Special Special difference operators and the constants in the classical Jackson-type theorems

Yu.V. Kryakin and A.G. Babenko

Wroclaw and Ekaterinburg

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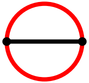
Problem. Jackson's Inequality



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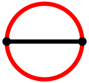

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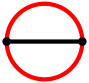

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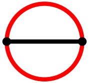

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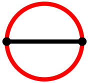

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$$\omega_m(f, \delta) := sup_{x, 0<h<\delta} \left|\widehat{\Delta}^m_h f(x)\right|$$

$$\widehat{\Delta}^m_h f(x) = \sum_{j=0}^{m} (-1)^{m+j} \binom{m}{j} f(x + (j - m/2)h)$$

Sharp results. Steklov, Favard, Kornejchuk

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Favard–Zhuk–Shalaev-Pichugov

$$\frac{1}{2}\left([1/2\alpha]^2+1\right)\,(1-\frac{1}{2n}) \le J(2,\alpha) \le \frac{1}{2}\left((1/2\alpha)^2+1\right)$$

Bohr–Favard Inequality

$$T_{n-1} := \{\tau : \tau(x) = \sum_{j=-(n-1)}^{n-1} \alpha_j \exp(ijx), \quad \alpha_j = \overline{\alpha}_{-j}\}$$

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$$\mathcal{K}_0 = 1,\ \mathcal{K}_1 = \frac{\pi}{2},\ \mathcal{K}_2 = \frac{\pi^2}{8},\ \mathcal{K}_3 = \frac{\pi^3}{24},$$

$$\mathcal{K}_4 = \frac{5\pi^4}{384}, \quad \mathcal{K}_6 = \frac{61\,\pi^6}{46080}, \quad \mathcal{K}_8 = \frac{277\,\pi^8}{2064384}.$$

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where $\mathcal{K}_r$ are the Favard constants:

$$\mathcal{K}_r := \frac{4}{\pi}\sum_{j=-\infty}^{\infty}(4j+1)^{-r-1}$$

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$$J_a(2k,2) < 10, \qquad n > 2k(2k-1)$$

Whitney's constants

For $f \in C(J),\ J = [0,1]$ *and* $\int_0^{j/m} f = 0,\ j = 1, \dots, m$

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One can consider the condition

$$\int_0^{j/m} f = 0, \quad j=1,\dots,m$$

as the Bohr–Favard type condition: $f \in StepF_m^{\perp}$

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My personal way : Beurling–Whitney (1957) $\to$ Bohr–Favard (1935) (variant with differences) $\to$ Steklov (1922) $\to$ Jackson–Stechkin (trig, 1951) $\to$ Jackson–Freud–Brudnyi (alg. 1959,1968)

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Start point : Two my preprints, 2005

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Exact norm of difference Boman-Shapiro operator: $3 \to 2.669962\ldots$

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Beurling used it to prove the Whitney theorem for $f \in C \cap L(\mathbb{R})$. Now, if we suppose that $f \in T_{n-1}^{\perp}$ then

$$\|f * \chi_h\| \le c_2(h)\|f\|, \quad c_2 = c_2(h) < 1$$

and we have Bohr–Favard inequality for differences

$$\|f\| \le \frac{1}{1 - c_2}\ sup_x W_2(f, x, \chi_h) \le \frac{1}{2}\frac{1}{1 - c_2}\omega_2(f, h/2)$$

Sharp value of constant $c_2(2\pi/n)$

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In general, the problem of the integral approximation of characteristic function and the problem of integral approximation of convolution powers of characteristic function are important for the uniform approximation of continuous functions. It is interesting that this problem is related to the classical Korkin–Zolotarev problem in L and the best approximation of χ_h is the linear combination of the roots of Bernstein – Szegö – Geronimus polynomials (1935)

$$Re(z^{n-1}(z-q)^2), \quad q \in (-1, 1)$$

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This gives the sharp results (for the special values of h) in terms of

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We have for $f \in C(\mathbb{T})$

$$W_{2k}(f, x, h) = f(x) - (f * \Lambda_{2k,h})(x) \qquad (ie)$$

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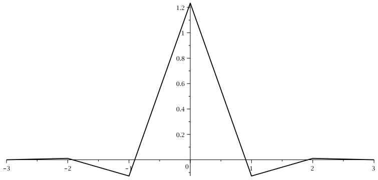
$$\Lambda_{2k,h}(x) = 2\sum_{j=1}^{k} (-1)^{j+1} a_{j,k} \chi_{jh}^2, \qquad a_{j,k} = \binom{2k}{k+j} \Big/ \binom{2k}{k}$$

Special Kernels



Kernel $W_{6,1}$

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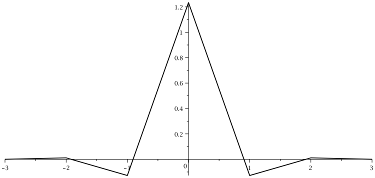

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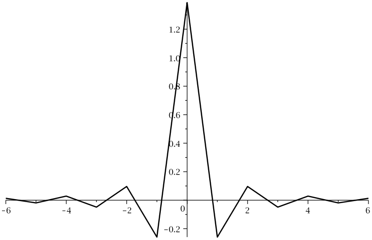

Kernel $W_{\infty,1}$

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We have the following important property:

$$\|W_{2k,h}(f,h)\| \le \|W_{\infty,h}(f,h)\| \le c_* \|f\|$$

with

$$c_* = 2.6699263\ldots$$

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Let $f \in C(\mathbb{T})$ and $\alpha > 1$, $k, n \in \mathbb{N}$. Then

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Proof. The main idea is to use the following telescoping identity

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$$\sum_{j=0}^{\infty} E_{n-1}(\Lambda_{2k,h}^{j*})_L \le \sum_{j=0}^{\infty} \mathcal{K}_{2j}\alpha^{-2j} = \sec(\pi/2\alpha), \quad \text{for } h = \alpha\pi/n \quad \square$$

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Bernstein–Nikolsky–Stechkin inequality for W_{2k} implies the classical Bernstein–Markov inequality for even derivatives:

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3. Using the boundedness of $\|W_{2k}\|$

$$\|W_{2k}(f,h)\| \le 3\|f\|$$

Remark This works if $n \ge 2k(2k-1)$

$$(n-2k)!/n! < 2n^{-2k}, \quad n > 2k(2k-1), \quad k \ge 2$$

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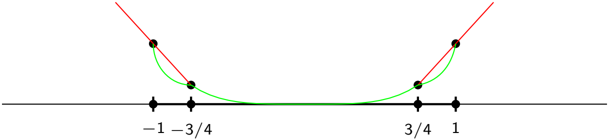
Extension and constants $J_a(2k, \alpha)$



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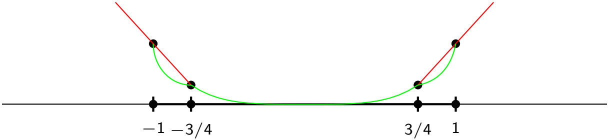

$$k = 1, \;\; h = 1/8, \;\; W_2, \;\; \omega_2$$

Let $f \in C(\mathbb{I}), \; k \in \mathbb{N}, \; 0 < h < (2k)^{-1}$. Then there exists $g_f := g_{f,k,h}$, equal to f on $\mathbb{I}$, continuous on $\mathbb{R} \setminus \mathbb{I}$, such that

$$\|W_{2k}(g_f, \cdot, h)\| \le c^* \omega_{2k}(f, h), \qquad c^* = 3 \cdot (2 + \exp(-2))$$

and we can not take $c_k^* \to 0, \;\; k \to \infty$, instead of c^*

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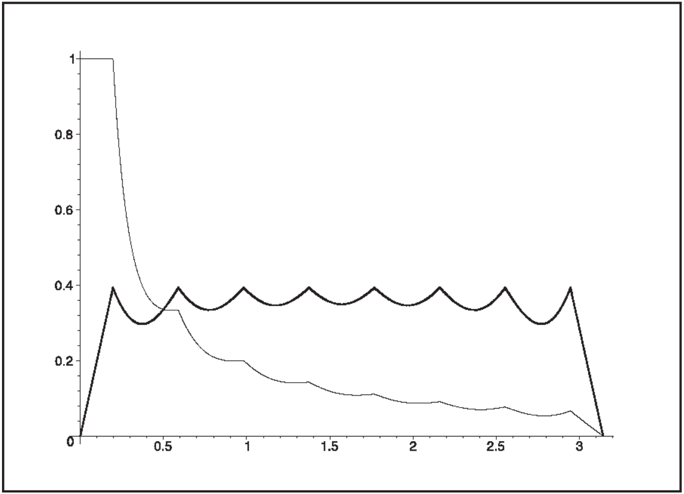
Best approximation of χ_h and $\overline{\chi}_h = h \cdot \chi_h$



The best trigonometric integral approximation $E_7(\chi_{2h})$, $E_7(\overline{\chi}_{2h})$, $\quad h \in [0, \pi]$

ONE-SIDE BEST APPROXIMATION OF $\overline{\chi}_h$

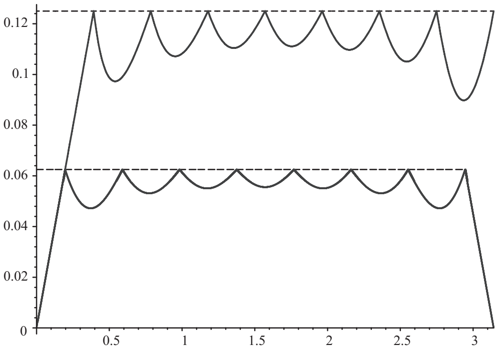


Figure shows graphs of the functions $(2\pi)^{-1}\, E_n^-(\overline{\chi}_{2h})_L$ and $(2\pi)^{-1}\, E_n(\overline{\chi}_{2h})$ of the variable $h \in (0, \pi]$ as well as graphs of the constant functions $\frac{1}{n+1}$ and $\frac{1}{2(n+1)}$ (the dashed lines) for $n = 7$;

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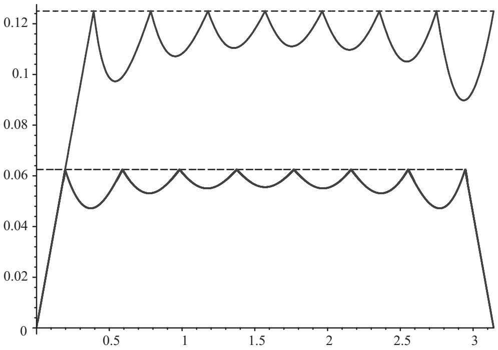

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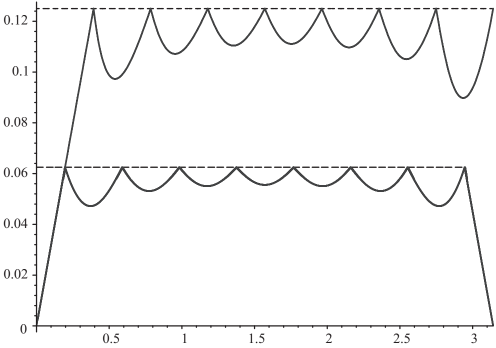

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One-side approximation. Number Theory, works by Beurling, Selberg, Vaaler, see for example
H. L. Montgomery *Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis*, 1994.

People and hints

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Hint: cherchez la femme as look for the integral equation

$$W_{2k}(f, x, h) = f(x) - (f * \Lambda_{2k,h})(x) \qquad (W)$$