Behavior of maximal functios in $\mathbf{R}^n$ for large n

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1. Introduction

Let M denote the standard maximal function representing the supremum of averages taken over balls in $\mathbf{R}^n$, that is,

$$M(f)(x) = M^{(n)}f(x) = \sup_{0<r} c_n \frac{1}{r^n} \int_{|y|\leqq r} |f(x-y)|\, dy,$$

where c_n^{-1} is the volume of the unit ball. It has recently been proved (see [2]), that the L^p bounds for $M, p>1$, can be taken to be independent of n. Namely one has

Theorem A. *We have*

$$\|M^{(n)}(f)\|_p \leqq A_p \|f\|_p, \quad 1<p\leqq\infty, \tag{1.1}$$

with a constant A_p *independent of* n.

What is noteworthy here is that any of the usual covering arguments lead only to a weak-type (1,1) bound which grows exponentially in n, and thus by interpolation one obtains by this method (1.1) with A_p replaced by a bound which increases exponentially in n.

Thus the following further questions now present themselves:

(1) Does $M^{(n)}$ have a weak-type (1, 1) bound independent of n?

(2) What can be said when the usual balls are replaced by dilates of more general sets?

We give here some partial answers to these questions:

(a) First, let B be any bounded, open, convex, and symmetric set in $\mathbf{R}^n$, and let $B^r = \{x | r^{-1}x \in B\}$, $r>0$. Define $M = M_B$ by

$$M_B(f)(x) = \sup_{r>0} (m(B^r))^{-1} \int_{B^r} |f(x-y)|\, dy.$$

Then M_B has a weak-type bound majorized by $cn \log n$. (Here c is a constant which is of course independent of n and B.) The main idea of the proof of this result (Theorem 1) is a rather complicated variant of the Vitali covering idea. One can also obtain by rather simpler arguments an L^p estimate (Theorem 2); the result is $\|M_B(f)\|_p < cn(p/(p-1))\|f\|_p$. This is optimal as far as the behaviour of the bound when $p \to 1$, but not necessarily best possible when $n \to \infty$.

(b) When B is the usual unit ball in $\mathbf{R}^n$, we can show by different arguments that the weak-type bound can be taken to be cn (Theorem 3), and the L^p bound can be taken to be $cn^{\frac{1}{2}}(p/(p-1))$ (Theorem 4). Here one relies on the abstract version of the maximal ergodic theorem, and the maximal theorem for symmetric diffusion semi-groups.

Finally in an appendix we give the details of the proof of theorem A, since these have not appeared before.

2. The case of general B

Suppose B is an open, bounded, convex, and symmetric set in $\mathbf{R}^n$. We denote by B^r its dilate by the factor r i.e. $B^r = \{x | r^{-1}x \in B\}$. Let

$$M(f)(x) = \sup_{r>0} \frac{1}{m(B^r)} \int_{B^r} |f(x-y)|\, dy.$$

Theorem 1. *There exists a constant c, independent of B and $n, n>1$, so that:*

$$m\{x | M(f)(x) > \lambda\} \leq \frac{c}{\lambda} n \log n \|f\|_1, \quad \lambda > 0. \tag{2.1}$$

We shall denote by $|x|_B$ the norm on $\mathbf{R}^n$ induced by B, i.e. $|x|_B = \inf\{r | r^{-1}x \in B\}$.

We shall also need the following terminology. The ball of radius r with center x_0, $B^r(x_0)$, is the set $\{x | x - x_0 \in B^r\}$. Suppose B is any ball (with radius r and center x_0), then we denote by B^* the ball with radius nr and the same center. (Later we shall also have occasion to use the balls B^{**} and B^{***}, both having the same center x_0, but with radius respectively $(n+1)r$, and $(n+2)r$.)

The theorem will be a consequence of the following lemma

Lemma. *Let $\{B_\alpha\}_\alpha$ be any finite collection of balls. Then we can find a subcollection $B_1, B_2, \ldots, B_N$ with the following properties. If we denote by I_k the "increment" of B_k with respect to $B_1 \cup \ldots \cup B_{k-1}$, i.e. $I_k = B_k \setminus (B_1 \cup \ldots \cup B_{k-1})$,* *then:*

$$m\Big(\bigcup_{\alpha} B_\alpha\Big) \leqq c_1 m\left(\bigcup_{j=1}^{N} B_j\right) \tag{1}$$

$$\sum_{j=1}^{N} \frac{m(I_j)}{m(B_j^*)} \chi_{B_j^*} \leqq c_2 n \log n. \tag{2}$$

Let us first show how the lemma implies the theorem. We shall assume that $f \geqq 0$. Instead of M we consider $\tilde{M}$ defined by $(\tilde{M}f)(x) = \sup_{B \ni x} \frac{1}{m(B^*)} \int_{B^*} f(y)\, dy$. It is obvious that $\tilde{M}f(x) \geqq Mf(x)$ (and in fact it is also easy to see that $\tilde{M}f(x) \leqq eMf(x)$), and we shall prove (2.1) with $\tilde{M}$ in place of M.

We let $E_\lambda = \{x | \tilde{M}f(x) > \lambda\}$, and K any compact set so that $K \subset E_\lambda$. For each $x \in K$, there exists a ball $B(x)$ with $x \in B(x)$, so that

$$\frac{1}{m(B^*(x))} \int_{B^*(x)} f(y)\, dy > \lambda.$$

By compactness of K we can select a finite collection (call it $\{B_\alpha\}_\alpha$) of balls $B(x)$ which cover K. Now let $B_1, \ldots, B_n$ be the sub-collection whose existence is guaranteed by the Lemma. We have

$$m(K) \leqq m\Big(\bigcup_{\alpha} B_\alpha\Big) \leqq c_1 m\left(\bigcup_{j=1}^{N} B_j\right);$$

however

$$m\left(\bigcup_{j=1}^{N} B_j\right) = m\left(\bigcup_{j=1}^{N} I_j\right) = \sum_{j=1}^{N} m(I_j),$$

since the I_j are mutually disjoint. Moreover

$$m(I_j) = \frac{m(I_j)}{m(B_j^*)} m(B_j^*), \quad \text{and} \quad m(B_j^*) < (1/\lambda) \int_{B_j^*} f(y)\, dy.$$

Thus $$\sum_{j=1}^{N} m(I_j) \leqq \frac{1}{\lambda} \int \sum_{j=1}^{N} \frac{m(I_j)}{m(B_j^*)} \chi_{B_j^*}(y) f(y)\, dy = \frac{c_2}{\lambda} n \log n f(y)\, dy.$$

This proves the inequality $m(K) \leqq \frac{c}{\lambda} n \log n \|f\|_1$, with $c = c_1 c_2$. If we take the supremum over all $K \subset E$, we get (2.1).

Proof of lemma. We describe the method of picking $B_1, \ldots, B_N$. Pick B_1 to have maximal radius. Assume now $B_1, \ldots, B_{k-1}$ are already picked (this of course defines the increment sets $I_1, \ldots, I_{k-1}$). Pick B_k to have the maximal radius among all balls whose centers y_k satisfy.

$$\sum_{j=1}^{k-1} \frac{m(I_j)}{m(B_j^*)} \chi_{B_j^{**}}(y_k) \leqq 1. \tag{2.2}$$

Recall that B_j^{**} is the ball with the same center as B_j but whose radius is expanded by the factor $n+1$.

First we prove conclusion (1) of the lemma.

Suppose B_α is a ball not in the collection picked. We claim that

$$\sum_{j=1}^{N} \frac{m(I_j)}{m(B_j^*)} \chi_{B_j^{***}}(x) > 1, \quad \text{for} \quad x \in B_\alpha. \tag{2.3}$$

In comparing (2.3) with (2.2) we should recall that B_j^{***} is the ball with the same center as B_j, but whose radius is expanded by the factor $n+2$. To see (2.3) let r_α be the radius of B_α, and y_α its center, and consider those balls B_j (with radius r_j), for which $r_j \geqq r_\alpha$. Observe that if $y_\alpha \in B_j^{**}$, and $x \in B_\alpha$, then $x \in B_j^{***}$. (Because $|y_\alpha - y_j|_B < (n+1) r_j$, and $|x - y_\alpha|_B < r_\alpha$ implies $|x - y_j| < (n+2) r_j$.) Therefore since

$$\sum_{r_j > r_\alpha} \frac{m(I_j)}{m(B_j^*)} \chi_{B_j^{**}}(y_\alpha) > 1$$

(because the ball B_α was not picked) we get

$$\sum_{r_j > r_\alpha} \frac{m(I_j)}{m(B_j^*)} \chi_{B_j^{***}}(x) > 1$$

for all $x \in B_\alpha$, and (2.3) is proved. By integrating both sides of (2.3) over the union of the balls not picked we get

$$m\Big(\bigcup_{\alpha \text{ not picked}} B_\alpha\Big) < \Sigma m(I_j) \frac{m(B_j^{***})}{m(B_j^*)} = (\Sigma m(I_j)) \left(\frac{n+2}{n}\right)^n \leqq e^2 \Sigma m(I_j) = e^2 m(\cup B_j).$$

Thus conclusion (2) is proved with $c_1 = e^2 + 1$.

We next turn to conclusion (2) of the lemma. Suppose $x \in \mathbf{R}^n$ is such that

$$\sum \frac{m(I_j)}{m(B_j^*)} \chi_{B_j^*}(x) > 0.$$

Then there is a smallest radius r_j, (which we denote by r_k), so that $\chi_{B_j^*}(x) > 0$ (i.e. where $x \in B_j^*$). Now after suitable translation and dilations we may assume that $x = 0$, and $r_k = 1$. So we have $r_j \geqq 1$, for all radii that matter, and

$$\begin{cases} 0 \in B_k^*, \quad \text{i.e.} \quad |y_k|_B < n. \\ y_k \in B_j^{**} \Leftrightarrow |y_k - y_j|_B < (n+1) r_j. \\ 0 \in B_j^* \quad \Leftrightarrow |y_j|_B < n r_j. \end{cases} \tag{2.4}$$

We write

$$\sum \frac{m(I_j)}{m(B_j^*)} \chi_{B_j^*}(0) = \mathrm{I} + \mathrm{II}$$

where

$$\mathrm{I} = \sum_{r_j \geqq n} \frac{m(I_j)}{m(B_j^*)} \chi_{B_j^*}(0), \quad \text{and} \quad \mathrm{II} = \sum_{1 \leqq r_j < n}.$$

Observe that the jth term in I is non-zero, only when $0 \in B_j^*$, which by (2.4) implies that $y_k \in B_j^{**}$. (This is because $|y_k - y_j|_B \leqq |y_k|_B + |y_j|_B < n + nr_j \leqq (n+1) r_j$, if $r_j \geqq 1$.) Since

$$\sum_{r_j > 1} \frac{m(I_j)}{m(B_j^*)} \chi_{B_j^{**}}(y_k) \leqq 1$$

(the ball B_k was picked), we get

$$\mathrm{I} = \sum_{r_j \geqq n} \frac{m(I_j)}{m(B_j^*)} \chi_{B_j^*}(0) \leqq 1. \tag{2.5}$$

We next estimate

$$\sum_{a \leqq r_j < b} \frac{m(I_j)}{m(B_j^*)} \chi_{B_j^*}(0), \tag{2.6}$$

where $1 \leqq a < b$.

Observe that in the sum $m(B_j^*) \geqq m(B)(na)^n$, where $m(B)$ is the measure of the unit ball. Also the sets I_j are mutually disjoint and are each contained in a ball with radius $< b$, with center y_j, and therefore their union is contained in the ball of radius $(n+1)b$, (centered at the origin). Thus by (2.4),

$$\sum_{r_j \leqq b} m(I_j) \chi_{B_j^*}(0) \leqq m(B)\big((n+1)b\big)^n.$$

Hence we get $(1+1/n)^n (b/a)^n \leqq e(b/a)^n$, as an estimate for (2.6). Finally we write

$$\mathrm{II} = \sum_{1 \leqq r_j < n} = \sum_{l=1}^{m} \mathrm{II}_l,$$

where II_l is the sum taken over radii r, with $(1+1/n)^{l-1} \leqq r_j < (1+1/n)^l$. So we use the estimate just gotten for (2.6) with $a = (1+1/n)^{l-1}$, $b = (1+1/n)^l$, giving

$$\mathrm{II}_l \leqq e(1+1/n)^n \leqq e^2.$$

To conclude the proof of the lemma note that for appropriate $c_0 > 0$, the inequality $(1+1/n)^{c_0 n \log n} \geqq n$ holds, and so with $m = c_0 n \log n$ we have

$$\mathrm{II} = \sum_{l=1}^{m} \mathrm{II}_l \leqq e^2 c_0 n \log n.$$

Since the lemma is now established, so is Theorem 1.

We now turn to L^p estimates for M_B in a general setting. Here B will be an open, bounded, and radial set; it can be written as $B = \{x | x = t\theta \text{ with } 0 \leqq t < \varrho(\theta), \theta \in S^{n-1}\}$, where S^{n-1} denotes the unit sphere in $\mathbf{R}^n$, and ϱ is a positive bounded function on S^{n-1}.

Theorem 2. *With B as above,*

$$\|M_B(f)\|_p \leqq cn\big(p/(p-1)\big)\|f\|_p, \quad 1 < p \leqq \infty$$

where c is independent of n and B.

Proof. We use the method of "rotations". For any $\theta \in S^{n-1}$ denote by M^θ the maximal function in the direction θ given by

$$(M^\theta)f(x) = \sup_{r>0}\left\{\frac{\int_0^r |f(x-t\theta)|\,t^{n-1}\,dt}{\int_0^r t^{n-1}\,dt}\right\}.$$

We assume now that $f \geqq 0$. Then

$$\int_{B^r} f(x-y)\,dy = \int_{S^{n-1}}\int_0^{r\varrho(\theta)} f(x-t\theta)\,t^{n-1}\,dt\,d\theta \leqq r^n \int_{S^{n-1}}\left\{M^\theta(f)(x)\int_0^{\varrho(\theta)} t^{n-1}\,dt\right\}d\theta.$$

Thus

$$\sup_{r>0}\frac{1}{m(B^r)}\int_{B^r} f(x-y)\,dy \leqq \frac{1}{m(B)}\int_{S^{n-1}}\left\{M^\theta(f)(x)\int_0^{\varrho(\theta)} t^{n-1}\,dt\right\}d\theta.$$

The crucial point is that

(2.7) $$\|M^\theta(f)\|_p \leqq cn\big(p/(p-1)\big)\|f\|_p,$$

which follows from the one-dimensional maximal theorem since

$$\sup_{T>0}\frac{\int_0^T f(x-t)\,t^{n-1}\,dt}{\int_0^T t^{n-1}\,dt} \leqq n\sup_{T>0}\frac{1}{T}\int_0^T f(x-t)\,dt.$$

With (2.7) we get

$$\|M_B(f)\|_p \leqq cn\big(p/(p-1)\big)\|f\|_p\cdot\frac{1}{m(B)}\cdot\int_{S^{n-1}}\int_0^{\varrho(\theta)} t^{n-1}\,dt\,d\theta;$$

but since $\int_{S^{n-1}}\int_0^{\varrho(\theta)} t^{n-1}\,dt\,d\theta = m(B)$, the proof of the theorem is complete.

3. The case when B is the standard ball in $\mathbf{R}^n$

We now return to the special case when B is the standard unit ball in $\mathbf{R}^n$, and show how the results in Theorems 1 and 2 can then be improved.

Theorem 3. $$m\{x \mid M(f)(x) > \lambda\} \leqq \frac{cn}{\lambda}\|f\|_1, \quad \lambda > 0.$$

To prove this consider the heat-diffusion semi-group on $\mathbf{R}^n$ given by $T^t(f) = f * h_t$, with

$$h_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}.$$

We observe that $\|T^t f\|_1 \leqq \|f\|_1$, $\|T^t f\|_\infty \leqq \|f\|_\infty$, $T^t(1)=1$, with $T^t f \geqq 0$, for $f \geqq 0$. So the semi-group satisfies all the assumptions at the Hopf abstract maximal ergodic theorem (see [1], VIII. 6 and 7), and hence we see that

$$m\left\{x \,\middle|\, \sup_{s>0} \frac{1}{s}\int_0^s (T^t f)(x)\,dt > \lambda\right\} \leqq 1/\lambda \|f\|_1, \quad \lambda > 0.$$

(The bound here is of course independent of n.) We take $f \geqq 0$, and we shall prove the theorem by comparing $Mf(x)$ with $a_n \sup_{s>0} \frac{1}{s} \int_0^s (T^t f)(x)\,dt$, for suitable a_n. To do this it suffices to find an appropriate s_0 so that

$$m(B)^{-1}\chi_B(x) \leqq a_n \frac{1}{s_0}\int_0^{s_0} h_t(x)\,dt. \tag{3.1}$$

Dilating both sides of (3.1) would then give the majorization

$$Mf(x) \leqq a_n \sup_{s>0} \frac{1}{s}\int_0^s T^t(f)(x)\,dt.$$

If we observe that both $\chi_B(x)$ and $h_t(x)$ are decreasing functions of $|x|$, it is clear that (3.1) is equivalent to

$$m(B)^{-1} \leqq a_n \frac{1}{s_0}\int_0^{s_0} h_t\,dt \tag{3.2}$$

with $h_t = (4\pi t)^{-n/2} e^{-1/4t}$. It turns out that an optimal choice in (3.2) can be made if we take s_0 slightly larger than $1/2n$. To simplify the calculation it would suffice for us to make the cruder choice $s_0 = 1/n$. Now

$$\int_0^\infty h_t\,dt = \pi^{-n/2}\int_0^\infty (4t)^{-n/2} e^{-1/4t}\,dt = \frac{\pi^{-n/2}}{4}\int_0^\infty u^{n/2-2} e^{-u}\,du = \frac{\pi^{-n/2}}{4}\Gamma(n/2-1).$$

However

$$\int_{s_0}^\infty h_t\,dt = \frac{\pi^{-n/2}}{4}\int_0^{1/(4s_0)} u^{n/2-2} e^{-u}\,du \leqq e^{-n/4}(4\pi)^{-n/2} n^{n/2-1}, \quad (n \text{ large}).$$

This last quantity is $o\left(\pi^{-n/2}\Gamma(n/2-1)\right)$, as $n \to \infty$, by Stirling's formula and so $\int_0^s h_t\,dt \geqq c\pi^{-n/2}\Gamma(n/2-1)$. However $m(B)^{-1} = 1/2\pi^{-n/2} n\Gamma(n/2)$, and thus (3.2) is proved with $a_n = c'n$ which implies Theorem 3.

In the same spirit we shall obtain an L^p estimate.

Theorem 4. $\|M(f)\|_p \leqq C\left(p/(p-1)\right) n^{1/2} \|f\|_p$, $\quad 1 < p \leqq \infty$.

Several remarks about this result are in order. The theorem is of no interest for p fixed, when compared with Theorem A. However the theorem gives the right behaviour in p as $p \to 1$, with however a sacrifice resulting from a growth in n; but this growth is smaller than that given by Theorem 2 (valid for more general "balls"). The result is also better than one would obtain by applying the Marcinkiewicz interpolation theorem to Theorem 3.

To prove Theorem 4 we shall use the maximal theorem for symmetric diffusion semi-groups (see [4], and p. 73). In fact, the heat semi-group $T^t(f)=f*h_t$ satisfies all the conditions for such semigroups (axions I, II, III, and IV in [4]), so we obtain

$$\left\|\sup_{t>0} T^t f\right\|_p \leqq A_p \|f\|_p, \quad 1 < p \leqq \infty,$$

with a bound A_p of course independent of n. Now the second proof of this maximal theorem (given in [4], Chapter 4) reduces matters to the martingale maximal theorem, leading to the bound $A_p \leqq C\big(p/(p-1)\big)$. Thus in analogy to the previous theorem we need only determine suitable b_n and t_0 so that

$$m(B)^{-1}\chi_B(x) \leqq b_n h_{t_0}(x) \tag{3.3}$$

which, as before, is equivalent to

$$m(B)^{-1} \leqq b_n (4\pi t_0)^{-n/2} e^{-1/(4t_0)}. \tag{3.4}$$

Now take $t_0 = 1/2n$. Then the right side of (3.4) equals $b_n(2\pi/n)^{-n/2}e^{-n/2}$, while the left-side equals $1/2\pi^{-n/2}n\Gamma(n/2)$. So by Stirling's formula we have (3.4) if $b_n = cn^{1/2}$, for some suitably large constant c. Theorem 4 is therefore proved.

4. Appendix

We shall now give a detailed proof of Theorem A. The result was initially given in [2], but there only a bare outline of the argument was presented.

The idea of the proof can be understood by examining the reasoning of Theorem 2. We observe that if there were a weak point in that proof (the introduction of the factor n) it would have come when one used the essentially one-dimensional result (2.7). The utilization of the k-dimensional spherical maximal function will overcome this difficulty.

Proof of Theorem A. We shall obtain the theorem as a consequence of a series of assertions. First we let $\mathcal{M}_k$ denote the spherical maximal function in $\mathbf{R}^k$, i.e.

$$\mathcal{M}_k(f)(x) = \sup_{\varrho>0} \frac{1}{\omega_{k-1}} \int_{S^{k-1}} |f(x-\varrho y')|\, d\sigma(y')$$

where $d\sigma$ is the usual measure on S^{k-1} (the unit sphere in $\mathbf{R}^k$), and ω_{k-1} is its total mass.

Proposition 1. $\|\mathcal{M}_k(f)\|_p \leqq A_{k,p}\|f\|_p$,

for $p>k/(k-1)$, *and* $k\geqq 3$.

This is just Theorem 1 of [3]. Next, we define the weighted maximal function $M_{k,m}$ on $\mathbf{R}^k$ by

$$M_{k,m}(f)(x) = \sup_{r>0}\left\{\frac{\int_{|y|\leqq r}|f(x-y)|\,|y|^m\,dy}{\int_{|y|\leqq r}|y|^m\,dy}\right\}$$

$$= \sup_{r>0}\frac{m+k}{\omega_{k-1}r^{m+k}}\int_{|y|\leqq r}|f(x-y)|\,|y|^m\,dy, \quad m \geqq 0.$$

Proposition 2. *One has the pointwise majorization*

$$M_{k,m}(f)(x) \leqq \mathcal{M}_k(f)(x),$$

for all $k\geqq 1$, $m\geqq 0$.

Proof. Using polar coordinates we can write

$$\int_{|y|\leqq r}|f(x-y)|\,|y|^m\,dy = \int_{S^{k-1}}\int_0^r |f(x-\varrho y')|\,\varrho^{m+k-1}\,d\varrho\,d\sigma(y')$$

$$\leqq \mathcal{M}_k(f)(x)\omega_{k-1}\int_0^r \varrho^{m+k-1}\,d\varrho = \mathcal{M}_k(f)(x)\,\omega_{k-1}\frac{r^{m+k}}{m+k},$$

and the result follows.

Proposition 3. *If* $k\geqq 3$, *and* $k>p/(p-1)$, *then*

$$\|M_{k,m}(f)\|_p \leqq A_{k,p}\|f\|_p,$$

with the constant $A_{k,p}$ *independent of* m.

This follows immediately from Propositions 1 and 2. We now consider $\mathbf{R}^n$, with $n\geqq 3$, and write it as $\mathbf{R}^n=\mathbf{R}^k\times\mathbf{R}^{n-k}$. So we shall denote an $x\in\mathbf{R}^n$ as a pair $x=(x_1, x_2)$ with $x_1\in\mathbf{R}^k$, $x_2\in\mathbf{R}^{n-k}$; similarly for $y=(y_1, y_2)\in\mathbf{R}^n$, with $y_1\in\mathbf{R}^k$, $y_2\in\mathbf{R}^{n-k}$. We let τ denote an arbitrary element of $O(n)$, a rotation of $\mathbf{R}^n$ about the origin. For each such τ we define M_k^τ, (acting on functions defined in $\mathbf{R}^n$) as

$$(M_k^\tau f)(x) = \sup_{r>0}\frac{\int_{|y_1|\leqq r}|f(x-\tau(y_1, 0))|\,|y_1|^m\,dy_1}{\int_{|y_1|\leqq r}|y_1|^m\,dy_1}$$

with $m=n-k$.

Proposition 4. $\|M_k^\tau(f)\|_p \leqq A_{k,p}\|f\|_p$ *where*

$k\geqq 3$, *and* $k>p/(p-1)$.

By rotation invariance it suffices to prove this when τ is the identity rotation. In that case we use the decomposition $\mathbf{R}^n=\mathbf{R}^k\times\mathbf{R}^{n-k}$, with $x=(x_1, x_2)$. For each fixed $x_2 \in \mathbf{R}^{n-k}$ one applies Proposition 3 and then an additional integration in x_2 (after raising both sides to the pth power) gives the result.

Finally, we let $d\tau$ denote the Haar measure on the group $O(n)$, normalized so that its total measure is 1.

Proposition 5. *We have*

$$\sup_{r>0} \frac{1}{m(B)^r} \int_{B^r} |f(x-y)|\, dy \leqq \int_{O(n)} M_k^\tau(f)(x)\, d\tau.$$

The proposition depends on the following integration formula (valid for non-negative measurable functions on $\mathbf{R}^n$)

$$\frac{\int_{|y|<r} f(y)\, dy}{\int_{|y|\leqq r} dy} = \frac{\int_{O(n)} \int_{|y_1|<r} f(\tau(y_1, 0)) |y_1|^{n-k}\, dy_1\, d\tau}{\int_{|y_1|<r} |y_1|^{n-k}\, dy_1}. \tag{4.1}$$

Here $y=(y_1, y_2) \in \mathbf{R}^n = \mathbf{R}^k \times \mathbf{R}^{n-k}$, with $y_1 \in \mathbf{R}^k$. To verify (4.1) it suffices to do so for f of the form $f(y)=f_0(|y|) f_1(y')$, where $y' \in S^{n-1}$, and $y=|y|y'$, since linear combinations of such functions are dense. Then for such f the left-side of (4.1) is clearly

$$\int_0^r f_0(t)\, t^{n-1}\, dt \cdot \int f_0(y')\, d\sigma(y') \cdot nr^{-n} \cdot \omega_{n-1}^{-1}.$$

To evaluate the right-side, write $y_1 = |y_1| y_1'$, where $y_1' \in S^{k-1}$. Then $f(\tau(y_1, 0)) = f_0(|y_1|) f_1(\tau(y_1'))$ and the quotient on the right-side of (4.1) equals

$$\int_0^r f_0(t)\, t^{n-1}\, dt \cdot \int_{O(n)} \int_{S^{k-1}} f(\tau(y_1'))\, d\sigma(y_1')\, d\tau \cdot nr^{-n} \omega_{k-1}^{-1}.$$

So matters are reduced to checking that

$$\frac{1}{\omega_{n-1}} \int_{S^{n-1}} f_0(y')\, d\sigma(y') = \frac{1}{\omega_{k-1}} \int_{O(n)} \int_{S^{k-1}} f_0(\tau(y_1'))\, d\sigma(y_1')\, d\tau. \tag{4.2}$$

In fact (4.2) holds because $d\sigma(y')$ is up to a constant multiple the unique measure on S^{n-1} which is rotation invariant, and clearly the right-side of (4.2) induces such an invariant measure on S^{n-1}; moreover both sides of (4.2) are normalized so as to agree on constants. With (4.1) now established we have ($|f(x-y)|$ replaces $f(y)$)

$$\frac{1}{m(B^r)} \int_{B^r} |f(x-y)|\, dy = \int_{O(n)} \int_{|y_1| \leqq r} |f(x-\tau(y_1, 0))| \, |y|^{n-k}\, dy_1\, d\tau$$

$$\div \int_{|y_1| \leqq r} |y_1|^{n-k}\, dy_1 \leqq \int_{O(n)} M_k^\tau(f)(x)\, d\tau,$$

with $m=n-k$, and the proposition is proved.

We can now prove the theorem. Suppose p is given, $1<p\leqq\infty$, and keep p fixed. When $n\leqq p/(p-1)$, or $n\leqq 2$, we use the usual estimates to prove (1.1) for that range. Now when $n>p/(p-1)$ and $n\geqq 3$, then write n as $n=k+m$, where k is the smallest integer greater than $p/(p-1)$ and 2. Then our theorem follows from Propositions 4 and 5.

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