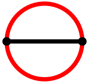
Special Special difference operators and the constants in the classical Jackson-type theorems

Yu.V. Kryakin and A.G. Babenko

Wroclaw and Ekaterinburg

May 3, 2017

Problem. Jackson Inequality



$\mathbb{T} = [0, 2\pi)$ $\qquad\qquad$ $\mathbb{I} = [-1, 1]$

τ_{n-1} $\qquad\qquad$ p_{n-1}

$$f \in C(A)$$

$$E_{n-1}(f) := \inf_{\tau_{n-1}} \|f - \tau_{n-1}\| \le J(m, \alpha)\,\omega_m(f, \frac{\alpha\pi}{n}), \quad \alpha > 0$$

$$E^a_{n-1}(f) := \inf_{p_{n-1}} \|f - p_{n-1}\| \le J_a(m, \alpha)\,\omega_m(f, \frac{\alpha\pi}{n}), \quad n > m$$

$$\omega_m(f, \delta) := sup_{x, 0<h<\delta} \left|\widehat{\Delta}^m_h f(x)\right|$$

Sharp results. Steklov, Favard, Kornejchuk

Kornejchuk

$$\frac{1}{2}\left([1/\alpha]+1\right)\,\left(1-\frac{1}{2n}\right) \le J(1,\alpha) \le \frac{1}{2}(1/\alpha+1), \quad \alpha > 0$$

Favard–Zhuk–Shalaev-Pichugov

$$\frac{1}{2}\left([1/2\alpha]^2+1\right)\,(1-\frac{1}{2n}) \le J(2,\alpha) \le \frac{1}{2}\left((1/2\alpha)^2+1\right)$$

Bohr–Favard Inequality

$$T_{n-1} := \{\tau : \tau(x) = \sum_{j=-(n-1)}^{n-1} \alpha_j \exp(ijx), \quad \alpha_j = \overline{\alpha}_{-j}\}$$

$$T_{n-1}^{\perp} := \{f \in L : (f,\tau) = 0,\ \tau \in T_{n-1}\}$$

$$\|g\| \le \frac{\mathcal{K}_r}{n^r}\,\|g^{(r)}\|, \quad g \in C^r(\mathbb{T}), \quad g \in T_{n-1}^{\perp}$$

$$\mathcal{K}_0 = 1,\ \mathcal{K}_1 = \frac{\pi}{2},\ \mathcal{K}_2 = \frac{\pi^2}{8},\ \mathcal{K}_3 = \frac{\pi^3}{24},$$

$$\mathcal{K}_4 = \frac{5\pi^4}{384}, \quad \mathcal{K}_6 = \frac{61\,\pi^6}{46080}, \quad \mathcal{K}_8 = \frac{277\,\pi^8}{2064384}.$$

Jackson–Stechkin Constansts, $m = 2k$

$$\frac{1}{\binom{2k}{k}} \le J(2k,\alpha) \le \frac{1}{\binom{2k}{k}} \cdot \sec(\pi/2\alpha), \qquad \alpha > 1$$

$$\sec(\pi/2\alpha) \asymp \frac{2}{\pi}\,\frac{1}{\alpha - 1}, \qquad \alpha \to 1$$

$$\alpha = 2, \quad 2\pi/n, \quad \sqrt{2}$$

$$\sec(\pi/2\alpha) = \sum_{j=0}^{\infty} \mathcal{K}_{2j}\alpha^{-2j}$$

$$\mathcal{K}_r := \frac{4}{\pi} \sum_{j=-\infty}^{\infty} (4j+1)^{-r-1}$$

Jackson–Brudnyi Constants, $m = 2k$

$$\frac{1}{2} \le J_a(2k,\alpha) \le (2 + 1/e^2)\cdot 3 \cdot (2\sec(\pi/2\alpha) - 1 - \mathcal{K}_2/\alpha^2)$$

$$\alpha > 1, \quad n > 2k(2k-1)$$

$2 + 1/e^2$ – estimate of Whitney's constant (J.Gilewicz , I.Shevchuk, K)

3 – estimate of Boman–Shapiro operators norm

$\mathcal{K}_2 := \frac{\pi^2}{8}$ – second Favard's constant

$$J_a(2k,2) < 10, \qquad n > 2k(2k-1)$$

Whitney's constants

For $f \in C(J),\ J = [0,1]$ *and* $\int_0^{j/m} f = 0,\ j = 1, \dots, m$

$$\|f\| \le (2 + 1/e^2)\,\omega_m(f, 1/m)$$

$$\int_0^{j/m} f = 0, \quad j = 1, \dots, m$$

as the Bohr–Favard type condition: $f \in StepF_m^{\perp}$

People

K, Shadrin, Babenko, Zhuk, Vinogradov, Facourt, Staszak, Dolmatova, 2005–2017

My personal way : Beurling–Whitney (1957) $\rightarrow$ Bohr–Favard (1935) (variant with differences) $\rightarrow$ Steklov (1922) $\rightarrow$ Jackson–Stechkin (trig, 1951) $\rightarrow$ Jackson–Brudnyi (alg. 1968)

Start point : Two my preprints, 2005

Results

$$\frac{1}{\binom{2k}{k}} \le J(2k,\alpha) \le \frac{1}{\binom{2k}{k}} \cdot \sec(\pi/2\alpha), \qquad \alpha > 1$$

$$\frac{1}{2} \le J_a(2k,\alpha) \le (2+1/e^2)\cdot 3 \cdot (2\sec(\pi/2\alpha) - 1 - \mathcal{K}_2/\alpha^2)$$

$$\alpha > 1, \quad n > 2k(2k-1)$$

Remarks
Sendov-K conjecture: $2 + 1/e^2 \to 1$
Exact norm of the difference Boman-Shapiro operator: $3 \to 2.669962\ldots$

How to prove it?

Main idea on Whitney inequality is due to ... Arne Beurling and Vladimir Steklov.
Consider the simple case $k = 1$, $\omega_2(f, \delta)$ and the Steklov-Boman-Shapiro operator

$$W_2(f, x, \chi_h) := f(x) - \frac{1}{h} \int_{-h/2}^{h/2} f(x + t)\, dt = (f - f * \chi_h)(x)$$

Beurling used it to prove the Whitney theorem for $f \in C \cap L(\mathbb{R})$. Now, if we suppose that $f \in T_{n-1}^{\perp}$ then

$$\|f * \chi_h\| \leq c_2(h)\|f\|, \quad c_2 = c_2(h) < 1$$

and we have Bohr–Favard inequality for differences

$$\|f\| \leq \frac{1}{1 - c_2}\, sup_x W_2(f, x, \chi_h) \leq \frac{1}{2}\, \frac{1}{1 - c_2} \omega_2(f, h/2)$$

Sharp value of constant $c_2(2\pi/n)$

This is my problem (2005, preprint). The problem was solved in 2009 in joint work with A.G. Babenko. We found the best integral approximation of characteristic function for all values of $h > 0$. I conjectured (2005) that $\lim_{n\to\infty} c_2(2\pi/n) = \frac{4}{\pi^2} = 0.40\ldots$, but this is not correct.

$$\lim_{n\to\infty} c_2(2\pi/n) = 0.3817\cdots = 1 - 2x_1$$

where x_1 is the root of equation

$$\cos(\pi x) = \frac{2x}{1+x^2}, \quad x \in (0, 1/2)$$

In general, the problem of the integral approximation of characteristic function and the problem of integral approximation of convolution powers of characteristic function are important for the uniform approximation of continuous functions. It is interesting that this problem is related to the classical Korkin–Zolotarev problem in L and the best approximation of χ_h is the linear combination of the roots of Bernstein – Szegö – Geronimus polynomials (1935)

$$Re(z^{n-1}(z-q)^2), \quad q \in (-1, 1)$$

C. Neumann series

The next step to good estimates of $J(2k, \alpha)$ is the using convolution series (Cambridge, May, 2006, K., Shadrin) In the case $k = 1$ one can use instead of Steklov–Favard decomposition

$$f = f - f * \chi_h + f * \chi_h, \qquad \int_{\mathbb{R}} \chi_h(t)\, dt = 1$$

the Nuemann series

$$f = f - f * \chi_h + f * \chi_h - f * \chi_h * \chi_h + \cdots = \sum_{j=0}^{\infty} (f - f * \chi_h) * \chi_h^{j*}$$

This gives the sharp results in terms of

$$W_2(f, x, \chi_h) := (f - f * \chi_h)(x)$$

Special Boman–Shapiro difference operators

For fixed $h > 0$ and $k \in \mathbb{N}$, consider the operator W_{2k} :

$$W_{2k}(f, x, h) := (-1)^k \frac{1}{\binom{2k}{k}} \int_{\mathbb{R}} \Delta_t^{2k} f(x)\, \chi_h^{2*}(t)\, dt$$

Here

$$\Delta_t^{2k} f(x) = \sum_{j=-k}^{k} (-1)^{j+k} \binom{2k}{k+j} f(x + jt)$$

$$\chi_h^{2*}(x) := \begin{cases} \frac{1}{h}(1 - \frac{|x|}{h}), & x \in (-h, h) \\ 0, & x \notin (-h, h) \end{cases}$$

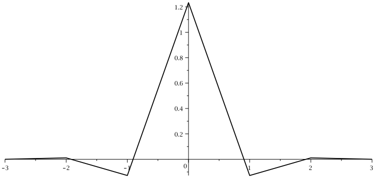
We have for $f \in C(\mathbb{T})$

$$W_{2k}(f, x, h) = f(x) - (f * \Lambda_{2k,h})(x) \qquad (ie)$$

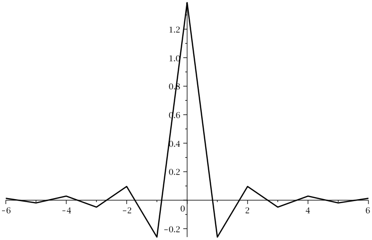
with

$$\Lambda_{2k,h}(x) = 2 \sum_{j=1}^{k} (-1)^{j+1} a_{j,k} \chi_{jh}^2, \qquad a_{j,k} = \binom{2k}{k+j} \Big/ \binom{2k}{k}$$

Special Kernels



Kernel $W_{6,1}$



Kernel $W_{\infty,1}$

Norms of W_{2k}

Note that

$$W_{2k}(f,h) := \|W_{2k}(f,\cdot,h)\| \le \frac{1}{\binom{2k}{k}}\omega_{2k}(f,h)$$

We have the following important property:

$$\|W_{2k,h}(f,h)\| \le \|W_{\infty,h}(f,h)\| \le c_* \|f\|$$

with

$$c_* = 2.6699263\ldots$$

Jackson–Stechkin inequality. New version

Let $f \in C(\mathbb{T})$ and $\alpha > 1$, $k, n \in \mathbb{N}$. Then

$$E_{n-1}(f) \le \sec\left(\frac{\pi}{2\alpha}\right) \; W_{2k}\left(f, \frac{\alpha\pi}{n}\right)$$

Proof. The main idea is to use the following telescoping identity

$$f = \sum_{j=0}^{\infty} (f - f * \Lambda_{2k,h}) * \Lambda_{2k,h}^{j*}$$

We have

$$E_{n-1}(f) \le \left(\sum_{j=0}^{\infty} E_{n-1}(\Lambda_{2k,h}^{j*})_L \right) W_{2k}(f, h)$$

and

$$\sum_{j=0}^{\infty} E_{n-1}(\Lambda_{2k,h}^{j*})_L \le \sum_{j=0}^{\infty} \mathcal{K}_{2j} \alpha^{-2j} = \sec(\pi/2\alpha), \quad \text{for } h = \alpha\pi/n \quad \square$$

Bernstein–Nikolsky–Stechkin inequlity

If $\tau \in T_n$, $k, n \in \mathbb{N}$, $c_n(x) := \cos(nx)$, then

$$\|\tau^{(2k)}\| \le \frac{n^{2k}}{W_{2k}(c_n, h)} W_{2k}(\tau, h), \quad h \in (0, 2\pi/n]$$

Bernstein–Nikolsky–Stechkin inequality for W_{2k} implies the classical Bernstein–Markov inequality for even derivatives:

$$\|\tau^{(2k)}\| \le n^{2k}\|\tau\|$$

Algebraic approximation and constants $J_a(2k,\alpha)$

1. Extension by polynomials of best approximation and Whitney's theorem with constant $2+\exp(-2)$.
2. Favard-type inequality for algebraic approximation (Sinwel, 1981):

$$E^a_{n-1}(f) \le \mathcal{K}_{2k} \frac{\|f^{(2k)}\|(n-2k)!}{n!}$$

3. Using the boundedness of $\|W_{2k}\|$

$$\|W_{2k}(f,h)\| \le 3\|f\|$$

Remark This works if $n \ge 2k(2k-1)$

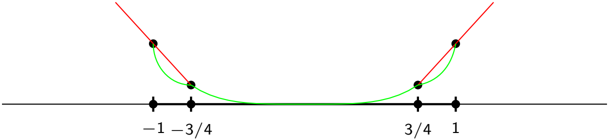
$$(n-2k)!/n! < 2n^{-2k}, \quad n > 2k(2k-1), \quad k \ge 2$$

Truncated Neumann convolution series

$$g_f = \sum_{j=0}^{k-1} (g_f - g_f * \Lambda_{2k}) * \Lambda_{2k}^{j*} + g_f * \Lambda_{2k}^{k*}$$

$$\Lambda_{2k}^{j*} := \Lambda_{2k}^{j-1\,*} * \Lambda_{2k}, \quad \Lambda_{2k}^{0*} = 1$$

Extension and constants $J_a(2k,\alpha)$



$$k = 1, \;\; h = 1/8, \;\; W_2, \;\; \omega_2$$

Let $f \in C(\mathbb{I}), \; k \in \mathbb{N}, \; 0 < h < (2k)^{-1}$. Then there exists $g_f := g_{f,k,h}$, equal to f on $\mathbb{I}$, continuous on $\mathbb{R} \setminus \mathbb{I}$, such that

$$\|W_{2k}(g_f, \cdot, h)\| \leq c^* \, \omega_{2k}(f, h), \qquad c^* = 3 \cdot (2 + \exp(-2))$$

and we can not take $c_k^* \to 0, \;\; k \to \infty$, instead of c^*

Open Problems

1. Prove the modified Sendov's conjecture

$$\textit{If } \int_0^{j/m} f(t)\,dt = 0,\ j = 1, \dots, m, \textit{ then}$$

$$\|f\| \le \omega_m(f, 1/m)$$

This problem is solved for $m = 1, \dots, 8$ (K., Zhelnov)

2. Find the best constant $c_\alpha = c_\alpha(k)$ in the inequality

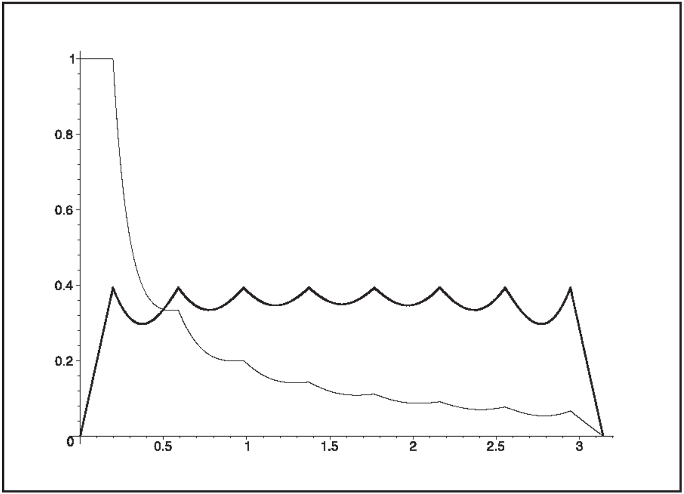
$$E_{n-1}(f) \le c_\alpha W_{2k}\left(f, \frac{\alpha\pi}{n}\right), \quad \alpha > 0$$

Problem is solved only for $\alpha = 1, 3, \dots$ (Favard's case) and in the case $k = 1$ for $\alpha \in (0, 1)$

3. Find the best integral trigonometric approximation $E_{n-1}(\chi_h^{m*})_L$, $h > 0$

Problem is solved only for $m = 1$ and for $0 < h < \pi/n$ (unpublished)

Best approximation of χ_h and $\overline{\chi}_h = h \cdot \chi_h$



The best trigonometric integral approximation $E_7(\chi_{2h}),\ E_7(\overline{\chi}_{2h}), \quad h \in [0, \pi]$

One-side best approximation of $\overline{\chi}_h$

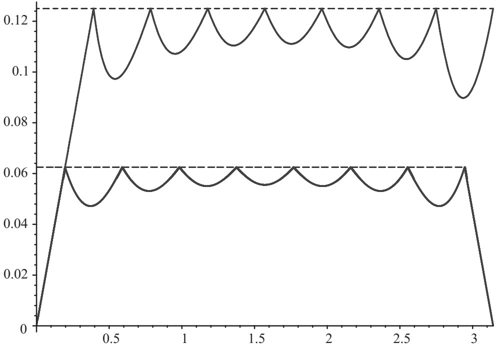


Figure shows graphs of the functions $(2\pi)^{-1}\,E_n^-(\overline{\chi}_{2h})_L$ and $(2\pi)^{-1}\,E_n(\overline{\chi}_{2h})$ of the variable $h \in (0, \pi]$ as well as graphs of the constant functions $\frac{1}{n+1}$ and $\frac{1}{2(n+1)}$ (the dashed lines) for $n = 7$;

V.A.Yudin, A.G. Babenko, K, 2011.

One-side approximation. Number Theory, works by Beurling, Selberg, Vaaler, see for example

H. L. Montgomery *Ten Lectures on the Interface between Analytic*

People and hints

1. C. Neumann (1832 –1925) (mathematical physics, integral equation):

$$(I - A)^{-1} = I + A + A^2 + \dots$$

2. A.V. Steklov (1863 – 1826) (mathematical physics, theoretical mechanics):

$$f = f - f * \chi_h + f * \chi_h$$

3. J. Favard (1902 – 1965) (analysis):

$$\|f\| \le \mathcal{K}_m n^{-m} \|f^{(m)}\|, \quad f \in T_{n-1}^{\perp}$$

4. A. Beurling (1905 – 1986) (harmonic analysis, potential theory):

$$\|f\|_{\mathbb{R}} \le \binom{2k}{k}^{-1} \sup_{0<h<\infty} \|\widehat{\Delta}_h^{2k} f\|_{\mathbb{R}}, \quad f \in C \cap L(\mathbb{R})$$

Hint: cherchez la femme as look for the integral equation

$$W_{2k}(f, x, h) = f(x) - (f * \Lambda_{2k,h})(x) \qquad (W)$$