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Best constants of harmonic approximation on classes associated with the Laplace operator

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Abstract

We compute the best constants of approximation by entire functions of spherical type and by trigonometric polynomials of spherical degree on classes of functions *f* satisfying the condition $\|\Delta^k f\|_{L_p} \leqslant 1$, where $p=1$ or 2 and Δ is the Laplace operator.

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1. Introduction

Let $\mathbb{R}^m$ be the *m*-dimensional Euclidean space, $Q_a := \{x \in \mathbb{R}^m : |x_j| \leqslant a,\ 1 \leqslant j \leqslant m\}$ the *m*-dimensional cube, $V(r) := \{x \in \mathbb{R}^m : |x| \leqslant r\}$ the ball in $\mathbb{R}^m$ of radius *r*, and $\mathbb{Z}^m$ the set of all vectors $\alpha = (\alpha_1, \ldots, \alpha_m)$ with integral coordinates. Next, let $L_p(\Omega),\ 1 \leqslant p \leqslant \infty$, be a Banach space of all measurable functions *f* defined on a measurable set $\Omega \subseteq \mathbb{R}^m$ with the finite norm

$$\|f\|_{L_p(\Omega)} := \begin{cases} \left(\int_\Omega |f|^p\,dx\right)^{1/p}, & 1 \leqslant p < \infty, \\ \text{ess sup}_\Omega |f|, & p = \infty, \end{cases}$$

and let $L_p^*,\ 1 \leqslant p \leqslant \infty$, be a Banach space of all functions f^* 2π-periodic in each variable with the finite norm $\|f^*\|_{L_p^*} := \|f^*\|_{L_p(Q_\pi)}$.

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We say that an entire function g of m variables has spherical type σ, if for any $\varepsilon > 0$ there exists a constant $C(\varepsilon, g)$ such that for any complex vector $z = (z_1, \dots, z_m)$,

$$|g(z)| \leqslant C(\varepsilon, g) \exp\left((1+\varepsilon)\sigma \left(\sum_{j=1}^{m} |z_j|^2\right)^{1/2}\right).$$

Let $B_{\sigma,m}$ be the set of all entire functions of spherical type $\sigma > 0$ and let $\mathcal{T}_{n,m}$ be the set of all trigonometric polynomials $T(x) = \sum_{\alpha \in \mathbb{Z}^m \cap V(n)} c_\alpha e^{i(\alpha,x)}$ of spherical degree n, where $(\alpha, x) := \sum_{j=1}^{m} \alpha_j x_j$. Note that $n \geqslant 0$ is not necessarily an integer. We denote the errors of best approximation of a locally integrable function f or a function $f^* \in L_p^*$ by functions from $B_{\sigma,m}$ or $\mathcal{T}_{n,m}$ by

$$A_{\sigma,m}(f)_p := \inf_{g \in B_{\sigma,m}} \|f - g\|_{L_p(\mathbb{R}^m)}, \quad E_{n,m}(f^*)_p := \inf_{T \in \mathcal{T}_{n,m}} \|f^* - T\|_{L_p^*}.$$

The corresponding best constants of approximation on a class $K \subset L_p(\mathbb{R}^m)$ or $K \subset L_p^*$ are denoted by

$$A_{\sigma,m}[K]_p := \sup_{f \in K} A_{\sigma,m}(f)_p, \quad E_{n,m}[K]_p := \sup_{f^* \in K} E_{n,m}(f^*)_p.$$

Let $\tilde{W}_p^r$ or W_p^r be the periodic or non-periodic class of all r times differentiable functions f^* or f of a single variable such that $\|f^{*(r)}\|_{L_p^*} \leqslant 1$ or $\|f^{(r)}\|_{L_p(\mathbb{R}^1)} \leqslant 1$, respectively.

The problems of finding best constants of approximation on classes $\tilde{W}_p^r$ and W_p^r were solved in [2,7,18,20] for $p = \infty$ and $p = 1$, see also [26, Section 5.5, 1, p. 241, 17, Sections 4.2 and 7.2, 4, Sections 7.4 and 11.4, 13] for generalizations and discussions. In particular, the constant $E_{n,1}[\tilde{W}_\infty^r]_\infty$ was found independently by Akhiezer and Krein [2] and Favard [7], while a non-periodic analogue of this result was established by Krein [18]. Nikolskii [20] (see also [4, p. 215]) showed that

$$E_{n,1}[\tilde{W}_1^r]_1 = \frac{4}{\pi n^r} \sum_{k=0}^{\infty} \frac{(-1)^{k(r+1)}}{(2k+1)^{r+1}}, \tag{1.1}$$

and using his method of dual relations [20], it is easy to establish the following non-periodic analogue of (1.1):

$$A_{\sigma,1}[W_1^r]_1 = \frac{4}{\pi \sigma^r} \sum_{k=0}^{\infty} \frac{(-1)^{k(r+1)}}{(2k+1)^{r+1}}. \tag{1.2}$$

The problems of finding $A_{\sigma,m}[K]_1$ for some classes K of multivariate convolutions with integrable kernels, such as Bessel potentials and Gauss–Weierstrass and Poisson integrals were considered by the author in [9,11,12]. More recent results on approximation by entire functions of exponential type were obtained by Ditzian [5].

In this paper, we discuss best constants of approximation by entire functions of spherical type and by trigonometric polynomials of spherical degree on some classes associated with the iterated Laplace operator. For a locally integrable function f, we define the k-iterated Laplace operator as $\Delta^k f(x) := (\sum_{j=1}^{m} \partial^2/\partial x_j^2)^k f(x)$, where $f^{(\alpha)}(x) := \partial^{\lceil \alpha \rceil} f(x)/\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}$ is a derivative of the distribution f of order $\lceil \alpha \rceil := \sum_{j=1}^{m} \alpha_j$. Let $D_{k,m,p}$ be the class of all locally integrable functions f on $\mathbb{R}^m$ such that $\Delta^k f$ is a function with a bounded support satisfying the condition $\|\Delta^k f\|_{L_p(\mathbb{R}^m)} \leqslant 1$ for $2 \leqslant m < 2k,\ 1 \leqslant p \leqslant \infty$. We denote by $D^*_{k,m,p}$ the class of all functions f^* 2π-periodic in each variable satisfying the condition $\|\Delta^k f^*\|_{L^*_p} \leqslant 1$ for $2 \leqslant m < 2k,\ 1 \leqslant p \leqslant \infty$.

The fundamental function (the fundamental solution) for $\Delta^k,\ 2 \leqslant m < 2k$, is given by

$$\mathcal{E}^{2k}_m(x) = \mathcal{E}^{2k}_m(x, A) = \begin{cases} c_{m,2k}|x|^{2k-m} & \text{if } m \text{ is odd}, \\ b_{m,2k}|x|^{2k-m}\log|x| + A|x|^{2k-m} & \text{if } m \text{ is even},\ m > 2, \\ B_k|x|^{2k-2}\log|x| + A|x|^{2k-2} & m = 2, \end{cases} \tag{1.3}$$

where

$$c_{m,2k} := -\Gamma(m/2)\left[2(m-2)\pi^{m/2}\prod_{l=1}^{k-1}(2l)\prod_{l=2}^{k}(2l-m)\right]^{-1},$$

$$b_{m,2k} := c_{m,m-2}\left[(m-2)\prod_{l=1}^{k-m/2}(2l)\prod_{l=m/2}^{k-1}(2l)\right]^{-1},$$

$$B_k := -[2^{2k-1}\pi((k-1)!)^2]^{-1},$$

and $A \in \mathbb{R}^1$ [23, p. 51]. In the definition of $\mathcal{E}^{2k}_m$ for even m, A is usually set to be zero since $\Delta^k|x|^{2k-m} = 0$. Nevertheless, the form of $\mathcal{E}^{2k}_m$ given in (1.3) is more suitable for our purposes.

In the following theorem, we find $A_{\sigma,m}[D_{k,m,1}]_1$ and $A_{\sigma,m}[D_{k,m,2}]_2$ and compute the bivariate best constant of approximation in $L_1(\mathbb{R}^2)$.

Theorem 1. (a) *For* $\sigma > 0$ *and* $2 \leqslant m < 2k$,

$$A_{\sigma,m}[D_{k,m,1}]_1 = A_{\sigma,m}(\mathcal{E}^{2k}_m)_1. \tag{1.4}$$

(b) *For* $\sigma > 0$ *and* $2 \leqslant m < 2k$,

$$A_{\sigma,m}[D_{k,m,2}]_2 = (2\pi)^{m/2}\sigma^{-2k}. \tag{1.5}$$

(c) *For* $\sigma > 0,\ k > 1$, *and* $m = 2$,

$$A_{\sigma,2}[D_{k,2,1}]_1 = \frac{2^{2k-3}(2k-1)!}{((k-1)!)^2\sigma^{2k}}\sum_{s=0}^{\infty}\frac{1}{(2s+1)^{2k+1}}. \tag{1.6}$$

The proof of the theorem is given in Section 3 and based on some properties of $\mathcal{E}^{2k}_m$ discussed in Section 2. The proof of the following periodic analogue of Theorem 1 is given in Section 4.

Theorem 2. (a) *For* $2 \leqslant m < 2k$,

$$\lim_{n\to\infty} n^{2k}E_{n,m}[D^*_{k,m,1}]_1 = A_{1,m}(\mathcal{E}^{2k}_m)_1. \tag{1.7}$$

(b) *For* $2 \leqslant m < 2k$,

$$\lim_{n\to\infty} n^{2k}E_{n,m}[D^*_{k,m,2}]_2 = (2\pi)^{m/2}. \tag{1.8}$$

(c) *For* $k > 1$ *and* $m = 2$,

$$\lim_{n\to\infty} n^{2k} E_{n,2}[D^*_{k,2,1}]_1 = \frac{2^{2k-3}(2k-1)!}{((k-1)!)^2} \sum_{s=0}^{\infty} \frac{1}{(2s+1)^{2k+1}}. \tag{1.9}$$

Remark 1. Note that unlike the univariate case, the direct computation of $E_{n,m}[D^*_{k,m,1}]_1$ for $m \geqslant 2$ appears to be a difficult problem. The proof of statement (a) of Theorem 2 is based on a limit relation between $E_{n,m}[D^*_{k,m,1}]_1$ and $A_{1,m}[D_{k,m,1}]_1$. A general approach to such relations was developed in [10,12].

Remark 2. Comparing relations (1.4) and (1.5) with (1.7) and (1.8), we see that for $m \geqslant 2$ and $p = 1$ or $p = 2$,

$$\lim_{n\to\infty} n^{2k} E_{n,m}[D^*_{k,m,p}]_p = A_{1,m}[D_{k,m,p}]_p. \tag{1.10}$$

It is known that this relation is valid for $m = 1$ and $p = 1, 2, \infty$ (see (1.1) and (1.2) for $p = 1$). We do not know if (1.10) holds for $m \geqslant 2$ and $p = \infty$ as well.

In the paper, we use the following additional notation. Let χ_E denote the characteristic function of $E \subset \mathbb{R}^m$. The direct and inverse Fourier transforms of a function or a distribution f are denoted by $\mathcal{F}(f)$ and $\mathcal{F}^{-1}(f)$, respectively. In particular for $f \in L_p(\mathbb{R}^m)$, $1 \leqslant p \leqslant 2$,

$$\mathcal{F}(f)(y) := (2\pi)^{-m/2} \int_{\mathbb{R}^m} f(x) e^{-ixy}\, dx,$$

$$\mathcal{F}^{-1}(f)(y) := (2\pi)^{-m/2} \int_{\mathbb{R}^m} f(x) e^{ixy}\, dx.$$

Throughout the paper C is a positive constant independent of essential parameters. The same symbol does not necessarily denote the same constant in different occurrences.

The following two multivariate versions of the Paley–Wiener theorem are frequently used in the proofs.

Proposition 1. $g \in B_{\sigma,m} \cap L_2(\mathbb{R}^m)$, $\sigma > 0$, *if and only if* $\operatorname{supp} \mathcal{F}(g) \subseteq V(\sigma)$.

The proof of this proposition can be found in [24, Theorem 3.4.9].

Proposition 2. (a) *Let h be a distribution supported in $V(\sigma)$. Then both $\mathcal{F}(h)$ and $\mathcal{F}^{-1}(h)$ belong to $B_{\sigma,m}$ and satisfy the growth estimates*

$$\max(|\mathcal{F}(h)(x)|, |\mathcal{F}^{-1}(h)(x)|) \leqslant C(1+|x|)^N, \quad x \in \mathbb{R}^m,$$

for some constants C and N.

(b) *Conversely, if $g \in B_{\sigma,m}$ and g satisfies the condition*

$$|g(x)| \leqslant C(1+|x|)^N, \quad x \in \mathbb{R}^m, \tag{1.11}$$

then there exist distributions h_1 and h_2 with their supports in $V(\sigma)$ such that $g = \mathcal{F}(h_1)$ and $g = \mathcal{F}^{-1}(h_2)$.

Proposition 2(a) is proved in [22, Theorem 7.23(a)]. Proposition 2(b) follows from [22, Theorem 7.23(b)] if we note that for any $g \in B_{\sigma,m}$ satisfying (1.11) and any complex vector $z = (z_1, \dots, z_m)$, the following inequality holds [16, Lemma 2]:

$$|g(z)| \leqslant 2^{m/2} C(1 + |z|)^{2N} \exp(\sigma |\mathrm{Im}\, z|).$$

2. Properties of $\mathcal{E}_m^{2k}$

It is well known [19, Theorem 1.4.2] that for every function $f \in D_{k,m,p}$, the fundamental function $\mathcal{E}_m^{2k}$ generates the following representation: $f(y) = (\mathcal{E}_m^{2k} * \Delta^k f)(y)$. Note that the convolution $(\mathcal{E}_m^{2k} * \Delta^k f)(y)$ exists in the classical sense for every $y \in \mathbb{R}^m$ because $\Delta^k f$ has a bounded support. Since $\mathcal{E}_m^{2k} \notin L_1(\mathbb{R}^m)$, we give a more classical representation by replacing $\mathcal{E}_m^{2k}$ with $\varphi = \mathcal{E}_m^{2k} - g$, where $g \in B_{\varepsilon,m}$ is a radial function such that $\varphi \in L_1(\mathbb{R}^m)$. The details are provided in the following two lemmas.

Lemma 1. *For $2 \leqslant m < 2k$, any $\varepsilon > 0$, and every $A \in \mathbb{R}^1$, the following statements hold*:

(a) $A_{\varepsilon,m}(\mathcal{E}_m^{2k})_1 < \infty$.

(b) *There exists a radial function $g(x) = g(x, A) = g_\varepsilon(|x|) \in B_{\varepsilon,m}$, where $g_\varepsilon \in B_{\varepsilon,1}$ is even, such that the following relations are valid*:

$$|g_\varepsilon(|x|)| \leqslant C(1 + |x|^2)^N, \quad x \in \mathbb{R}^m, \tag{2.1}$$

$$|\mathcal{E}_m^{2k}(x) - g_\varepsilon(|x|)| \leqslant C(1 + |x|^2)^{-m}, \quad x \in \mathbb{R}^m, \tag{2.2}$$

$$\mathcal{F}(\varphi)(x) = (-1)^k |x|^{-2k}, \quad |x| \geqslant \varepsilon, \tag{2.3}$$

where $N \geqslant 0$ is an integer and $\varphi(x) := \mathcal{E}_m^{2k}(x) - g_\varepsilon(|x|)$.

Proof. We first note that statement (a) follows immediately from (b). We also remark that it suffices to prove (b) for some fixed $A = A^*$. Indeed, this is trivial for an odd m since in this case $\mathcal{E}_m^{2k}(x, A)$ is independent of A. If m is even and statement (b) is proved for some $A^* \in \mathbb{R}^1$, then due to the identity $\mathcal{E}_m^{2k}(x, A) - \mathcal{E}_m^{2k}(x, A^*) = (A - A^*)|x|^{2k-m}$, $2k - m > 0$, $A \in \mathbb{R}^1$, statement (b) holds for any $A \in \mathbb{R}^1$ with $g(x, A) := g(x, A^*) + (A - A^*)|x|^{2k-m} \in B_{\sigma,m}$. Let us now introduce the function $(0 < a < b)$

$$\psi_{a,b}(x) := \begin{cases} 0, & |x| \leqslant a, \\ c \int_a^{|x|} \exp(-(u-a)^{-2}(b-u)^{-2})\, du, & a < |x| < b, \\ 1, & |x| \geqslant b, \end{cases}$$

where $c := (\int_a^b \exp(-(u-a)^{-2}(b-u)^{-2})\, du)^{-1}$. Then $\psi_{\varepsilon/2,\varepsilon}$ is a radial infinitely differentiable function on $\mathbb{R}^m$ and $\psi_{\varepsilon/2,\varepsilon}(x) = 0$ for $|x| \leqslant \varepsilon/2$ and $\psi_{\varepsilon/2,\varepsilon}(x) = 1$ for $|x| \geqslant \varepsilon$.

Next, it is known [15, Section 2.3.3] that there exists $A = A^* \in \mathbb{R}^1$ such that the Fourier transform of the tempered distribution $\mathcal{E}_m^{2k}(x, A^*)$ for $2 \leqslant m < 2k$ is

$$\mathcal{F}(\mathcal{E}_m^{2k})(y) = (-1)^k |y|^{-2k}, \quad y \in \mathbb{R}^m. \tag{2.4}$$

Then the function $h := \psi_{\varepsilon/2,\varepsilon} \mathcal{F}(\mathcal{E}_m^{2k})$ satisfies the conditions

$$h^{(\alpha)}(x) \in L_1(\mathbb{R}^m), \quad 0 \leqslant \lceil \alpha \rceil \leqslant 2m.$$

Therefore, the inverse Fourier transform $\mathcal{F}^{-1}(h)$ satisfies the inequality

$$|\mathcal{F}^{-1}(h)(x)| \leqslant C(1+|x|^2)^{-m}, \quad x \in \mathbb{R}^m. \tag{2.5}$$

Further, the function $H := \mathcal{F}(\mathcal{E}_m^{2k}) - h$ is a tempered distribution with the support in the ball $V(\varepsilon)$. Then by Proposition 2(a), the function $g := \mathcal{F}^{-1}(H)$ belongs to $B_{\varepsilon,m}$ and has polynomial growth on $\mathbb{R}^m$, that is,

$$|g(x)| \leqslant C(1+|x|^2)^N \tag{2.6}$$

for some integer $N \geqslant 0$. Moreover, g is invariant under the group $D(m)$ of all rotations of $\mathbb{R}^m$ (about the origin). To prove this statement, we use the following fact [15, Section 2.3.1]: if a tempered distribution f of m variables is invariant under $D(m)$, that is, f satisfies the condition $f(sx) = f(x)$ for all $s \in D(m)$, then $\mathcal{F}(f)$ and $\mathcal{F}^{-1}(f)$ are invariant under $D(m)$. Hence the tempered distributions $\mathcal{F}(\mathcal{E}_m^{2k})$, h, and H are invariant under $D(m)$, consequently $g = \mathcal{F}^{-1}(H)$ is invariant under $D(m)$ as well.

Next by [14, Proposition 6.1], $g(x) = g_\varepsilon(|x|)$, where $g_\varepsilon \in B_{\varepsilon,1}$ is an even function. Since $\mathcal{F}^{-1}(h)(x) = \mathcal{E}_m^{2k}(x) - g_\varepsilon(|x|)$, inequalities (2.1), (2.2), and (2.3) follow from (2.6), (2.5) and (2.4), respectively. □

Lemma 2. *Let $g_\varepsilon(|x|) \in B_{\varepsilon,m}$ be a function from Lemma* 1. *Then for every $f \in D_{k,m,p}$, $2 \leqslant m < 2k$, $1 \leqslant p \leqslant \infty$, the function*

$$G_\varepsilon(y) := f(y) - \int_{\mathbb{R}^m} (\mathcal{E}_m^{2k}(y-x) - g_\varepsilon(|y-x|))\Delta^k f(x)\,dx \tag{2.7}$$

belongs to $B_{\varepsilon,m}$.

Proof. Since $f(y) = (\mathcal{E}_m^{2k} * \Delta^k f)(y)$ [19, Theorem 1.4.2], we note that

$$G_\varepsilon(y) = \int_{\mathbb{R}^m} g_\varepsilon(|y-x|)\Delta^k f(x)\,dx, \tag{2.8}$$

where $\Delta^k f(x)$ is an integrable function with a bounded support. Next, it follows from (2.1) and (2.8) that $g_\varepsilon(|y|)$ and $G_\varepsilon(y)$ have polynomial growth on $\mathbb{R}^m$. Then by Proposition 2(b), the support of the distribution $\mathcal{F}(G_\varepsilon)$ belongs to $V(\varepsilon)$. Therefore by Proposition 2(a), $G_\varepsilon \in B_{\varepsilon,m}$. □

Next we compute $A_{\sigma,2}(\mathcal{E}_2^{2k})_1$, $k > 1$, where the explicit expression for $\mathcal{E}_2^{2k}(x) = e_{2k}(|x|)$, $x \in \mathbb{R}^2$ is given by

$$e_{2k}(t) := B_k t^{2k-2}\log|t| + At^{2k-m}, \quad t \in \mathbb{R}^1, \quad B_k = -[2^{2k-1}\pi((k-1)!)^2]^{-1}.$$

Here, $A \in \mathbb{R}^1$ is a fixed number.

Lemma 3. *For $k > 1$,*

$$A_{\sigma,2}(\mathcal{E}_2^{2k})_1 = \frac{2^{2k-3}(2k-1)!}{((k-1)!)^2\sigma^{2k}} \sum_{s=0}^{\infty} \frac{1}{(2s+1)^{2k+1}}. \tag{2.9}$$

Proof. This consists of three steps.

Step 1: It follows from Lemma 1(a) that $A_{\sigma,2}(\mathcal{E}_2^{2k})_1 < \infty$. Next by Lemma 1(b), there exists an even $g_\varepsilon \in B_{\varepsilon,1}$ such that $\mathcal{E}_2^{2k}(x) - g_\varepsilon(|x|) \in L_1(\mathbb{R}^2)$ and

$$|g_\varepsilon(t)| \leqslant C(1+t^2)^N, \quad t \in \mathbb{R}^1, \tag{2.10}$$

$$|t(e_{2k}(t) - g_\varepsilon(t))| \leqslant C(1+t^2)^{-1}, \quad t \in \mathbb{R}^1. \tag{2.11}$$

Since the function $\mathcal{E}_2^{2k}(x) - g_\varepsilon(|x|)$ is radial and integrable on $\mathbb{R}^2$, Theorem 6.1 in [14] shows that there is an even function $Q_\sigma \in B_{\sigma,1}$, such that for $\sigma > \varepsilon$,

$$\begin{aligned} A_{\sigma,2}(\mathcal{E}_2^{2k})_1 &= A_{\sigma,2}(\mathcal{E}_2^{2k}(x) - g_\varepsilon(|x|))_1 = \|\mathcal{E}_2^{2k}(x) - g_\varepsilon(|x|) - Q_\sigma(|x|)\|_{L_1(\mathbb{R}^2)} \\ &= \pi A_{\sigma,1,t}(e_{2k} - g_\varepsilon)_1 := \pi \inf_{g \in B_{\sigma,1}} \int_{\mathbb{R}^1} |e_{2k}(t) - g_\varepsilon(t) - g(t)|\,|t|\,dt. \end{aligned} \tag{2.12}$$

This shows that the bivariate approximation problem is reduced to a univariate one in a weighted L_1-space. Next, we have

$$A_{\sigma,1,t}(e_{2k} - g_\varepsilon)_1 = A_{\sigma,1}(te_{2k}(t) - tg_\varepsilon(t))_1, \tag{2.13}$$

where $te_{2k}(t) - tg_\varepsilon(t)) \in L_1(\mathbb{R}^1)$, by (2.11). Indeed, the inequality

$$A_{\sigma,1,t}(e_{2k} - g_\varepsilon)_1 \geqslant A_{\sigma,1}(te_{2k}(t) - tg_\varepsilon(t))_1 \tag{2.14}$$

is trivial. Next let $H_\sigma \in B_{\sigma,1}$ satisfies the equality

$$A_{\sigma,1}(te_{2k}(t) - tg_\varepsilon(t))_1 = \|t(e_{2k}(t) - g_\varepsilon(t)) - H_\sigma(t)\|_{L_1(\mathbb{R}^1)}.$$

Without loss of generality we can assume that H_σ is an odd function. Then $H_\sigma^*(t) := H_\sigma(t)/t \in B_{\sigma,1}$ and we have

$$A_{\sigma,1}(te_{2k}(t) - tg_\varepsilon(t))_1 = \int_{\mathbb{R}^1} |e_{2k}(t) - g_\varepsilon(t) - H_\sigma^*(t)|\,|t|\,dt \geqslant A_{\sigma,1,t}(e_{2k} - g_\varepsilon)_1. \tag{2.15}$$

Thus (2.14) and (2.15) imply (2.13).

Step 2: Next we find the Fourier sin-transform of the integrable function $te_{2k}(t) - tg_\varepsilon(t)$. Namely, we prove that for $|y| > \varepsilon$,

$$\Lambda(y) := \int_{\mathbb{R}^1} (te_{2k}(t) - tg_\varepsilon(t)) \sin ty\,dt = \pi(-1)^k B_k (2k-1)!|y|^{-2k} \operatorname{sign} y. \tag{2.16}$$

Let $S(\mathbb{R}^1)$ be the Schwartz class of all rapidly decreasing functions on $\mathbb{R}^1$. Then for any $h \in S(\mathbb{R}^1)$ with its support outside of $[-\varepsilon, \varepsilon]$ we have

$$\begin{aligned} \int_{\mathbb{R}^1} (te_{2k}(t) - tg_\varepsilon(t))\mathcal{F}(h)(t)\,dt &= \int_{\mathbb{R}^1} (B_k t^{2k-1} \log|t| - A^* t^{2k-1})\mathcal{F}(h)(t)\,dt \\ &\quad - \int_{\mathbb{R}^1} (tg_\varepsilon(t) - (A + A^*)t^{2k-1})\mathcal{F}(h)(t)\,dt, \end{aligned} \tag{2.17}$$

where $A^* := B_k \psi(2k)$, $\psi(t)$ is the psi-function and the second integral in the right-hand side of (2.17) exists by (2.10). Next using Proposition 2(b) again, we have

$$\int_{\mathbb{R}^1} (tg_\varepsilon(t) - (A + A^*)t^{2k-1})\mathcal{F}(h)(t)\,dt = 0. \tag{2.18}$$

Further, it is known [15, Eq. (2.3.13)] that

$$\int_{\mathbb{R}^1} (B_k t^{2k-1} \log|t| - A^* t^{2k-1}) \mathcal{F}(h)(t)\, dt = \int_{\mathbb{R}^1} \Phi(t) h(t)\, dt$$
$$= \int_{|t|>\varepsilon} \Phi(t) h(t)\, dt, \tag{2.19}$$

where Φ is the function defined on the right-hand side of (2.16). Thus (2.17), (2.18), and (2.19) imply the equality

$$\int_{\mathbb{R}^1} (te_{2k}(t) - tg_\varepsilon(t)) \mathcal{F}(h)(t)\, dt = \int_{|t|>\varepsilon} \Phi(t) h(t)\, dt. \tag{2.20}$$

Finally choosing h in (2.20) as a peak delta-like function from $S(\mathbb{R}^1)$ supported in the interval $[y-\delta, y+\delta]$ with $0 < \delta < |y| - \varepsilon$ and letting $\delta \to 0$, we arrive at (2.16).

Step 3: Finally, we prove (2.9). It follows from (2.11) and (2.16) that the function $\Psi(t) := (-1)^k (te_{2k}(t) - tg_\varepsilon(t))$ satisfies the following conditions of the Sz-Nagy criterion [25,1, Section 88] for approximation in $L_1(\mathbb{R}^1)$ by entire functions of exponential type σ: Ψ is an odd function, satisfying $|\Psi(t)| \leqslant C(1+t^2)^{-1}$, and the following inequalities hold for its sin-transform $\Lambda_1(y) := (-1)^{k+1}\Lambda(y)$, where Λ is defined in (2.16):

$$\Lambda_1(y) > 0, \quad \Lambda_1'(y) \leqslant 0, \quad \Lambda_1''(y) \geqslant 0, \quad y > \varepsilon.$$

Then for $\sigma > \varepsilon$,

$$A_{\sigma,1}(te_{2k}(t) - tg_\varepsilon(t))_1 = \frac{4}{\pi} \sum_{s=0}^{\infty} \frac{\Lambda_1((2s+1)\sigma)}{2s+1}$$
$$= (4/\pi)|B_k|(2k-1)!\sigma^{-2k} \sum_{s=0}^{\infty} (2s+1)^{-2k-1}. \tag{2.21}$$

Thus (2.9) follows from (2.12), (2.13), and (2.21). □

Remark 3. A general approach to Markov-type theorems in $L_1(\mathbb{R}^m)$, $m \geqslant 2$, for radial integrable functions was developed in [9,11]. Note that the Sz-Nagy criterion cannot be applied for $m > 2$, and computation of $A_{\sigma,m}(\mathcal{E}_m^{2k})_1$ appears to be a difficult problem for $m > 2$. In addition, we remark that $A_{\sigma,2}(|x|^\lambda)_1$, $\lambda > 0$, was computed in [14, Theorem 6.2] similarly to the proof of Lemma 3.

3. Proof of Theorem 1

Proof of statement (a). Let $f \in D_{k,m,1}$. Then by Lemma 2, the following representation holds:

$$f(y) = G_\varepsilon(y) + \int_{\mathbb{R}^m} (\mathcal{E}_m^{2k}(y-x) - g_\varepsilon(|y-x|)) \Delta^k f(x)\, dx, \tag{3.1}$$

where $\varepsilon > 0$, $G_\varepsilon \in B_{\varepsilon,m}$, $\|\Delta^k f\|_{L_1(\mathbb{R}^m)} \leqslant 1$, and $g_\varepsilon \in B_{\varepsilon,1}$ is the function from Lemma 1.

Let $H_\sigma \in B_{\sigma,m} \cap L_1(\mathbb{R}^m)$ be a function of best approximation in $L_1(\mathbb{R}^m)$ to $\varphi(x) := \mathcal{E}_m^{2k}(x) - g_\varepsilon(|x|)$. Then by the Nikolskii inequality [26, p. 235], $H_\sigma \in B_{\sigma,m} \cap L_2(\mathbb{R}^m)$ (see [6] for new

Nikolskii-type inequalities). Therefore using Proposition 1, we conclude that $\mu_\sigma := H_\sigma * \Delta^k f \in B_{\sigma,m}$. Hence for $0 < \varepsilon < \sigma$, (3.1) implies that

$$A_{\sigma,m}(f)_1 \leqslant \|f - G_\varepsilon - \mu_\sigma\|_{L_1(\mathbb{R}^m)} \leqslant \|\varphi - H_\sigma\|_{L_1(\mathbb{R}^m)} \|\Delta^k f\|_{L_1(\mathbb{R}^m)} \leqslant A_{\sigma,m}(\mathcal{E}_m^{2k})_1.$$

Therefore,

$$A_{\sigma,m}[D_{k,m,1}]_1 \leqslant A_{\sigma,m}(\mathcal{E}_m^{2k})_1. \tag{3.2}$$

To prove the inequality

$$A_{\sigma,m}[D_{k,m,1}]_1 \geqslant A_{\sigma,m}(\mathcal{E}_m^{2k})_1, \tag{3.3}$$

we consider an infinitely differentiable function θ_δ satisfying the conditions: $\theta_\delta \geqslant 0$ on the ball $V(\delta)$, $\delta > 0$; $\theta_\delta = 0$ outside of $V(\delta)$ and $\int_{\mathbb{R}^m} \theta_\delta(x)\,dx = 1$. Then the convolution $T_\delta(y) := (\mathcal{E}_m^{2k} * \theta_\delta)(y)$ is infinitely differentiable on $\mathbb{R}^m$ and $\Delta^k T_\delta = \theta_\delta$. Hence $T_\delta \in D_{k,m,1}$. Next we estimate $A_{\sigma,m}(T_\delta)_1$. We first note that

$$A_{\sigma,m}(T_\delta)_1 = A_{\sigma,m}(T_\delta^*)_1, \tag{3.4}$$

where $T_\delta^* := (\varphi - H_\sigma) * \theta_\delta$. Further,

$$\begin{aligned} A_{\sigma,m}(T_\delta^*)_1 &\geqslant A_{\sigma,m}(\varphi - H_\sigma)_1 - A_{\sigma,m}(\varphi - H_\sigma - T_\delta^*)_1 \\ &\geqslant A_{\sigma,m}(\mathcal{E}_m^{2k})_1 - \|\varphi - H_\sigma - T_\delta^*\|_{L_1(\mathbb{R}^m)}. \end{aligned} \tag{3.5}$$

Furthermore,

$$\begin{aligned} &\|\varphi - H_\sigma - T_\delta^*\|_{L_1(\mathbb{R}^m)} \\ &\quad\leqslant \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} |(\varphi(y-x) - H_\sigma(y-x)) - (\varphi(y) - H_\sigma(y))|\theta_\delta(x)\,dx\,dy \\ &\quad= \int_{V(\delta)}\int_{\mathbb{R}^m} |(\varphi(y-x) - H_\sigma(y-x)) - (\varphi(y) - H_\sigma(y))|\,dy\,\theta_\delta(x)\,dx \\ &\quad\leqslant \omega(\varphi - H_\sigma, \delta)_1, \end{aligned} \tag{3.6}$$

where $\omega(F,\delta)_1 := \sup_{|x|\leqslant\delta} \|F(\cdot + x) - F(\cdot)\|_{L_1(\mathbb{R}^m)}$ is the integral modulus of continuity of F in the $L_1(\mathbb{R}^m)$-metric. Therefore (3.4), (3.5), and (3.6) imply the inequality

$$A_{\sigma,m}[D_{k,m,1}]_1 \geqslant A_{\sigma,m}(T_\delta)_1 \geqslant A_{\sigma,m}(\mathcal{E}_m^{2k})_1 - \omega(\varphi - H_\sigma, \delta)_1. \tag{3.7}$$

Since $\lim_{\delta\to 0} \omega(\varphi - H_\sigma, \delta)_1 = 0$, (3.3) follows from (3.7). Finally, (3.2) and (3.3) yield (1.4). □

Proof of statement (b). Let $f \in D_{k,m,2}$. Then using Lemma 2 and Proposition 1, we have for $0 < \varepsilon < \sigma$

$$\begin{aligned} A_{\sigma,m}(f)_2 = A_{\sigma,m}(\varphi * \Delta^k f)_2 &= \inf_{g\in B_{\sigma,m}} \left(\int_{\mathbb{R}^m} |\mathcal{F}(\varphi * \Delta^k f)(x) - \mathcal{F}(g)(x)|^2\,dx\right)^{1/2} \\ &= (2\pi)^{m/2}\left(\int_{|x|>\sigma} |\mathcal{F}(\varphi)(x)\mathcal{F}(\Delta^k f)(x)|^2\,dx\right)^{1/2}, \end{aligned}$$

where $\varphi(x) := \mathcal{E}_m^{2k}(x) - g_\varepsilon(|x|)$. Hence by (2.3), we get

$$A_{\sigma,m}[D_{k,m,2}]_2 = (2\pi)^{m/2} \sup_{|x|\geqslant\sigma} |\mathcal{F}(\varphi)(x)| = (2\pi)^{m/2}\sigma^{-2k}. \qquad \square$$

Proof of statement (c). This statement follows directly from statement (a) and Lemma 3. Thus Theorem 1 is established. $\square$

4. Proof of Theorem 2

This consists of four steps.

Step 1: We first prove that $f^* \in D^*_{k,m,p}$, $1\leqslant p\leqslant\infty$, $2\leqslant m < 2k$, if and only if f^* is a 2π-periodic convolution of the form

$$f^*(y) = a_0 + (\mathcal{P}_m^{2k} \star \mu^*)_{2\pi}(y) := a_0 + \int_{Q_\pi} \mathcal{P}_m^{2k}(y-x)\mu^*(x)\,dx, \tag{4.1}$$

where $\mathcal{P}_m^{2k}$ is the kernel defined by

$$\mathcal{P}_m^{2k}(x) := \frac{(-1)^k}{(2\pi)^m} \sum_{\alpha\in\mathbb{Z}^m\setminus\{0\}} \frac{e^{i(\alpha,x)}}{|\alpha|^{2k}}, \tag{4.2}$$

$\mu^* \in L_p^*$ is a function satisfying the conditions $\int_{Q_\pi}\mu^*(x)\,dx = 0$ and $\|\mu^*\|_{L_p^*}\leqslant 1$, and $a_0 := (2\pi)^{-m}\int_{Q_\pi} f^*(y)\,dy$. It is easy to see that the series in (4.2) uniformly converges on $\mathbb{R}^m$ for $m < 2k$ since it can be majorized by $\sum_{\alpha\in\mathbb{Z}^m\setminus\{0\}}|\alpha|^{-2k}$.

Let $f^* \in D^*_{k,m,p}$ and let $\sum_{\alpha\in\mathbb{Z}^m\setminus\{0\}} c_\alpha e^{i(\alpha,x)}$ be the Fourier series of $\Delta^k f^*$, where

$$c_\alpha := (2\pi)^{-m}\int_{Q_\pi} \Delta^k f^*(y) e^{-i(\alpha,y)}\,dy, \quad \alpha \neq 0. \tag{4.3}$$

Integrating by parts in this integral, we see that the Fourier series of f^* is $a_0+\sum_{\alpha\in\mathbb{Z}^m\setminus\{0\}} c_\alpha|\alpha|^{-2k} e^{i(\alpha,x)}$. Then setting $\mu^* = \Delta^k f^*$, we see that (4.1) follows from (4.3) and (4.2).

Conversely, if f^* allows representation (4.1) with the Fourier series of μ^* given by $\sum_{\alpha\in\mathbb{Z}^m\setminus\{0\}} c_\alpha e^{i(\alpha,x)}$, then for any infinitely differentiable and 2π-periodic function $\psi(y) = \sum_{\alpha\in\mathbb{Z}^m} d_\alpha e^{i(\alpha,y)}$ we have

$$\int_{Q_\pi} f^*(y)\Delta^k\psi(y)\,dy = (-1)^k(2\pi)^m \sum_{\alpha\in\mathbb{Z}^m} c_\alpha d_\alpha = (-1)^k \int_{Q_\pi}\mu^*(y)\psi(y)\,dy.$$

Therefore $\Delta^k f^* = \mu^*$, so $f^* \in D_{k,m,p}$. Note that representation (4.1) for $f^* \in D^*_{k,2,p}$ was given in [3].

Now statement (b) of the theorem follows from integral representation (4.1) and the Parseval identity. Since statement (c) is a direct consequence of (a) and Lemma 3, it remains to prove statement (a). The proof of this statement is given in Steps 2–4.

Step 2: We first consider the following $2b$-periodization linear operator $P_b : L_1(\mathbb{R}^m) \to L_1(Q_b)$ with $\|P_b\|\leqslant 1$ defined by $P_b f(x) := \sum_{\alpha\in\mathbb{Z}^m} f(x+2b\alpha)$ for $f \in L_1(\mathbb{R}^m)$. It is well known [24, Theorem 7.2.4] that this series converges in $L_1(Q_b)$, its sum $P_b f \in L_1(Q_b)$ is $2b$-periodic and the Fourier series of $P_b f$ is $(2b)^{-m}\sum_{\alpha\in\mathbb{Z}^m}\mathcal{F}(f)((\pi/b)\alpha)e^{i(\pi/b)(\alpha,x)}$. Moreover, if

f is continuous on $\mathbb{R}^m$ and if for all $x \in \mathbb{R}^m$ and some $\gamma > 0$, $|f(x)| \leqslant C(1+|x|)^{-m-\gamma}$ and $|\mathcal{F}(f)(x)| \leqslant C(1+|x|)^{-m-\gamma}$, then [24, Corollary 7.2.6]

$$P_b f(x) = (2b)^{-m} \sum_{\alpha\in\mathbb{Z}^m} \mathcal{F}(f)((\pi/b)\alpha)e^{i(\pi/b)(\alpha,x)}. \tag{4.4}$$

In particular, the function $\varphi(x) := \mathcal{E}_m^{2k}(x) - g_\varepsilon(|x|)$ satisfies the conditions $|\varphi(x)| \leqslant C(1+|x|)^{-2m}$ and $|\mathcal{F}(\varphi)(x)| \leqslant C(1+|x|)^{-m-\gamma_1}$ for all $x \in \mathbb{R}^m$ and $\gamma_1 := 2k - m > 0$. Indeed, the first estimate is given in (2.2), while the second one follows from the construction of the function $g_\varepsilon(|x|)$. Namely, it was proved in the proof of Lemma 1 that

$$\mathcal{F}(\varphi)(x) = \begin{cases} 0, & |x| \leqslant \varepsilon/2, \\ (-1)^k |x|^{-2k}, & |x| \geqslant \varepsilon, \end{cases} \tag{4.5}$$

and $\mathcal{F}(\varphi)(x)$ is bounded for $\varepsilon/2 < |x| < \varepsilon$. Hence $|\mathcal{F}(\varphi)(x)| \leqslant C(1+|x|)^{-m-\gamma_1}$ for all $x \in \mathbb{R}^m$, where $\gamma_1 = 2k - m$. Therefore taking account of (4.5) and using (4.4), we have

$$P_b\varphi(x) = T_b(x) + \frac{(-1)^k}{(2b)^m(\pi/b)^{2k}} \sum_{\alpha\in\mathbb{Z}^m,\ |\alpha|\geqslant \varepsilon b/\pi} \frac{e^{i(\pi/b)(\alpha,x)}}{|\alpha|^{2k}}, \tag{4.6}$$

where $T_b(x) := \sum_{\alpha\in\mathbb{Z}^m\cap V(b\varepsilon/\pi)} c_\alpha e^{i(\pi/b)(\alpha,x)}$ is a trigonometric polynomial. In particular, (4.6) implies the relation

$$P_b\varphi(bx/\pi) = (b/\pi)^{2k-m}\mathcal{P}_m^{2k}(x) + T^*(x), \tag{4.7}$$

where $T^* \in \mathcal{T}_{\varepsilon b/\pi,m}$. In addition, note that if $b = \pi$ and $0 < \varepsilon < 1/2$, then (4.4) and (4.5) imply the identity

$$P_\pi\varphi(x) = \mathcal{P}_m^{2k}(x). \tag{4.8}$$

Further, if $F_n \in B_{n,m} \cap L_1(\mathbb{R}^m)$, then $P_b F_n \in \mathcal{T}_{n,m}$. Indeed, by the Nikolskii inequality [26, p. 235], $F_n \in B_{n,m} \cap L_2(\mathbb{R}^m)$. Therefore by Proposition 1, we conclude that $\mathcal{F}(F_n)(x) = 0$ for $|x| > n$. This shows that the Fourier series of $P_\pi F_n(x)$ is the finite sum $(2\pi)^{-m}\sum_{\alpha\in\mathbb{Z}^m\cap V(n)} \mathcal{F}(F_n)(\alpha)e^{i(\alpha,x)}$. Thus $P_b F_n \in \mathcal{T}_{n,m}$.

Next we consider the periodization of a convolution. If $h \in L_1(\mathbb{R}^m)$ and $\mu \in L_1(\mathbb{R}^m)$, then $h * \mu \in L_1(\mathbb{R}^m)$ and

$$\begin{aligned} &P_b(h*\mu)(y) \\ &\quad = \int_{\mathbb{R}^m} (P_b h)(y-x)\mu(x)\,dx = \int_{\mathbb{R}^m} h(y-x)(P_b\mu)(x)\,dx \\ &\quad = \sum_{\alpha\in\mathbb{Z}^m} \int_{Q_b - 2\alpha b} (P_b h)(y-x)\mu(x)\,dx = \int_{Q_b} (P_b h)(y-x) \sum_{\alpha\in\mathbb{Z}^m} \mu(x+2b\alpha)\,dx \\ &\quad = \int_{Q_b} (P_b h)(y-x)(P_b\mu)(x)\,dx = (P_b h \star P_b\mu)_{2b}(y). \end{aligned} \tag{4.9}$$

In other words, the P_b-image of a convolution is a $2b$-periodic one. It follows from (4.9) that

$$\|h * P_b\mu\|_{L_1(Q_b)} \leqslant \|P_b h\|_{L_1(Q_b)} \|P_b\mu\|_{L_1(Q_b)} \leqslant \|h\|_{L_1(\mathbb{R}^m)} \|\mu\|_{L_1(\mathbb{R}^m)}. \tag{4.10}$$

Step 3: Here, we establish the upper estimate

$$\limsup_{n\to\infty} n^{2k} E_{n,m}[D^*_{k,m,1}]_1 \leqslant A_{1,m}(\mathcal{E}_m^{2k})_1. \tag{4.11}$$

For a function $f^* \in D^*_{k,m,1}$, we set $\mu^* := \Delta^k f^*$. Then $\mu^* \in L_1^*$ and $\|\mu^*\|_{L_1^*} \leqslant 1$. Next, the function $\mu := \chi_{Q_\pi}\mu^*$ has a bounded support and $\|\mu\|_{L_1(\mathbb{R}^m)} \leqslant \|\mu^*\|_{L_1^*} \leqslant 1$. Moreover, the following relation holds for a. e. $x \in \mathbb{R}^m$:

$$\mu^* = P_\pi \mu(x). \tag{4.12}$$

Hence the convolution $f(y) := (\mathcal{E}_m^{2k} * \mu)(y)$ exists for every $y \in \mathbb{R}^m$ and $\Delta^k f = \mu$ [19, Theorem 1.4.2]. Therefore $f \in D_{k,m,1}$. Next by Lemma 2, there exists $G_\varepsilon \in B_{\varepsilon,m}$, $0 < \varepsilon < 1/2$, such that

$$f(y) - G_\varepsilon(y) = \int_{\mathbb{R}^m} (\mathcal{E}_m^{2k}(y-x) - g_\varepsilon(|y-x|))\mu(x)\,dx. \tag{4.13}$$

Further using (4.1), (4.8), (4.9), (4.12), and (4.13), we get

$$\begin{aligned} P_\pi(f - G_\varepsilon) &= \int_{\mathbb{R}^m} \varphi(y-x) P_\pi \mu(x)\,dx \\ &= \int_{Q_\pi} \mathcal{P}_m^{2k}(y-x)\mu^*(x)\,dx = f^*(y) - a_0. \end{aligned} \tag{4.14}$$

Furthermore, if $F_n \in B_{n,m}$ is a function of best approximation to $f - G_\varepsilon$ in the metric of $L_1(\mathbb{R}^m)$, then $P_\pi F_n \in \mathcal{T}_{n,m}$ (see Step 2). Hence (4.14) implies that

$$\begin{aligned} E_{n,m}(f^*)_1 &\leqslant \|f^* - a_0 - P_\pi F_n\|_{L_1^*} = \|P_\pi(f - G_\varepsilon - F_n)\|_{L_1^*} \\ &\leqslant \|f - G_\varepsilon - F_n\|_{L_1(\mathbb{R}^m)} = A_{n,m}(f)_1. \end{aligned} \tag{4.15}$$

Since f^* is an arbitrary function from $D^*_{k,m,1}$ and $f \in D_{k,m,1}$, we get from (4.15)

$$n^{2k} \sup_{f^* \in D^*_{k,m,1}} E_{n,m}(f^*)_1 \leqslant n^{2k} \sup_{f \in D_{k,m,1}} A_{n,m}(f)_1. \tag{4.16}$$

It remains to note that for every $f \in D_{k,m,1}$,

$$n^{2k} A_{n,m}(f)_1 = A_{1,m}(h)_1, \tag{4.17}$$

where the function $h(x) := n^{2k-m} f(x/n)$ belongs to $D_{k,m,1}$. Then (4.16), (4.17), and Theorem 1(a) imply (4.11).

Step 4: Now we prove the lower estimate

$$\liminf_{n\to\infty} n^{2k} E_{n,m}[D^*_{k,m,1}]_1 \geqslant A_{1,m}(\mathcal{E}_m^{2k})_1. \tag{4.18}$$

We first set $h^{[s]} := \chi_{Q_{\pi s/2}} h$ for $h \in L_1(\mathbb{R}^m)$ and $s > 0$.

Next, let $f \in D_{k,m,1}$ and $\varepsilon > 0$. Then by Lemma 2, there exist $G_\varepsilon \in B_{\varepsilon,m}$ and a function $\mu \in L_1(\mathbb{R}^m)$ with a bounded support S such that $\|\mu\|_{L_1(\mathbb{R}^m)} \leqslant 1$ and $f - G_\varepsilon = \varphi * \mu$.

It is easy to see that for any $\delta > 0$ there exists $n_0 = n_0(\varphi, \mu, \delta)$ such that $S \subset Q_{\pi n_0/2}$ and

$$\|\varphi^{[n_0]} - \varphi\|_{L_1(\mathbb{R}^m)} < \delta. \tag{4.19}$$

Further, let us set for $n > 0$

$$f_\delta := \varphi^{[n_0]} * \mu, \quad f^*_{\delta,n} := \varphi^{[n_0]} * P_{\pi n}\mu, \quad f^*_n := \varphi * P_{\pi n}\mu.$$

Note that $f^*_{\delta,n}$ and f^*_n are $2\pi n$-periodic functions. In addition, note that the supports of $\varphi^{[n_0]}$ and μ are contained in $Q_{\pi n_0/2}$ so the support of f_δ is a proper subset of $Q_{\pi n}$ for $n > n_0$. Since by (4.9), $f^*_{\delta,n}$ is the $2\pi n$-periodization of f_δ, we observe that

$$f_\delta(x) = f^*_{\delta,n}, \quad x \in Q_{\pi n}, \quad n > n_0. \tag{4.20}$$

Next, it follows from (4.19) and (4.10) that for any $n > 0$,

$$\|f - G_\varepsilon - f_\delta\|_{L_1(\mathbb{R}^m)} < \delta, \tag{4.21}$$

$$\|f^*_{\delta,n} - f^*_n\|_{L_1(Q_{\pi n})} < \delta. \tag{4.22}$$

Let $\mathcal{T}^*_{n,m}$ be the class of all $2\pi n$-periodic trigonometric polynomials $T(x) = \sum_{\alpha\in\mathbb{Z}^m\cap V(n)} c_\alpha \exp(i(\alpha, x/n))$. It is easy to see that for all $n > 0$, $\mathcal{T}^*_{n,m} \subset B_{1,m}$. Next let us set $E^*_{n,m}(h)_1 := \inf_{T\in\mathcal{T}^*_{n,m}} \|h - T\|_{L_1(Q_{\pi n})}$, where $h \in L_1(Q_{\pi n})$ is a $2\pi n$-periodic function.

Further, let $T_n \in \mathcal{T}^*_{n,m}$ be a polynomial of best approximation to $f^*_{\delta,n}$ in $L_1(Q_{\pi n})$, that is, $E^*_{n,m}(f^*_{\delta,n})_1 = \|f^*_{\delta,n} - T_n\|_{L_1(Q_{\pi n})}$. Then for small enough $\delta > 0$ we get from (4.22)

$$\|T_n\|_{L_1(Q_{\pi n})} \leqslant 2\|f^*_{\delta,n}\|_{L_1(Q_{\pi n})} \leqslant 2(\|f^*_n\|_{L_1(Q_{\pi n})} + \delta) \leqslant 4\|\varphi\|_{L_1(\mathbb{R}^m)}. \tag{4.23}$$

Furthermore using the Nikolskii inequality [26, p. 231], we have

$$\|T_n\|_{L_1(Q_{\pi n})} = n^m \int_{Q_\pi} \left| \sum_{\alpha\in\mathbb{Z}^m\cap V(n)} c_\alpha e^{i(\alpha,x)} \right| dx \geqslant C\|T_n\|_{L_\infty(Q_{\pi n})}, \tag{4.24}$$

where C is independent of n. Combining (4.23) and (4.24), we get the estimate $\sup_{n>0} \|T_n\|_{L_\infty(\mathbb{R}^m)} < \infty$. Since the set of all polynomials $T_n \in B_{1,m},\ n > 0$, is uniformly bounded on $\mathbb{R}^m$, we can use the compactness theorem for $B_{1,m}$ [21, Theorem 3.3.6, 8, Lemma 2, 14, Lemma 2.1] that for any sequence $\{n_j\}_{j=1}^\infty$ there exist a subsequence $\{n_{j_k}\}_{k=1}^\infty$ and a function $g \in B_{1,m}$ such that

$$\lim_{k\to\infty} T_{n_{j_k}}(x) = g(x) \tag{4.25}$$

uniformly on any compact in $\mathbb{R}^m$.

We can assume without loss of generality that $\sup_{n>0} E^*_{n,m}(f^*_{\delta,n})_1 < \infty$. In addition, we assume that $\{n_j\}_{j=1}^\infty$ satisfies the condition

$$\liminf_{n\to\infty} E^*_{n,m}(f^*_{\delta,n})_1 = \lim_{j\to\infty} E^*_{n_j,m}(f^*_{\delta,n_j})_1. \tag{4.26}$$

Then taking into account (4.20), (4.25), and (4.26), we have for any compact $E \subset \mathbb{R}^m$

$$\begin{aligned}\liminf_{n\to\infty} E^*_{n,m}(f^*_{\delta,n})_1 &= \lim_{k\to\infty} \|f_{\delta,n_{j_k}} - T_{n_{j_k}}\|_{L_1(Q_{\pi n_{j_k}})} \\ &\geqslant \lim_{k\to\infty} \|f_{\delta,n_{j_k}} - T_{n_{j_k}}\|_{L_1(E)} = \|f_\delta - g\|_{L_1(E)}.\end{aligned} \tag{4.27}$$

Since E is an arbitrary compact, we get from (4.27) and (4.21)

$$\liminf_{n\to\infty} E^*_{n,m}(f^*_{\delta,n})_1 \geqslant \|f_\delta - g\|_{L_1(\mathbb{R}^m)} \geqslant A_{1,m}(f_\delta)_1 \geqslant A_{1,m}(f)_1 - \delta. \tag{4.28}$$

So the following inequality is a consequence of (4.28) and (4.22):

$$\liminf_{n\to\infty} E^*_{n,m}(f^*_n)_1 \geqslant A_{1,m}(f)_1 - 2\delta. \tag{4.29}$$

Next, we study some properties of f^*_n. We first note that choosing $b = \pi n$ in (4.7), we get for any $\varepsilon \in (0, 1)$

$$P_{\pi n}\varphi(nx) = n^{2k-m}\mathcal{P}^{2k}_m(x) + T^*(x), \tag{4.30}$$

where $T^* \in \mathcal{T}_{n,m}$. Therefore by (4.9) and (4.30),

$$\begin{aligned} f^*_n(ny) = (\varphi * P_{\pi n}\mu)(ny) &= \int_{Q_{\pi n}} (P_{\pi n}\varphi)(x)(P_{\pi n}\mu)(ny - x)\,dx \\ &= \int_{Q_\pi} (n^{2k}\mathcal{P}^{2k}_m(x) + n^m T^*(x))(P_\pi\mu(n\cdot))(y - x)\,dx. \end{aligned} \tag{4.31}$$

Since $\|P_\pi\mu(n\cdot)\|_{L_1(Q_\pi)} \leqslant n^{-m}$, we obtain from (4.31) that there exists $T^{**} \in \mathcal{T}_{n,m}$ such that $n^{m-2k} f^*_n(ny) - T^{**}(y) \in D^*_{k,m,1}$. Hence

$$\begin{aligned} \liminf_{n\to\infty} E^*_{n,m}(f^*_n)_1 = n^m \liminf_{n\to\infty} E_{n,m}(f^*_n(n\cdot))_1 &= \liminf_{n\to\infty} n^{2k} E_{n,m}(n^{m-2k} f^*_n(n\cdot) - T^{**})_1 \\ &\leqslant \liminf_{n\to\infty} n^{2k} E_{n,m}[D^*_{k,m,1}]_1. \end{aligned} \tag{4.32}$$

Combining (4.29) and (4.32), we get

$$\liminf_{n\to\infty} n^{2k} E_{n,m}[D^*_{k,m,1}]_1 \geqslant A_{1,m}(f)_1 - 2\delta. \tag{4.33}$$

Since f is an arbitrary function from $D_{k,m,1}$ and $\delta > 0$ is any small enough number, (4.33) implies the relation

$$\liminf_{n\to\infty} n^{2k} E_{n,m}[D^*_{k,m,1}]_1 \geqslant A_{1,m}[D_{k,m,1}]_1. \tag{4.34}$$

Then (4.18) follows from (4.34) and (1.4). Finally, (1.7) is a consequence of (4.11) and (4.18). □

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