BEST CONSTANTS FOR UNCENTERED MAXIMAL FUNCTIONS

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Abstract. We precisely evaluate the operator norm of the uncentered Hardy-Littlewood maximal function on $L^p(\mathbb{R}^1)$. Consequently, we compute the operator norm of the "strong" maximal function on $L^p(\mathbb{R}^n)$, and we observe that the operator norm of the uncentered Hardy-Littlewood maximal function over balls on $L^p(\mathbb{R}^n)$ grows exponentially as $n \to \infty$.

For a locally integrable function f on $\mathbb{R}^n$, let

$$(\mathcal{M}_n f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all closed balls B that contain the point x. $\mathcal{M}_n f$ is called the uncentered Hardy-Littlewood maximal function of f on $\mathbb{R}^n$. In this paper we find the precise value of the operator norm of $\mathcal{M}_1$ on $L^p(\mathbb{R}^1)$. It turns out that this operator norm is the solution of an equation. Our main result is the following:

Theorem. *For $1 < p < \infty$, the operator norm of $\mathcal{M}_1 : L^p(\mathbb{R}^1) \to L^p(\mathbb{R}^1)$ is the unique positive solution of the equation*

$$(p-1)\ x^p - p\ x^{p-1} - 1 = 0. \tag{1}$$

1991 Mathematics Classification Number 42B25

*Research partially supported by the NSF.

In order to prove our Theorem, we fix a nonnegative f and we introduce the left and right maximal functions:

$$(M_L f)(x) = \sup_{a<x} \frac{1}{x-a} \int_a^x f(t)\,dt \qquad \text{and} \qquad (M_R f)(x) = \sup_{b>x} \frac{1}{b-x} \int_x^b f(t)\,dt.$$

For the proof of the next result, known popularly as the "sunrise lemma", we refer the reader to Lemma (21.75) (i), Ch VI in [**2**].

Lemma 1. *Let $f \geq 0$ be in $L^1(\mathbb{R}^1)$. For each $\lambda > 0$, let $C_\lambda = \{x : (M_L f)(x) > \lambda\}$ and $D_\lambda = \{x : (M_R f)(x) > \lambda\}$. Then*

$$\lambda |C_\lambda| = \int_{C_\lambda} f\,dt \qquad \text{and} \qquad \lambda |D_\lambda| = \int_{D_\lambda} f\,dt. \tag{2}$$

Now we are ready to prove the main lemma that leads to our Theorem. This next result may be viewed as the "correct" weak type estimate for the maximal function $\mathcal{M}_1$.

Lemma 2. *Let $f \geq 0$ be in $L^1(\mathbb{R}^1)$. For each $\lambda > 0$, let $A_\lambda = \{x : (\mathcal{M}_1 f)(x) > \lambda\}$ and $B_\lambda = \{x : f(x) > \lambda\}$. Then*

$$\lambda(|A_\lambda| + |B_\lambda|) \leq \int_{A_\lambda} f\,dt + \int_{B_\lambda} f\,dt. \tag{3}$$

To prove (3), first note that

$$\sup(M_L, M_R) = \mathcal{M}_1. \tag{4}$$

For, clearly $\sup(M_L, M_R) \leq \mathcal{M}_1$. On the other hand, it is easy to see that for each real number x, $(\mathcal{M}_1 f)(x)$ is bounded by a convex combination of $(M_L f)(x)$ and $(M_R f)(x)$.

Now we add the two equalities in (2). Then using the fact that $A_\lambda = C_\lambda \cup D_\lambda$ which follows from (4), we obtain

$$\lambda(|A_\lambda| + |C_\lambda \cap D_\lambda|) = \int_{A_\lambda} f\,dt + \int_{C_\lambda \cap D_\lambda} f\,dt. \tag{5}$$

Clearly $B_\lambda - (C_\lambda \cap D_\lambda)$ is a set of measure zero, and $f \le \lambda$ on $(C_\lambda \cap D_\lambda) - B_\lambda$. Therefore

$$\int_{(C_\lambda \cap D_\lambda) - B_\lambda} f\, dt \le \lambda |(C_\lambda \cap D_\lambda) - B_\lambda|. \tag{6}$$

Equations (5) and (6) now imply equation (3), as required.

To prove the inequality in our Theorem, we require the following fact.

Lemma 3. *Let f and g be nonnegative functions on $\mathbb{R}^1$. Then if $p > 1$, we have*

$$\int_0^\infty \lambda^{p-2} \int_{g(t)>\lambda} f(t)\, dt\, d\lambda = \frac{1}{p-1} \int_{\mathbb{R}^1} f g^{p-1}\, dt,$$

and if $p > 0$, we have

$$\int_0^\infty \lambda^{p-1} |\{g > \lambda\}|\, d\lambda = \frac{1}{p} \int_{\mathbb{R}^1} g^p\, dt.$$

The first equality is easily proved, since by Fubini's theorem, the left hand side is

$$\int_{-\infty}^\infty f(t) \int_0^{g(t)} \lambda^{p-2}\, d\lambda\, dt,$$

which is readily seen to equal the right hand side. The second equality is the special case of the first when $f = 1$.

We now continue the proof of our Theorem. Multiplying (3) by $\lambda^{(p-2)}$, integrating λ from 0 to ∞, and applying Lemma 3, we obtain

$$\frac{1}{p}\|\mathcal{M}_1 f\|_p^p + \frac{1}{p}\|f\|_p^p \le \frac{1}{p-1}\|f\|_p^p + \frac{1}{p-1}\int_{\mathbb{R}^1} f(x)[(\mathcal{M}_1 f)(x)]^{p-1}\, dx,$$

that is,

$$(p-1)\|\mathcal{M}_1 f\|_p^p - \int_{\mathbb{R}^1} f(x)[(\mathcal{M}_1 f)(x)]^{p-1}\, dx - \|f\|_p^p \le 0.$$

Applying Hölder's inequality with exponents p and $p/(p-1)$ to the second term, we obtain

$$(p-1)\Big(\frac{\|\mathcal{M}_1 f\|_p}{\|f\|_p}\Big)^p - p\Big(\frac{\|\mathcal{M}_1 f\|_p}{\|f\|_p}\Big)^{p-1} - 1 \le 0, \tag{7}$$

from which we conclude that $\frac{\|\mathcal{M}_1 f\|_p}{\|f\|_p} \le c_p$, where c_p is the unique positive solution of (1).

To show that c_p is in fact the operator norm of $\mathcal{M}_1$ on $L^p(\mathbb{R}^1)$, we give an example. Note that equality in (3) is satisfied when f is even symmetrically decreasing and equality in (7) is satisfied when $\mathcal{M}_1 f$ is a multiple of f. We are therefore led to the following example. Let $f_{\varepsilon,N}(t) = |t|^{-\frac{1}{p}}\chi_{\varepsilon,N}(|t|)$, where $\chi_{\varepsilon,N}$ is the characteristic function of the interval $[\varepsilon, N]$. It can be easily seen that

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}\frac{\|\mathcal{M}_1 f\|_p}{\|f\|_p} = \mathcal{M}_1(f_0)(1), \tag{8}$$

where $f_0(t) = |t|^{-\frac{1}{p}} \in L^1_{\text{loc}}$. An easy calculation gives that

$$\mathcal{M}_1(f_0)(1) = \frac{p}{p-1}\,\frac{\gamma^{\frac{1}{p'}}+1}{\gamma+1}, \tag{9}$$

where γ is the unique positive solution of the equation

$$\frac{p}{p-1}\,\frac{\gamma^{\frac{1}{p'}}+1}{\gamma+1} = \gamma^{-\frac{1}{p}}. \tag{10}$$

Using (9) and (10), it is a matter of simple arithmetic to now show that $\mathcal{M}_1(f_0)(1)$ is the unique positive root of equation (1). This completes the proof of our Theorem.

Before we conclude, we would like to make some remarks. Denote by $x = (x_1, \ldots, x_n)$ points in $\mathbb{R}^n$. For a locally integrable function f on $\mathbb{R}^n$, define

$$(\mathcal{N}_n f)(x) = \sup_{\substack{a_1<x_1\\ b_1>x_1}} \cdots \sup_{\substack{a_n<x_n\\ b_n>x_n}} \frac{1}{(b_1-a_1)\cdots(b_n-a_n)}\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} f(y_1,\ldots,y_n)\,dy_n\cdots dy_1.$$

$\mathcal{N}_n$ is called the "strong" maximal function on $\mathbb{R}^n$. Clearly $\mathcal{N}_1 = \mathcal{M}_1$. Observe that

$$\mathcal{N}_n \leq \mathcal{M}_1^{(1)} \circ \cdots \circ \mathcal{M}_1^{(n)},$$

where $\mathcal{M}_1^{(j)}$ denotes the maximal operator $\mathcal{M}_1$ applied to the x_j coordinate. This shows that the operator norm of $\mathcal{N}_n$ on $L^p(\mathbb{R}^n)$ is less than or equal to c_p^n. By considering the function

$$g(x) = \prod_{j=1}^{n} f_{\epsilon,N}(x_j),$$

where $f_{\epsilon,N}$ is as above, we obtain the converse inequality. We have therefore proved the following:

Corollary. *For $1 < p < \infty$, the operator norm of $\mathcal{N}_n : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is c_p^n, where c_p is the unique positive solution of equation* (1).

One can show that $\frac{p}{p-1} < c_p < \frac{2p}{p-1}$. This implies that the operator norm of $\mathcal{N}_n$ on $L^p(\mathbb{R}^n)$ grows exponentially with n, as $n \to \infty$. Next, we observe that the same is true for the uncentered maximal function $\mathcal{M}_n$. There are several ways to see this. One way is by considering the sequence of functions

$$h_{\epsilon,N}(x) = |x|^{-\frac{n}{p}} \chi_{\epsilon,N}(|x|).$$

Let U_n be the open unit ball in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, let $B_x = \frac{x}{2} + \frac{|x|}{2}\overline{U_n}$. Then $x \in B_x$ and

$$(11)\quad \big(\mathcal{M}_n(h_{\varepsilon,N})\big)(x) \geq \frac{1}{|B_x|} \int_{B_x} |y|^{-\frac{n}{p}} \chi_{\varepsilon,N}(|y|)\, dy = \frac{1}{|U_n|}\Big(\frac{2}{|x|}\Big)^n \int_{B_x} |y|^{-\frac{n}{p}} \chi_{\varepsilon,N}(|y|)\, dy.$$

Therefore for $1 < p < \infty$ and for all $\varepsilon, N > 0$ we have

$$\frac{\|\mathcal{M}_n(h_{\varepsilon,N})\|_{L^p}}{\|h_{\varepsilon,N}\|_{L^p}} \geq \frac{2^n}{\|h_{\varepsilon,N}\|_{L^p}|U_n|} \left\{ \int_{r=0}^{+\infty} \int_{S^{n-1}} \Big[\frac{1}{r^n} \int_{B_{r\phi}} |y|^{-\frac{n}{p}} \chi_{\epsilon,N}(|y|) dy\Big]^p d\phi\, r^n \frac{dr}{r} \right\}^{\frac{1}{p}}$$

$$(12)\quad = \frac{2^n}{\|h_{\epsilon,N}\|_{L^p}|U_n|} \left\{ \int_{r=0}^{+\infty} \int_{S^{n-1}} \Big[\frac{1}{r^n} \int_{t=0}^{r} \int_{S_\phi(\frac{t}{r})} t^{-\frac{n}{p}} \chi_{\epsilon,N}(t) t^n \frac{dt}{t} d\theta\Big]^p d\phi\, r^n \frac{dr}{r} \right\}^{\frac{1}{p}},$$

where $S_\phi(t) = \{\theta \in S^{n-1} : |t\theta - \frac{\phi}{2}| \le \frac{1}{2}\}$. By a change of variables (12) is equal to

$$\frac{2^n}{\|h_{\varepsilon,N}\|_{L^p}|U_n|}\left\{\int\limits_{S^{n-1}}\int\limits_{r=0}^{+\infty}\left[\int\limits_{t=0}^{1}\int\limits_{S_\phi(t)}\chi_{\varepsilon,N}(rt)\, t^{\frac{n}{p'}}\frac{dt}{t}d\theta\right]^p\frac{dr}{r}d\phi\right\}^{\frac{1}{p}}$$

$$=\frac{2^n}{|U_n|}\left\{\int\limits_{S^{n-1}}\left[\frac{\int\limits_{r=0}^{\infty}\left|(K_\phi * \chi_{\varepsilon,N})(r)\right|^p\frac{dr}{r}}{\int\limits_{r=0}^{\infty}\chi^p_{\varepsilon,N}(r)\frac{dr}{r}}\right]\frac{d\phi}{\omega_{n-1}}\right\}^{\frac{1}{p}}, \tag{13}$$

where $K_\phi(t) = t^{n/p'}\chi_{[0,1]}(t)\int_{S_\phi(t)}|\theta|_B^{-n/p}d\theta$, $\omega_{n-1} = |S^{n-1}| = \frac{(n-1)\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})}$, and $*$ denotes convolution on the multiplicative group $G = (\mathbb{R}^+, \frac{dt}{t})$. If $K \ge 0$ on G, the sequence of functions $\chi_{\epsilon,N}$ gives equality in the convolution inequality $\|g * K\|_{L^p(G)} \le \|K\|_{L^1(G)}\|g\|_{L^p(G)}$ as $\varepsilon \to 0$ and $N \to \infty$. Therefore, the expression inside brackets in (13) converges to $\|K_\phi\|^p_{L^1(G)}$ as $\epsilon \to 0$ and $N \to \infty$, and we obtain the estimate

$$\lim_{\substack{\epsilon\to 0\\ N\to\infty}}\frac{\|\mathcal{M}_n(h_{\varepsilon,N})\|_{L^p}}{\|h_{\varepsilon,N}\|_{L^p}} \ge \frac{2^n}{|U_n|}\left\{\int\limits_{S^{n-1}}\left[\int\limits_0^1 t^{\frac{n}{p'}}\int\limits_{S_\phi(t)}d\theta\,\frac{dt}{t}\right]^p\frac{d\phi}{\omega_{n-1}}\right\}^{\frac{1}{p}} = \frac{n2^n}{\omega_{n-1}}\int\limits_0^1 t^{\frac{n}{p'}}\int\limits_{\substack{S^{n-1}\\ \theta_1\ge t}}d\theta\,\frac{dt}{t}$$

$$= 2^n p'\frac{\omega_{n-2}}{\omega_{n-1}}\int\limits_0^1 s^{\frac{n}{p'}}(1-s^2)^{\frac{n-3}{2}}\,ds = 2^{n-1}p'\frac{\omega_{n-2}}{\omega_{n-1}}B(\tfrac{n}{2p'}-\tfrac{1}{2},\tfrac{n-3}{2}). \tag{14}$$

Stirling's formula gives that expression (14) is asymptotic to $\left\{4(\frac{1}{p'})^{\frac{1}{p'}}(\frac{1}{p'}+1)^{-\frac{1}{p'}-1}\right\}^{\frac{n}{2}}$ as $n \to \infty$, and since the number inside the braces above is bigger than 1 when $1 < p < \infty$, we get that the operator norm of $\mathcal{M}_n$ on $L^p(\mathbb{R}^n)$ grows exponentially as $n \to \infty$.

These remarks should be compared to the fact that for $1 < p < \infty$, the operator norm of the Hardy-Littlewood maximal function on $L^p(\mathbb{R}^n)$ is bounded above by some constant A_p independent of the dimension n (see [**3**] and [**4**]).

References

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