $\delta > 0$ are such, that $[x - m\delta, x + (k-m)\delta] \subset I$, then*

$$(-1)^{k-m} \binom{k}{m} f(x) = \int_0^1 \Delta_{t\delta}^k f(x - m\delta t)\,dt$$

$$-(-1)^{k-m} \frac{1}{\delta} \binom{k}{m} (\sigma_{k-m} - \sigma_m) F(x)$$

$$-\frac{1}{\delta} \sum_{j=0, j \ne m}^{k} (-1)^{k-j} \binom{k}{j} \frac{1}{j-m} F(x + (j-m)\delta),$$

*where*

$$\sigma_0 := 0, \quad \sigma_m := \sum_{j=1}^{m} \frac{1}{j}, \qquad m = 1, 2, \ldots.$$

The estimates of $F$ in Lemma 1 provide the following Zhuk–Natanson identity.

**Lemma 2** [31, 12]. *If $F(i/k) = 0$, $i = 1, \ldots, k$, then*

$$F(x) = A_k(x)\int_0^1 \Delta^k_{t/k} f(x(1-t))\,dt, \quad x \in I,$$

*where*

$$A_k(x) := \frac{k^k}{k!}x\left(x-\frac{1}{k}\right)\left(x-\frac{2}{k}\right)\dots\left(x-\frac{k}{k}\right) = x(-1)^k\prod_{j=1}^{k}\left(1-\frac{kx}{j}\right).$$

By combining Lemma 1 and Lemma 2 one can obtain inequality $W_k^* <$ Const . For relative small $k < 1000$ one can use PC for the estimates of constants. But for $k >> 1$ we need something else. The next Shevchuk's Lemma is the appropriate tool for the estimates for large $k$.

**Lemma 3** [5]. *Let* $g := f - q$. *Suppose that* $\omega_k(g) \le 1$, $m < k/2$, $x \in [m/k, (m+1)/k]$, $\delta := (1-x)/(k-m)$. *Then*

$$\binom{k}{m}|g(x)|$$

$$\le 1 + (k\delta)^k - (-1)^{k-m}\binom{k}{m}A_k'(x) + \frac{2}{\delta}\sum_{j=0}^{m-1}\binom{k}{j}\frac{1}{m-j}\left(|A_k(x+\delta(j-m))|\right).$$

Note, that modified Sendov's Lemma implies inequality $|g(x)| \le 1$ for $x \in [1/k, 1-1/k]$. So, to prove Theorem 1 we need only

**Lemma 4** [5]. *For* $x \in [0, 1/k)$ *we have*

$$(1-x)^k - (-1)^k A_k'(x) \le 1 + \frac{1}{e^2},$$

*and*

$$(1-x)^k - (-1)^k A_k'(x) \le 1, \quad k \le 82000.$$

Remark. Theorem 1 corrects an arithmetical mistake in [12], where it was claimed that $W_k^* \le 2$ for all $k$.

**Theorem 2** [11, 14, 29].

$$W_k^* = 1, \quad k \le 7.$$

It is not hard to prove that $W_k^* \ge 1, k \ge 1$ (see [10]). Example may be constructed by smoothing the function $f(x) = 0, x \ne 0,\ f(1) = 1$. Inequality

$W_1^* \le 1$ is trivial. With intention to make idea of the proof as clear as possible, consider the simple case $k = 2$. Put

$$G(x,y) := \frac{1}{y-x}\int_x^y g(t)\,dt, \quad G(x,x) := g(x)$$

and

$$\Delta_{h1,h2}^k G(x,y) := \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}G(x+jh1, y+jh2).$$

It easy to check, that

$$\Delta_{h1,h2}^k G(x,y) = \int_0^1 \Delta_{h1+t(h2-h1)}^k g(x+t(y-x))\,dt.$$

We need to prove that if $\int_0^{i/2} g = \int_0^{i/2}(f-q) = 0, \quad i = 1,2$ and $\omega_2(g) \le 1$ then $|g(x)| \le 1$, or in other notation: if $G(0, j/2) = 0, \quad j = 1,2$ and $|\Delta_{h1,h2}^2 G(x,y)| \le 1$, then we have $|g(x)| \le 1$.

Suppose that $\max|g(y)| = g(x)$ and $x < 1/3$ (the case of $x \in [1/3, 1/2]$ is more simple; in the case $\max|g(y)| = -g(x)$, we can consider the function $g_1 = -g$). We have identities:

$$g(x) = \Delta_{(1-x)/2,x}^2 G(x,x) - \frac{x}{1-3x}(6G(0,2x) - G(2x,3x) - 2G(1/2, 1/2 + x/2))$$

and

$$g(x) = -\frac{1}{2}\Delta_{x,0}^2 G(0,x) + G(0,2x).$$

First identity is a global estimate (with big step $h = (1-x)/2$). Second identity is a local estimate (with step $h = x$). Now from the local estimate we deduce

$$G(0,2x) \ge g(x) - 1/2.$$

Combining last inequality with the global estimate we find that

$$1 \ge \left|\Delta_{(1-x)/2,x}^2 G(x,x)\right|$$

$$\ge \left|g(x) + \frac{x}{1-3x}(6G(0,2x) - G(2x,3x) - 2G(1/2, 1/2 + x/2))\right|$$

$$\ge \left|g(x) + \frac{x}{1-3x}\left(6(g(x) - 1/2) - g(x) - 2g(x)\right)\right|$$

or

$$g(x) \le 1.$$

The proof of Theorem 2 for $2 < k < 8$ is not simple. The main idea of proofs, presented in [14,29], is due to H. Whitney [27]. Put $G(u) := \int_0^u g(t)\,dt$ and suppose, that $G(i/k) = 0,\ i = 1, \ldots, k$. Let $\max|g(y)| = g(x) > \omega_k(f)$. Consider the identity

$$(-1)^k \int_0^1 \Delta^k_{th+(1-t)\alpha x/2}\, g(x)\,dt - g(x)$$

$$= \frac{1}{h - \alpha x/2}\left(\sum_{j=1}^{k} \frac{(-1)^j\binom{k}{j}}{j} G(x + jh) - \sum_{j=1}^{k} \frac{(-1)^j\binom{k}{j}}{j} G(x + j\frac{\alpha x}{2})\right),$$

$$h = (1 - x)/k,\ \alpha : 0 < \frac{\alpha x}{2} < h.$$

Since the left hand part of this identity is non-positive then

$$M_\alpha(x) := -\frac{1}{x}\sum_{j=1}^{k} \frac{(-1)^j\binom{k}{j}}{j} G(x + j\frac{\alpha x}{2}) \le -\frac{1}{x}\sum_{j=1}^{k} \frac{(-1)^j\binom{k}{j}}{j} G(x + jh).$$

Lemma 2 implies ( see [14, 29] )

$$M_\alpha(x) \le \sigma_k - 1.$$

The kernel of the proof is the identity

$$Ag(x) = \sum a_i g_i + \sum b_j \Delta^k_j + \sum c_l M_{\alpha(l)}(x), \tag{3}$$

where $a_i, b_j \in R,\ c_k \in R_+,\ g_i := G(\frac{x(i-1)}{2}, \frac{xi}{2}),\ \Delta^k_j :=$ means of finite differences. We will use (3) for $x$ near origin (difficult case of Theorem 2). From other $x$ we can use some modification of (3) (see [14, 29]). We can suppose, that $|\Delta^k_j| \le 1$. To prove Theorem 2 it is sufficient to construct identity (3) with constraint

$$A > \sum |a_i| + \sum |b_j| + (\sigma_k - 1)\sum c_l.$$

**Identity for k = 2**, $x \in [0, 1/3]$**.**

$$g(x) = \frac{1}{12}(g_5 + g_6) - \frac{1}{2}\Delta^2_{x,0}G(0, x) + \frac{1}{3}M_2(x).$$

**Identity for k = 3**, $x \in [0, 1/6],\ g_0 := g(0)$**.**

$$Ag(x) = \sum_{i=0}^{12} a_i g_i + \sum_{j=1}^{4} b_j \Delta^3_j + cM_2(x),$$

$$\Delta_1^3 = \Delta_{x/2,0}^3 G(x/2, x), \quad \Delta_2^3 = \Delta_{x/2,0}^3 G(0, x),$$

$$\Delta_3^3 = \Delta_{x,x/2}^3 G(0, x/2), \quad \Delta_4^3 = \Delta_{2x,3x/2} G(0, 0).$$

$$A = 396/7, \quad c = 12,$$

$$\mathbf{a} = [4/3, 0, 0, 0, 148/7, 5/7, 5/7, 0, 0, 0, -4/9, -4/9, -4/9],$$

$$\mathbf{b} = [22/7, -22/7, 88/7, 4/3].$$

We have $\quad 396/7 = A > \sum |a_i| + \sum |b_j| + (\sigma_3 - 1) \cdot 12 = 388/7.$

**Identity for $\mathbf{k = 4}$,** $x \in [0, 1/12]$ .

$$Ag(x) = \sum_{i=1}^{22} a_i g_i + \sum_{j=1}^{12} b_j \Delta_j^4 + \sum_{l=1}^{4} c_l M_{(l+1)}(x),$$

$$\Delta_1^4 = \Delta_{\frac{x}{2},0}^4 G(0, x), \quad \Delta_2^4 = \Delta_{x,\frac{x}{2}}^4 G(0, \frac{x}{2}), \quad \Delta_3^4 = \Delta_{\frac{x}{2},\frac{x}{2}}^4 G(0, \frac{x}{2}),$$

$$\Delta_4^4 = \Delta_{x,x}^4 G(0, \frac{x}{2}), \quad \Delta_5^4 = \Delta_{\frac{3x}{2},\frac{3x}{2}} G(0, \frac{x}{2}), \quad \Delta_6^4 = \Delta_{\frac{x}{2},\frac{x}{2}}^4 G(\frac{x}{2}, x),$$

$$\Delta_7^4 = \Delta_{x,x}^4 G(\frac{x}{2}, x), \ \Delta_8^4 = \Delta_{\frac{3x}{2},\frac{3x}{2}} G(\frac{x}{2}, x), \ \Delta_9^4 = \Delta_{x,x}^4 G(3x, \frac{7x}{2}),$$

$$\Delta_{10}^4 = \Delta_{\frac{x}{2},\frac{x}{2}}^4 G(\frac{9x}{2}, 5x), \ \Delta_{11}^4 = \Delta_{\frac{3x}{2},\frac{3x}{2}}^4 G(\frac{9x}{2}, 5x), \ \Delta_{12}^4 = \Delta_{\frac{x}{2},\frac{x}{2}}^4 G(\frac{15x}{2}, 8x).$$

Coefficients $a_i, b_j, c_l$, for case $k = 4$ and identities for $x$, which are separated from the intervals endpoints one can find in [14]. Appropriate identities for $k = 5, 6, 7$ were constructed by O.Zhelnov [29].

**3.2. Estimates of $W_k'$.** The method, proposed by M.Takev [25], intermediate approximation by polynomials $q : \int_0^{i/k} (f - q) = 0, i = 1, \dots k$, and estimates of Lemma 3 led to the following Theorem.

**Theorem 3** [5].

$$W_k' \leq 3.$$

Since the inequality

$$|f(x) - L_{k-1}(f, x)| \leq \omega_k(f), \quad x \in [1/k, 1 - 1/k],$$

is known (see, for example, estimates in [13]), we only need to prove that

$$|f(x) - L_{k-1}(f, x)| \leq 3\, \omega_k(f), \quad x \in [0, 1/k).$$

By using the notation $g(x) := f(x) - q(x)$ we get

$$|f(x) - L_{k-1}(f;x)| \le |f(x) - q(x) - L_{k-1}(f,x) + q(x))|$$

$$\le |f(x) - q(x)| + |L_{k-1}(f - q, x)| = |g(x)| + \left| \sum_{m=0}^{k-1} g(x_m) l_{k-1,m}((k-1)x) \right|.$$

To estimate the value of $|g(x)|$ for $x \in [0, 1/k)$ and in points $x_m,\ m = 0, \dots, k-1$ we shall use Lemma 3.

For $x \in [0, 1/k)$ we have the inequality

$$|g(x)| \le 1 + (1-x)^k - (-1)^k A'_k(x).$$

To estimate

$$|L_{k-1}(g, x)| = \left| \sum_{m=0}^{k-1} g(x_m) l_{k-1,m}((k-1)x) \right|, \qquad x_m = \frac{m}{k-1},$$

we may use the following delicate Lemma 5.

**Lemma 5** [5]. *Suppose that $f \in C,\ \omega_k(f) \le 1$. Then for each $m = 0, \dots, k-1$ we have*

$$|g(x_m)| \le \binom{k-1}{m}^{-1} + 2\,(k-1)\,\sigma_{k-1}\,|A_k(x_m)|\,.$$

The proof of Lemma 5 is the most technical part of paper [5]. One can use Lemma 5 to deduce Lemma 6.

**Lemma 6** [5]. *Let $f \in C,\ k > 7,\ \omega_k(f) \le 1,\ x \in [0, 1/k)$ . Then*

$$|f(x) - L_{k-1}(f, x)| \le 2 + e\,(k-1)\,\sigma_{k-1}|A_{k-1}(x)|.$$

Since

$$e\,(k-1)\,\sigma_{k-1}|A_{k-1}(x)| \le 1$$

we have Theorem 3 for $k > 7$.

For $k \le 7$ second Sendov's conjecture follows from Theorem 2. It was proved by Danilenko ($k = 4$) and O. Zhelnov ( $k = 5, 6, 7$ ).

**Theorem 4** [6, 30].

$$W'_k \le 2, \quad k = 4, 5, 6, 7.$$

**References**

[1] P. Binev. $O(n)$ bound of Whitney constants, *C.R. Acad. Bulg. Sci.*, **38**, No 10, 1985, 1303-1305.

[2] Yu.A. Brudnyi. On a theorem on local best approximations, *Kazan Gos. Univ. Ucen. Zap.*, **124**, No 6, 1964, 43-49.

[3] Yu.A. Brudnyi. Whitney's constants, *in: Modern problems of the theory of functions*, Abstracts All-Union school-conference, Baku, 1989 [In Russian].

[4] H. Burkill. Cesaro–Perron almost periodic functions, *Proc. Lond. Math. Soc.,* **3**, No. 2, 1952, 150-174.

[5] J. Gilewicz, Yu. Kryakin, I. Shevchuk. Boundedness of the interpolation Whitney's constants by 3, submitted to *Journ. of Appr. Theory.*

[6] I.G. Danilenko. On the Sendov problem for the Whitney interpolation constant, *Ukr. Math. J.,* **50**, No.5, 1998, 831–833; *Ukr. Mat. Zh.*, **50**, No 5, 1998, 732-734 [Russian original].

[7] K. Ivanov, M. Takev. $O(n \ln(n))$ bounds Whitney constants, *C.R. Acad. Bulg. Sci.*, **38**, No 9, 1985, 1129-1131.

[8] Yu.V. Kryakin. On Whitney Constants, *in: Modern problems of the theory of functions*, Abstracts All-Union school-conference, Baku, 1989 [In Russian].

[9] Yu.V. Kryakin. Whitney constants, *Mat. Zam.*, **46**, No 2, 1989, 155-157 [In Russian].

[10] Yu.V. Kryakin. On the theorem of H. Whitney in spaces $L_p$, $1 \le p \le \infty$, *Math. Balkanika*, **4**, No 3, 1990, 258-271.

[11] Yu.V. Kryakin. On exact constants in the Whitney theorem, *Math. Notes*, **54**, 1993, 688-700; *Mat. Zametki.*, **53**, No 1, 1993, 34-51 [Russian original].

[12] Yu.V. Kryakin. On Whitney's theorem and constants, *Russ. Acad. Sci., Sb. Math.,* **81**, No 2, 1995, 281-295; *Math. Sb.*, **185**, No 3, 1994, 25-40 [Russian original].

[13] Yu.V. Kryakin, M.D. Takev. Whitney interpolation constants, *Ukr. Mat. Zh.*, **47**, No 8, 1995, 1038-1043 [In Russian].

[14] Yu.V. Kryakin. On functions with a bounded $n$-th difference, *Izv. Math.*, **61**, No. 2, 1997, 331-346; *Izv. Ross. Akad. Nauk. Ser.mat.*, **161**, No 2, 1997, 95-110 [Russian original].

[15] Yu.V. Kryakin. On approximation by splines in $L_p, p > 0$, *in: Fourier Series: Theory and Applications (Kamenets-Podol'sky, 1997). Pr. Inst.*

*Mat. Nats. Akad. Nauk Ukr. Mat. Zastos.*, **20**, Kiev 1998, 132–141 [ In Russian ].

[16] Yu.V. Kryakin. On the approximation by Chebyshev splines in the metric $L_p$, $p > 0$. *Ukr. Math. J.*, **50**, No 9, 1998, 1365-1375; *Ukr. Mat. Zh.*, **50**, No 9, 1998, 1193-1201 [Russian original].

[17] Bl. Sendov. On the constants of H. Whitney, *C.R. Acad. Bulg. Sci.*, **35**, 1982, 431-434.

[18] Bl. Sendov. A new proof of H.Whitney's theorem, *C.R. Acad. Bulg. Sci.,* **35**, 1982, 609-611.

[19] Bl. Sendov. Modifying Steklov means, *C.R. Acad. Bulg. Sci.,* **36**, 1983, 315-317. [in Russian]

[20] Bl. Sendov. The constants of H. Whitney are bounded, *C.R. Acad. Bulg. Sci.*, **38**, No 10, 1985, 1299-1302.

[21] Bl. Sendov. On a theorem of Whitney, *Sov. Math. Dokl.*, **34**, 1987, 603-606: *DAN SSSR*, **291**, 1986, 1296-1300 [Russian original].

[22] Bl. Sendov. On the theorem and constants of H.Whitney, *Constr. Approx.*, **3**, 1987, 1-11.

[23] Bl. Sendov, V. Popov. *The averaged moduli of smoothness*, Wiley, Chichester, 1988.

[24] E.A. Storozhenko, Yu.V. Kryakin. Whitney's theorem in the $L^p$–metric, $0 < p < \infty$. *Sb., Math.,*, **186**, No 3, 1995, 281-295; *Mat. Sbornik.*, **186**, No 3, 1995, 131-142 [Russian original].

[25] M.D. Takev. On the theorem of Whitney–Sendov, *in: Constructive theory of functions*, Sofia, 1992, 269-275.

[26] P.M. Tamrazov. Divided differences and moduli of smoothness of functions, function superposition and their application, *in: Constructive theory of functions*, Sofia, 1984, 840-851.

[27] H. Whitney. On functions with bounded $n$–differences, *J. Math. Pures. Appl.,* **36**, No 9, 1957, 67-95.

[28] Z. Wronicz. Chebyshevian splines, *Diss. math.*, **305**, 1990, 1-98.

[29] O. Zhelnov. Whitney constants are bounded by 1 for $k = 5, 6, 7$, *East Journal for Approximation,* submitted.

[30] O. Zhelnov. Whitney interpolation constants $W_k'$ are bounded by 2 for $k = 5, 6, 7$, *Ukr. Mat. Zh.*, to appear.

[31] V.V. Zhuk, G.I. Natanson. On the theory of cubic periodic splines with equidistant nodes, *Vest. Leningr. Univ.*, **37**, No 1, 1984, 5-11 [In Russian].

*Institute of Mathematics*
*Wrocław University*

*Received xx.xx.2002*

*Pl. Grunwaldzki 2/4*
*50-384 Wroclaw,*
*Poland*
*E-MAIL: kryakin@math.uni.wroc.pl*