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# Construction of Designs on the 2-Sphere

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Spherical $t$-designs are Chebyshev-type averaging sets on the $d$-dimensional unit sphere $S^{d-1}$ that are exact for all polynomials of degree at most $t$. The concept of such designs was introduced by Delsarte, Goethals and Seidel in 1977. The existence of spherical designs for every $t$ and $d$ was proved by Seymour and Zaslavsky in 1984. Although some sporadic examples are known, no general construction has been given. In this paper we give a simple construction of relatively small designs on $S^2$.

## 1. INTRODUCTION AND RESULTS

In order to construct spherical designs as defined below, we will use designs on the interval. Consider the following generalization of the Average Value Theorem:

THEOREM 1. *Let $V$ be a finite-dimensional vector space of continuous real functions defined on the interval $[a, b]$. Then there exists a positive integer $n_0$, such that for every integer $n \geqslant n_0$ there are $n$ points $x_1 < x_2 < \cdots < x_n$ all in $[a, b]$ for which*

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k) = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

*holds for every $f \in V$.*

This theorem was first proved by P. D. Seymour and T. Zaslavsky in 1984 in a very general context [13]. A second proof was given by J. Arias de Reyna in 1988 [1].

DEFINITION 1. The set $X = \{x_1, x_2, \ldots, x_n\}$ in Theorem 1 is called an averaging set for $V$ on the interval $[a, b]$. In particular, if $V$ contains the monomials $x^s$ for all $s = 0, 1, 2, \ldots, t$ then $X$ is called an interval $t$-design.

In other words, $X$ is a $t$-design on the interval, if the quadrature formula of Theorem 1 is exact for all polynomials of degree at most $t$. Note that the situation is invariant under a linear change of variable, so we may assume that we are working on the interval $[-1,1]$.

We will consider the following two quantities. Let $m(t)$ denote the size of a smallest $t$-design on $[-1, 1]$, and let $m'(t)$ denote the minimum integer such that for any $n \geqslant m'(t)$ an interval $t$-design exists having size $n$. The existence of $m'(t)$ (and hence of $m(t)$) follows from Theorem 1. The values of $m(t)$ are only known for $t \leqslant 9$: $m(1) = 1$, $m(2) = m(3) = 2$, $m(4) = m(5) = 4$, $m(6) = m(7) = 6$ and $m(8) = m(9) = 9$. Chebyshev posed the following question in 1873: For which values of $t$ is $m(t) \leqslant t$. This was answered in 1937 by S. N. Bernstein, who proved that $m(t) \leqslant t$ iff $t \leqslant 7$ or $t = 9$. For a proof, which also shows that $m(t) \geqslant 0.06t^2$, see [10, pp. 191–197]. In this paper we will give an upper bound for $m'(t)$ and hence for $m(t)$; namely, we will prove that $m'(t) = O(t^{4.5})$. For the more precise bound see Theorem 2.

The main purpose of this paper is to construct designs on the sphere. The following concept of spherical designs was introduced by Delsarte, Goethals and Seidel in 1977 [7].



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DEFINITION 2. Let $d$ and $t$ be positive integers and let $S^{d-1}$ be the unit sphere in $\mathbb{R}^d$. A finite subset $X$ of $S^{d-1}$ is called a spherical $t$-design, iff

$$\frac{1}{\sigma(S^{d-1})}\int_{S^{d-1}} f(x)\,\mathrm{d}\sigma(x)=\frac{1}{|X|}\sum_{x\in X} f(x)$$

holds for all polynomials $f(x)=f(x_1, x_2, \ldots, x_d)$ of degree at most $t$. Here $\sigma$ denotes the surface measure on $S^{d-1}$.

In 1984 Seymour and Zaslavsky proved the existence of spherical designs for any $t$ and $d$, but only for sufficiently large $N$, where $N$ is the number of nodes, cf. [13]. Although some sporadic examples of spherical designs are known for specific values of $t$, $d$ and $N$, no general construction has been given. For an explicit construction of spherical $t$-designs for $t\leqslant 7$ (and $d$ arbitrary), see [2], [3] and [11].

It is easy to see that the vertices of a regular $N$-gon with $N\geqslant t+1$ give a spherical $t$-design on $S^1$. (A $t$-design on the circle must have at least $t+1$ points.) In fact, a theorem of Y. Hong [9] from 1982 says that a spherical $t$-design on $S^1$ on $N$ modes:
(i) must be a regular $N$-gon if $t+1\leqslant N\leqslant 2t+1$;
(ii) is either a regular $N$-gon or a union of two regular $(t+1)$-gons if $N=2t+2$;
(iii) and there are infinitely many spherical $t$-designs on $S^1$ which are not the unions of regular polygons if $N\geqslant 2t+3$.

Now we will show how to obtain spherical designs on $S^2$ as direct products of interval designs on $[-1,1]$ and spherical designs on $S^1$. Suppose that the set $X=\{x_1, x_2, \ldots, x_n\}$ is an interval $t$-design on $[-1,1]$. The planes given by the equations $x=x_j$ intersect the sphere $x^2+y^2+z^2=1$ in $n$ circles. Let $m$ be a positive integer, with $m\geqslant t+1$. Place a regular $m$-gon on each of these $n$ circles. In Theorem 3 we will show that the resulting $nm$ nodes form a $t$-design on the sphere $S^2$.

From Theorem 3 we also obtain the upper bound $M(t)\leqslant(t+1)m(t)$, for $M(t)$, the minimum size of a $t$-design on the 2-sphere. The exact value of $M(t)$ is only known for $t=1,2,3$ and 5; namely a pair of antipodal points, the vertices of a regular tetrahedron, the regular octahedron and the regular icosahedron give minimal 1-, 2-, 3- and 5-designs, respectively.

As with interval designs, we are also interested in determining $M'(t)$, the smallest number such that for any integer $N\geqslant M'(t)$, $t$-designs on $S^2$ exist having size $N$. We will prove that

$$M'(t)\leqslant(2t+3)m'(t)+t^2+3t+2.$$

Combining this upper bound with Theorem 2 yields $M'(t)=O(t^{5.5})$.

## 2. An Upper Bound for the Number of Nodes in an Interval Design

In this section we will give an upper bound for $m'(t)$, the smallest number such that for any integer $n\geqslant m'(t)$, an interval $t$-design exists on $n$ nodes. We will use the definitions and notations of Section 1.

From J. Arias de Reyna's proof of Theorem 1 [1], we can see that if all functions in $V$ are differentiable, then $n_0$ can be chosen to be

$$4(b-a)ABC,$$

where

$$A=\sqrt{\sum_{k=1}^{m}\|g_k\|^2},\quad B=\sqrt{\sum_{k=1}^{m}(2\,\|g_k\|+1)^2}\quad\text{and}\quad C=\max\{\|g_k'\|: k=1,2,\ldots,m\}.$$

Here the set $\{1, g_1, g_2, \ldots, g_m\}$ is any orthonormal basis (in the $L^2$-norm) for $V$ (w.l.o.g. we may assume that $1 \in V$), and $\| \ \|$ denotes the supremum norm.

In particular, if $V$ is the set of all polynomials of degree at most $t$, then for $k = 1, 2, \ldots, t$ we can set $g_k(x)$ to be

$$g_k(x) = \sqrt{k + \tfrac{1}{2}} \cdot P_k(x),$$

where $P_k(x)$ is the Legendre polynomial of degree $k$,

$$P_k(x) = \frac{1}{2^k \cdot k!} \cdot \frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^2 - 1)^k.$$

Then the set $\{1, g_1, g_2, \ldots, g_t\}$ forms an orthonormal basis for $V$.

We use the well-known formulae

$$|P_k(x)| \leqslant 1 \quad \text{and} \quad |P_k'(x)| \leqslant k(k+1)/2 \qquad \text{for } -1 \leqslant x \leqslant 1,$$

to obtain

$$\|g_k\| \leqslant (k + 1/2)^{\frac{1}{2}} \qquad \text{and} \qquad \|g_k'\| \leqslant (k + 1/2)^{\frac{1}{2}} k(k+1)/2.$$

One can easily check the inequality

$$(k + 1/2)^{\frac{1}{2}} + (t - k + 3/2)^{\frac{1}{2}} \leqslant (2t + 4)^{\frac{1}{2}},$$

which yields

$$\sum_{k=1}^{t} \sqrt{k + \tfrac{1}{2}} \leqslant \frac{t}{2} \cdot \sqrt{2t + 4}$$

for every positive integer $t$.

Using these inequalities we obtain estimates for $A$, $B$ and $C$:

$$A = \sqrt{\sum_{k=1}^{t} \|g_k\|^2} \leqslant \sqrt{\sum_{k=1}^{t} (k + \tfrac{1}{2})} = \sqrt{\frac{t(t+2)}{2}},$$

$$\begin{aligned} B = \sqrt{\sum_{k=1}^{t} (2\,\|g_k\| + 1)^2} &\leqslant \sqrt{\sum_{k=1}^{t} (2\sqrt{k + \tfrac{1}{2}} + 1)^2} \\ &= \sqrt{\sum_{k=1}^{t} (4k + 3 + 4\sqrt{k + \tfrac{1}{2}})} \\ &\leqslant \sqrt{t(2t + 2\sqrt{2t+4} + 5)} \end{aligned}$$

and

$$C = \max\{\|g_k'\| \colon k = 1, 2, \ldots, t\} \leqslant \frac{t(t+1)}{2} \cdot \sqrt{t + \tfrac{1}{2}}.$$

Hence we have the following theorem:

THEOREM 2. *We can construct t-designs on the interval* $[-1, 1]$ *consisting of n points for every integer n for which*

$$n \geqslant 2t^2(t+1) \cdot \sqrt{(t+2)(2t+1)(2t + 2\sqrt{2t+4} + 5)}.$$

This means that $m'(t) \leqslant c \cdot t^{4.5}$, where $c$ is a constant independent of $t$, and asymptotically (as $t \to \infty$) we can take $c = 4$. Let us note again that Bernstein's lower bound for $m(t)$ (and hence for $m'(t)$) is $0.06t^2$ [10], and that the exact values of these quantities are not known for $t \geqslant 10$.

## 3. Designs on the 2-Sphere

To construct designs on the sphere, we will use the following equivalent definition, cf. [7]:

*A finite subset $X$ of $S^{d-1}$ is a spherical t-design, iff*

$$\sum_{x \in X} f(x) = 0$$

*for all homogeneous harmonic polynomials $f(x)$ with $1 \leqslant \deg f(x) \leqslant t$.*

A polynomial $f(x)$ is called harmonic if it satisfies Laplace's equation $\Delta f(x) = 0$. The set of all homogeneous, harmonic polynomials of degree $s$ forms a vector space $\mathrm{Harm}_d(s)$, with

$$\dim \mathrm{Harm}_d(s) = \binom{s+d-1}{d-1} - \binom{s+d-3}{d-1}.$$

Here we will fix $d = 3$, for which $\dim \mathrm{Harm}_3(s) = 2s + 1$. One can check that the following set $H_s$ of $2s + 1$ polynomials $f(x, y, z) \in \mathrm{Harm}_3(s)$,

$$H_s = \left\{ \mathrm{Re}(y + \mathrm{i}z)^a \frac{\mathrm{d}^a}{\mathrm{d}x^a} P_s(x),\ \mathrm{Im}(y + \mathrm{i}z)^b \frac{\mathrm{d}^b}{\mathrm{d}x^b} P_s(x) \colon a = 0, 1, \ldots, s \text{ and } b = 1, 2, \ldots, s \right\}$$

forms a basis for $\mathrm{Harm}_3(s)$, where $P_s(x)$ denotes the Legendre polynomial of degree $s$, cf. [2] or [8].

For a given positive integer $m$ and real number $x$ with $-1 \leqslant x \leqslant 1$, define the set

$$U_m(x) = \{(x, (1-x^2)^{\frac{1}{2}} \cos(2k\pi/m), (1-x^2)^{\frac{1}{2}} \sin(2k\pi/m)) \colon k = 1, 2, \ldots, m\}.$$

Then the points of $U_m(x)$ form a regular $m$-gon on the surface of $S^2$. We will construct the design on the 2-sphere as a union of these regular polygons:

Theorem 3. *Let $m$ be an integer with $m \geqslant t + 1$. Suppose the set $X = \{x_1, x_2, \ldots, x_n\}$ is an interval t-design on $[-1, 1]$. Then $U_m(X)$, the union of the sets $U_m(x_j)$, for $j = 1, 2, \ldots, n$, gives a spherical t-design on $S^2$.*

Proof. We need to prove that for every polynomial $f(x, y, z)$ from each set $H_s$ ($s = 1, 2, \ldots, t$) we obtain

$$\sum_{(x,y,z) \in U_m(X)} f(x, y, z) = 0.$$

Fix an $s \leqslant t$.

(a) If $a = 0$, then $f(x, y, z) = P_s(x)$, and we have

$$\sum_{U_m(X)} f(x, y, z) = m \cdot \sum_{j=1}^{n} P_s(x_j) = \frac{mn}{2} \int_{-1}^{1} P_s(x)\, \mathrm{d}x = 0,$$

using the assumption that $X$ is a $t$-design on $[-1, 1]$ and $P_s(x)$ is a polynomial of degree $s \leqslant t$.

(b) If $1 \leqslant a \leqslant s$ (and similarly for $1 \leqslant b \leqslant s$), then for every $x$ in $X$ we obtain

$$\sum_{U_m(x)} f(x, y, z) = (1-x^2)^{a/2} \cdot \frac{\mathrm{d}^a}{\mathrm{d}x^a} P_s(x) \cdot \sum_{k=1}^{m} \mathrm{Re}\left(\cos\frac{2k\pi}{m} + \mathrm{i}\sin\frac{2k\pi}{m}\right)^a = 0,$$

since $z + z^2 + \cdots + z^m = 0$ for every complex $m$th root of unity $z$ except for $z = 1$, and $1 \leqslant a \leqslant s \leqslant t < m$ implies that $z = \cos(2a\pi/m) + \mathrm{i}\sin(2a\pi/m) \neq 1$.

Our construction of spherical designs seems very economical, since we have the following:

COROLLARY. *The minimum size of a t-design on the 2-sphere is at most* $(t+1)$ *times the minimum size of an interval t-design:*

$$M(t) \leqslant (t+1)m(t).$$

Let $M_d(t)$ denote the minimum size of a spherical $t$-design on $S^{d-1}$ (so $M_3(t) = M(t)$). The lower bound

$$M_d(t) \geqslant \begin{cases} 2\begin{pmatrix} \frac{t-1}{2}+d-1 \\ d-1 \end{pmatrix} & \text{if } t \text{ is odd,} \\ \begin{pmatrix} \frac{t}{2}+d-1 \\ d-1 \end{pmatrix} + \begin{pmatrix} \frac{t}{2}+d-2 \\ d-1 \end{pmatrix} & \text{if } t \text{ is even,} \end{cases}$$

was given by Delsarte, Goethals and Seidel in [7]. The spherical design on exactly $M_d(t)$ nodes is called tight. The theory of tight spherical designs is well developed. Bannai and Damerell in [5] and [6] proved that tight spherical designs exist for $d \geqslant 3$ iff $t = 1, 2, 3, 4, 5, 7$ or $11$. All tight spherical designs are known, except for $t = 4, 5$ and $7$. For $d = 3$ the above inequality yields

$$M(t) \geqslant \begin{cases} \dfrac{t^2}{4} + t + \frac{3}{4} & \text{if } t \text{ is odd,} \\ \dfrac{t^2}{4} + t + 1 & \text{if } t \text{ is even.} \end{cases}$$

All tight designs are known on $S^2$:
(i) for $t = 1$, $M(1) = 2$, and a pair of antipodal points is a tight 1-design;
(ii) for $t = 2$, $M(2) = 4$, and the regular tetrahedron is a tight 2-design;
(iii) for $t = 3$, $M(3) = 8$, and the regular octahedron is a tight 3-design; and
(iv) for $t = 5$, $M(5) = 12$, and the regular icosahedron is a tight 5-design on $S^2$.

The cube is a 3-design (but not a 4-design), and the dodecahedron is a 5-design (but not a 6-design). For other values of $t$, $M(t)$ has not been determined. For small values of $t$, using Bernstein's result for $t \leqslant 9$ mentioned in Section 1, the Corollary above gives $M(4) \leqslant 20$, $M(6) \leqslant 42$, $M(7) \leqslant 48$, $M(8) \leqslant 81$ and $M(9) \leqslant 90$. By a paper of W. Neutsch [12], $M(9) \leqslant 60$ (and hence $M(8) \leqslant 60$ also).

Finally, noting that a disjoint union of $t$-designs is also a $t$-design, we will give an upper bound for $M'(t)$, the smallest integer such that for any integer $N \geqslant M'(t)$, $t$-designs on $S^2$ exist having size $N$.

THEOREM 4. *Suppose that N is an integer such that*

$$N \geqslant (2t+3)m'(t) + t^2 + 3t + 2.$$

*Then we can construct a t-design on* $S^2$ *having size N, as a disjoint union of two of the designs constructed in Theorem* 3.

PROOF. Let $N_1$ be the unique integer with $m'(t) \leqslant N_1 < m'(t) + t + 2$, for which the linear congruence $(t+1)N_1 \equiv N$ has a solution $\bmod(t+2)$. Such an $N_1$ exists because

$t+1$ and $t+2$ are relatively prime. Define the integer $N_2$ with $N=(t+1)N_1+(t+2)N_2$. The assumption on $N$ then ensures that $N_2 \geqslant m'(t)$ also; hence our construction in Theorem 3 yields two (disjoint) $t$-designs on $S^2$, of sizes $(t+1)N_1$ and $(t+2)N_2$, respectively. □

Combining Theorem 4 with the upper bound for $m'(t)$ in Theorem 2 gives $M'(t)=O(t^{5.5})$. The key to improving this upper bound lies in reducing the upper bound for $m'(t)$.

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