Special difference operators and the constants in the classical Jackson-type theorems

Yu.V. Kryakin and A.G. Babenko

Wroclaw and Ekaterinburg

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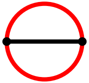
Problem. Jackson's Inequality



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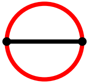

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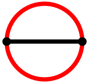

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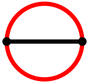

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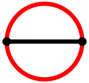

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$$\omega_m(f, \delta) := sup_{x, 0<h<\delta} \left| \widehat{\Delta}_h^m f(x) \right|$$

$$\widehat{\Delta}_h^m f(x) = \sum_{j=0}^{m} (-1)^{m+j} \binom{m}{j} f(x + (j - m/2)h)$$

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Favard–Zhuk–Shalaev–Pichugov

$$\frac{1}{2}\left([1/2\alpha]^2+1\right)\,(1-\frac{1}{2n})\le J(2,\alpha)\le\frac{1}{2}\left((1/2\alpha)^2+1\right)$$

Bohr–Favard Inequality

$$T_{n-1} := \{\tau : \tau(x) = \sum_{j=-(n-1)}^{n-1} \alpha_j \exp(ijx), \quad \alpha_j = \overline{\alpha}_{-j}\}$$

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$$\mathcal{K}_0 = 1, \ \mathcal{K}_1 = \frac{\pi}{2}, \ \mathcal{K}_2 = \frac{\pi^2}{8}, \ \mathcal{K}_3 = \frac{\pi^3}{24},$$

$$\mathcal{K}_4 = \frac{5\pi^4}{384}, \quad \mathcal{K}_6 = \frac{61\,\pi^6}{46080}, \quad \mathcal{K}_8 = \frac{277\,\pi^8}{2064384}.$$

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where $\mathcal{K}_r$ are the Favard constants:

$$\mathcal{K}_r := \frac{4}{\pi} \sum_{j=-\infty}^{\infty} (4j+1)^{-r-1}$$

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$$\frac{1}{2} \le J_a(2k,\alpha) \le (2 + 1/e^2) \cdot 3 \cdot (2\sec(\pi/2\alpha) - 1 - \mathcal{K}_2/\alpha^2)$$

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$$J_a(2k,2) < 10, \qquad n > 2k(2k-1)$$

Whitney's constants

For $f \in C(J),\ J = [0,1]$ *and* $\int_0^{j/m} f = 0,\ j = 1, \ldots, m$

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One can consider the condition

$$\int_0^{j/m} f = 0, \quad j = 1, \dots, m$$

as the Bohr–Favard type condition: $f \in StepF_m^{\perp}$

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My personal way : Beurling–Whitney (1957) $\to$ Bohr–Favard (1935) (variant with differences) $\to$ Steklov (1922) $\to$ Jackson–Stechkin (trig, 1951) $\to$ Jackson–Freud–Brudnyi (alg. 1959,1968)

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Start point : Two of my preprints, 2005

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Exact norm of difference Boman-Shapiro operator: $3 \to 2.669962\ldots$

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Consider the simple case $k = 1$, $\omega_2(f, \delta)$ and the Steklov-Boman-Shapiro operator

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Beurling used it to prove the Whitney theorem for $f \in C \cap L(\mathbb{R})$. Now, if we assume that $f \in T_{n-1}^{\perp}$ then

$$\|f * \chi_h\| \le c_2(h)\|f\|, \quad c_2 = c_2(h) < 1$$

and we have Bohr–Favard inequality for differences

$$\|f\| \le \frac{1}{1 - c_2}\, sup_x W_2(f, x, \chi_h) \le \frac{1}{2}\,\frac{1}{1 - c_2}\omega_2(f, h/2)$$

SHARP VALUE OF THE CONSTANT $c_2(2\pi/n)$

This is a problem posed by me (2005, preprint). The problem was solved in 2009 in joint work with A.G. Babenko. We found the best integral approximation of characteristic function for all values of $h > 0$. I conjectured (2005) that $\lim_{n\to\infty} c_2(2\pi/n) = \frac{4}{\pi^2} = 0.40\ldots$, but this is not correct.

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where x_1 is a root of the equation

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In general, the problem of the integral approximation of characteristic function and the problem of integral approximation of convolution powers of characteristic function are important for the uniform approximation of continuous functions. It is interesting that this problem is related to the classical Korkin–Zolotarev problem in L and the best approximation of χ_h is the linear combination of the roots of Bernstein – Szegö – Geronimus polynomials (1935)

$$Re(z^{n-1}(z-q)^2), \quad q \in (-1, 1)$$

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That gives a sharp results (for the special values of h) in terms of

$$W_2(f, x, \chi_h) := (f - f * \chi_h)(x)$$

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For fixed $h > 0$ and $k \in \mathbb{N}$, consider the operator W_{2k} :

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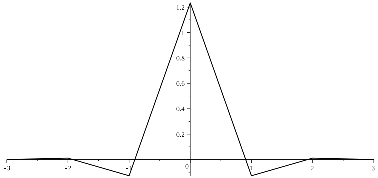
with

$$\Lambda_{2k,h}(x) = 2 \sum_{j=1}^{k} (-1)^{j+1} a_{j,k} \chi_{jh}^{2}, \qquad a_{j,k} = \binom{2k}{k+j} \Big/ \binom{2k}{k}$$

Special Kernels



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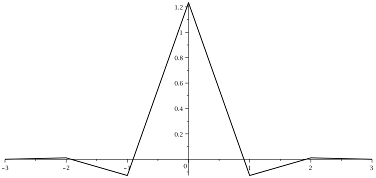

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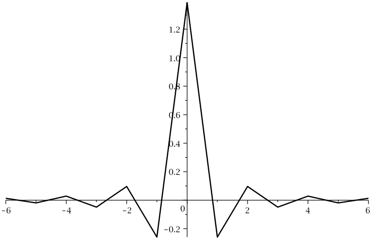

Kernel $W_{\infty,1}$

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We have the following important property:

$$\|W_{2k,h}(f,h)\| \le \|W_{\infty,h}(f,h)\| \le c_* \|f\|$$

where

$$c_* = 2.6699263\ldots$$

Jackson–Stechkin inequality. New version

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Let $f \in C(\mathbb{T})$ and $\alpha > 1$, $k, n \in \mathbb{N}$. Then

$$E_{n-1}(f) \le \sec\left(\frac{\pi}{2\alpha}\right) \; W_{2k}\left(f, \frac{\alpha\pi}{n}\right)$$

Jackson–Stechkin inequality. New version

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and

$$\sum_{j=0}^{\infty} E_{n-1}(\Lambda_{2k,h}^{j*})_L \le \sum_{j=0}^{\infty} \mathcal{K}_{2j} \alpha^{-2j} = \sec(\pi/2\alpha), \quad \text{for } h = \alpha\pi/n \quad \square$$

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If $\tau \in T_n$, $k, n \in \mathbb{N}$, $c_n(x) := \cos(nx)$, then

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Bernstein–Nikolsky–Stechkin inequality for W_{2k} implies the classical Bernstein–Markov inequality for even derivatives:

$$\|\tau^{(2k)}\| \le n^{2k}\|\tau\|$$

Algebraic approximation and constants $J_a(2k,\alpha)$

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3. Using the boundedness of $\|W_{2k}\|$

$$\|W_{2k}(f,h)\| \le 3\|f\|$$

Remark This works if $n \ge 2k(2k-1)$

$$(n-2k)!/n! < 2n^{-2k}, \quad n > 2k(2k-1), \quad k \ge 2$$

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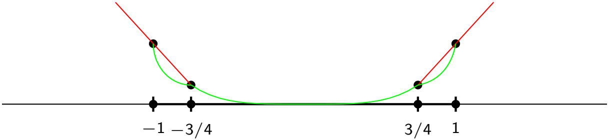
Extension and constants $J_a(2k, \alpha)$



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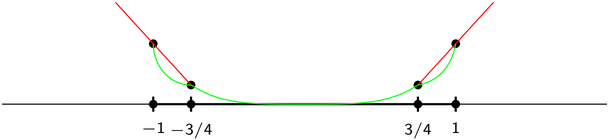

$$k = 1, \;\; h = 1/8, \;\; W_2, \;\; \omega_2$$

Let $f \in C(\mathbb{I})$, $k \in \mathbb{N}$, $0 < h < (2k)^{-1}$. Then there exists $g_f := g_{f,k,h}$, equal to f on $\mathbb{I}$, continuous on $\mathbb{R} \setminus \mathbb{I}$, such that

$$\|W_{2k}(g_f, \cdot, h)\| \le c^* \,\omega_{2k}(f, h), \quad c^* = 3 \cdot (2 + \exp(-2)) \,.$$

The above estimate false if instead of c^* one uses any $c_k^* \to 0$, $k \to \infty$.

Open Problems

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If $\int_0^{j/m} f(t)\,dt = 0,\ j = 1, \dots, m$, *then*

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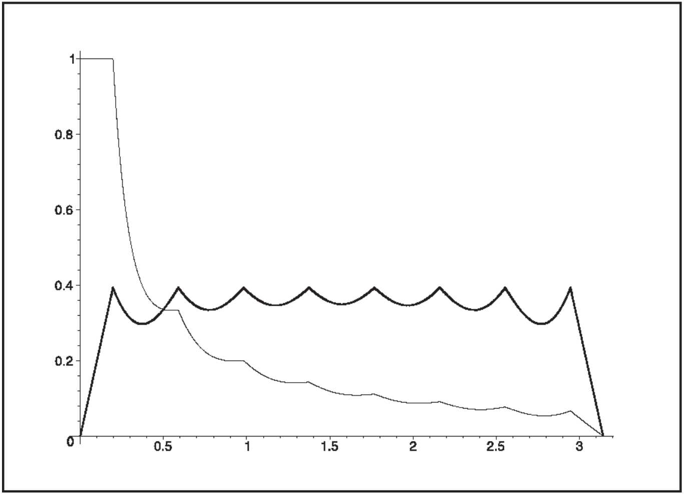
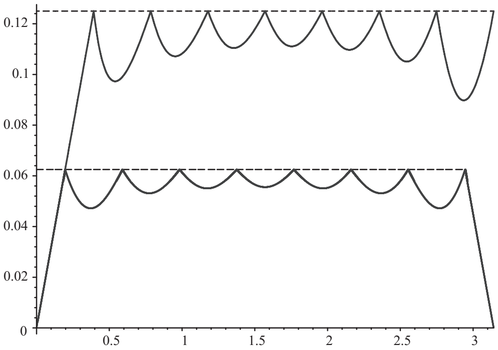
BEST APPROXIMATION OF χ_h AND $\overline{\chi}_h = h \cdot \chi_h$



The best trigonometric integral approximations $E_7(\chi_{2h}),\ E_7(\overline{\chi}_{2h}), \quad h \in [0, \pi]$

One-side best approximation of $\overline{\chi}_h$



The diagrams display the graphs of the functions $(2\pi)^{-1}\,E_n^-(\overline{\chi}_{2h})_L$ and $(2\pi)^{-1}\,E_n(\overline{\chi}_{2h})$ of the variable $h \in (0, \pi]$ as well as graphs of the constant functions $\frac{1}{n+1}$ and $\frac{1}{2(n+1)}$ (the dashed lines) for $n = 7$;

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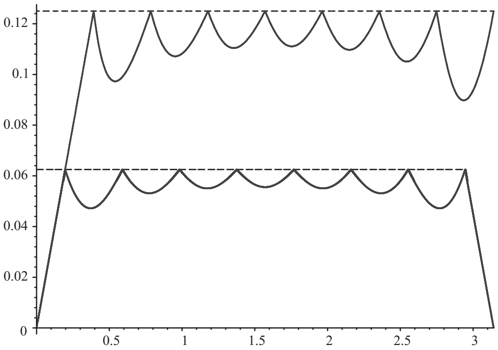

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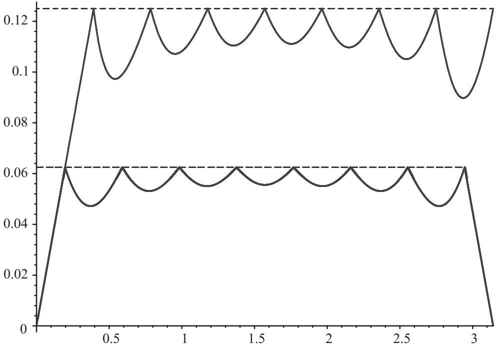

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One-side approximation. Number Theory, works by Beurling, Selberg, Vaaler, see for example
H. L. Montgomery *Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis*, 1994.

People and hints

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Hint: look for the integral equation (cherchez la femme)

$$W_{2k}(f, x, h) = f(x) - (f * \Lambda_{2k,h})(x) \qquad (W)$$