Constr. Approx. (2008) 27: 289–321
DOI: 10.1007/s00365-006-0670-3

CONSTRUCTIVE APPROXIMATION

Invariance Theorems in Approximation Theory and their Applications

Michael I. Ganzburg

Abstract. Let B be a closed linear subspace of a Banach space F and let $\{T_s\}_{s\in G}$ be a group of continuous linear operators $T_s : F \to F$, where G is a compact topological group. We prove that if $f \in F$ is invariant under $\{T_s\}_{s\in G}$, then under some conditions on f, F, B, and G, there exists an element $g^* \in B$ of best approximation to f that has the same property. As applications, we compute the bivariate Bernstein constant for L_1 polynomial approximation of $|x|^\lambda$ and solve a Braess problem on the exponential order of decay of the error of polynomial approximation of $|x-a|^{-\lambda}$. Other examples and applications are discussed as well.

1. Introduction

Let F be a Banach space with the norm $\|\cdot\|_F$ and let B be a closed linear subspace of F. We denote the error of best approximation of an element $f \in F$ by elements from B by

$$E(f, B, F) := \inf_{g\in B} \|f - g\|_F.$$

We say that $g_0 = g_0(f) \in B$ is an element of best approximation to $f \in F$ (or a best approximation of $f \in F$) if $E(f, B, F) = \|f - g_0\|_F$.

Let G be a topological group and let $\{T_s\}_{s\in G}$ be a group of continuous linear operators $T_s : F \to F$, $s \in G$, i.e., $T_e = I$, $T_{st} = T_s T_t$, $s \in G$, $t \in G$, where e is the identity element of G and I is the identity operator. We denote by F^G the set of all elements $f \in F$ satisfying the condition $T_s f = f$ for all $s \in G$. In other words, F^G is the subspace of F of all f which are *invariant under the group of operators* $\{T_s\}_{s\in G}$.

In this paper we discuss conditions on G, $\{T_s\}_{s\in G}$, F, and B that guarantee the implication

$$f \in F^G \quad \Rightarrow \quad E(f, B, F) = E(f, B \cap F^G, F). \tag{1.1}$$

In addition, if there exists a best approximation of $f \in F^G$, then under these conditions there exists a best approximation of f which is invariant under $\{T_s\}_{s\in G}$. These results

Date received: April 12, 2006. Date revised: October 5, 2006. Date accepted: November 1, 2006. Communicated by Edward B. Saff. Online publication: April 21, 2007.
AMS classification: 41A65, 41A10, 42A10, 41A63.
Key words and phrases: Invariance, Element of best approximation, Rotation group, Polynomial approximation, Entire functions of exponential type.

play a significant role in approximation theory, especially in multivariate approximation, because application of invariance theorems to multivariate problems often allows their reduction to univariate ones.

Invariance theorems like (1.1) are typically applied to a group of continuous operators $T_s\psi(x) = \psi(sx)$, $s \in G_m$, where G_m is a compact group of continuous transformations $s : \Omega_m \to \Omega_m$ on a subset Ω_m of the m-dimensional Euclidean space $\mathbf{R}^m$ and ψ belongs to a Banach space F of functions defined on Ω_m. In this case, we say that F^{G_m} is the subspace of all functions which are *invariant under* G_m.

It is well known that polynomials of best approximation to an even function on a symmetric set about the origin $\Omega_m \subseteq \mathbf{R}^m$ are even as well. In other words, if f is invariant under the group $G_m = \{-e, e\}$, where e is the identity transformation on Ω_m, then polynomials of best approximation to f inherit the same property. The author and Pichugov [23] extended this result to a general situation of a compact group of operators.

Theorem 1.1. *Let G be a compact group and let a closed subspace B of F and $\{T_s\}_{s\in G}$ satisfy the following conditions*:

(1) *for all $g \in B$ and all $s \in G$, $T_s g \in B$*;
(2) *for every $g \in B$, $T_s g$ is continuous in s as a function from G to B*; *and*
(3) $\|T_s\|_{F\to F} = 1$, $s \in G$.

Then, for $f \in F^G$,

$$E(f, B, F) = E(f, B \cap F^G, F). \tag{1.2}$$

Some special cases of the theorem were recently discussed in [2]. Andreev and Yudin [2], Xu [47], the author and Pichugov [23], and the author [17], [19] applied Theorem 1.1 to various problems of multivariate approximation.

Meinardus [28], [29, pp. 26–27] (see also Braess [9, pp. 4 and 195]) and Smoluk [38] developed a different approach to invariance theorems. Under some conditions on F, B and a linear operator $A : F \to F$ with $\|A\|_{F\to F} \le 1$, they showed that if $f \in F$ satisfies $Af = f$, and if there exists an element $g_0 \in B$ of best approximation to f, then there exists an element $g^* \in B$ of best approximation to f such that $Ag^* = g^*$. Since an efficient choice of A is possible only for some elementary examples, these results have not found interesting applications in approximation theory.

In this paper we establish new invariance theorems and apply them to some problems of approximation theory. The paper is organized as follows.

In Section 2 we first establish a different form of Theorem 1.1, which provides additional information on the existence of elements of best approximation (Theorem 2.1). Next, we introduce additional notation and discuss invariance theorems in function spaces with applications to approximation by trigonometric and algebraic polynomials and entire functions of exponential type in rearrangement-invariant spaces such as $L_p(\Omega_m)$, $C(\Omega_m)$, and also Lorentz, Marcinkewicz, and Orlicz spaces (Theorem 2.2 and Examples 2.2 and 2.3).

In Sections 4–6 we establish strengthened invariance theorems for approximation by algebraic polynomials and entire functions of exponential type finding the minimal system of homogeneous generators of the algebra $B \cap F^G$ for some groups of linear transformations of $\Omega_m \subseteq \mathbf{R}^m$ (Theorems 4.1, 5.1, 5.2, and 6.1). In addition, Sections 3–7 contain examples and applications of invariance theorems to several problems of univariate and multivariate approximation. In particular, in Section 3, classical estimates for coefficients of univariate trigonometric polynomials in the uniform and L_p, $1 \le p < \infty$, metrics are extended to a shift-invariant Banach space of 2π-periodic functions (Theorem 3.1).

In Sections 4 and 6 we discuss approximation of radial functions by algebraic polynomials and entire functions of exponential type. As an application, we compute the L_1-errors of best approximation of the function $f_\lambda(x) := |x|^\lambda$ in the bivariate case (Theorems 4.2 and 6.2). We also find the bivariate Bernstein constant for L_1-approximation. Note that the problem of univariate polynomial approximation of $|x|^\lambda$ has attracted much attention since the 1910s [4], [6], [31], [34], [35], [45], [20] and revisited recently in [22], [27], [24]. A criterion for the existence of the multivariate Bernstein constant was given in [16]. We show in Sections 4 and 6 that, for $\lambda > 0$, $\lambda \neq 2, 4, \dots$,

$$\text{(1.3)} \quad \lim_{n\to\infty} n^{\lambda+2} E(f_\lambda, \mathcal{P}_{n,2}, L_1(V_2)) = E(f_\lambda, B_{V_2}, L_1(\mathbf{R}^2)) = 8|\sin(\pi\lambda/2)|\Gamma(\lambda+2)\sum_{k=0}^{\infty}(2k+1)^{-\lambda-3},$$

where V_2 is the unit ball in $\mathbf{R}^2$, $\mathcal{P}_{n,2}$ is the set of bivariate polynomials of degree at most n, and B_{V_2} is the set of all bivariate entire functions of exponential type with the spectra in V_2.

Polynomial approximation on the unit ball V_m and the unit sphere S^{m-1} in $\mathbf{R}^m$ is discussed in Section 5. As an application, we solve a Braess problem [10] on the exponential order of decay of $E(f_{\lambda,a,2}, \mathcal{P}_{n,2}, C(V_2))$, where $f_{\lambda,a,m}(x) := |x-a|^{-\lambda}$, $x \in \mathbf{R}^m$, $a \in \mathbf{R}^m$, $\lambda > 0$, and $|a| > 1$. In addition, we extend this result to $f_{\lambda,a,m}$, $m \ge 2$, $\lambda \in \mathbf{R}^1$, $\lambda \neq 0, -2, \dots$, $a \in \mathbf{R}^m$, and to the function $\log|x-a|$. In particular, for $|a| > 1$, $m \ge 2$, and $\lambda \in \mathbf{R}^1$, $\lambda \neq 0, -2, \dots$, we prove the following inequalities (Theorem 5.3):

$$C_1(|a|,\lambda) n^{\lambda/2-1}|a|^{-n} \le E(f_{\lambda,a,m}, \mathcal{P}_{n,m}, C(V_m)) \le C_2(|a|,\lambda) n^{\mu-1}|a|^{-n},$$

where $\mu := \begin{cases} \lambda, & \lambda > 0, \\ \lambda/2, & \lambda < 0. \end{cases}$ Note that Braess [10] established weaker estimates under the conditions $m = 2$ and either $|a| \ge 3$, $\lambda > 0$ or $|a| > 1$, $0 < \lambda < 2$. We also prove the following estimates for $m \ge 2$, $\lambda < 0$, and $0 \le |a| \le 1$:

$$\text{(1.4)} \quad C_3(|a|,\lambda) n^{-\lambda} \le E(f_{\lambda,a,m}, \mathcal{P}_{n,m}, C(V_m)) \le C_4(|a|,\lambda) n^{-\lambda},$$

which are surprising for $|a| = 1$ since for $m = 1$ and $|a| = 1$, the lower estimate in (1.4) is not valid.

In Section 7 we show that if a continuous function $f \in C(\mathbf{R}^m)$ depends only on the variables $x_1, \dots, x_k$, $1 \le k < n$, then there exists an entire function of exponential type of best uniform approximation to f that has the same property.

Throughout the paper C is a positive constant independent of essential parameters and $C(q_1, \dots, q_d), C_1(q_1, \dots, q_d), C_2(q_1, \dots, q_d), \dots$ denote positive constants that depend only on the parameters $q_1, \dots, q_d$. The same symbol does not necessarily denote the same constant in different occurrences. In addition, throughout the paper $\mathbf{C}^m := \mathbf{R}^m + i\mathbf{R}^m$ is the m-dimensional complex space, and $[x]$ denotes the largest integer n such that $n \le x$.

2. Invariance Theorems for Banach Spaces

An Invariance Theorem for Compact Groups. The following invariance theorem holds.

Theorem 2.1. *Let G be a compact group and let a closed subspace B of a Banach space F and $\{T_s\}_{s\in G}$ satisfy the following conditions*:

(1) *for all $g \in B$ and all $s \in G$, $T_s g \in B$*;
(2) *for every $g \in B$, $T_s g$ is continuous in s as a function from G to B*; *and*
(3) $\|T_s\|_{F\to F} = 1$, $s \in G$.

If there exists an element $g_0 \in B$ of best approximation to $f \in F^G$, then there exists an element $g^* \in B$ of best approximation to f which is invariant under $\{T_s\}_{s\in G}$.

Proof. Since G is compact, there exists the Haar measure $\mu(s)$ on G with $\mu(G) = 1$. Then condition (2) implies the existence of the integral

$$g^* := \int_G T_s g_0 \, d\mu(s) \tag{2.1}$$

(see [37, Theorem 3.27]). Moreover, since B is a closed subspace of F, condition (1) shows that $g^* \in B$ [37, Theorem 3.27]. Next, for any $t \in G$, we have

$$T_t g^* = \int_G T_t T_s g_0 \, d\mu(s) = \int_G T_s g_0 \, d\mu(s) = g^*,$$

where the first equality is proved in [12, Theorem 3.2.19(c)] and the second one follows from the invariance of the Haar measure. Therefore, $g^* \in B \cap F^G$. Finally, using the generalized Minkowski inequality and condition (3), we obtain, for $f \in F^G$,

$$\begin{aligned} \|f - g^*\|_F &= \left\| \int_G T_s(f - g_0) \, d\mu(s) \right\|_F \le \int_G \|T_s(f - g_0)\|_F \, d\mu(s) \\ &\le \|f - g_0\|_F = E(f, B, F). \end{aligned}$$

This completes the proof of the theorem. ■

Remark 2.1. The proof of Theorem 2.1 is similar to that of Theorem 1.1 and is based on the existence of integral (2.1) in the case of a compact group G. In the case of a locally compact group G, integral (2.1) in general does not exist so the construction of an element $g^* \in B \cap F^G$ of best approximation to $f \in F^G$ is more technical and requires more conditions on G and $\{T_s\}_{s\in G}$. The corresponding general result will be discussed in other paper; however, an example of invariance theorems for locally compact groups is presented in Section 7.

Remark 2.2. It is possible to generalize Theorems 1.1 and 2.1 in the following way. Let $h : G \to \{-1, 1\}$ be a continuous homomorphism from a group G to the group $\{-1, 1\}$. If a group of operators $\{T_s\}_{s\in G}$ satisfies the conditions of Theorems 1.1 or 2.1, then the group of operators $\{h(s)T_s\}_{s\in G}$ satisfies the same conditions. Therefore, by these theorems, if

$f \in F_h^G := \{g \in F : g = h(s)T_s g,\ s \in G\}$, then $E(f, B, F) = E(f, B \cap F_h^G, F)$, and if there exists an element $g_0 \in B$ of best approximation to $f \in F_h^G$, then there exists an element $g^* \in B$ of best approximation to f which is invariant under $\{h(s)T_s\}_{s\in G}$.

These results are formally stronger than Theorems 1.1 and 2.1, which follow for $h(s) = 1$, $s \in G$. In addition, these statements give a wide generalization of the well-known result that polynomials of best approximation to an odd function on a symmetric set about the origin $\Omega_m \subseteq \mathbf{R}^m$ are odd as well. A different generalization of this result was established in [28], [29, Theorem 2.7].

Existence of Elements of Best Approximation. Here, we discuss some conditions on B that guarantee the existence of $g_0 \in B$ in Theorem 2.1. It is well known that if B is a finite-dimensional or approximately compact subspace, then for every $f \in F$ there exists an element $g_0 \in B$ of best approximation to f [43, p. 28], [14]. Below we define two other generalized compactness conditions (GCC and GCC*) on B, which are more suitable for our applications, and study their properties.

Definition 2.1. We say that a subspace B of a normed space F satisfies the GCC if there exists a sequence of semi-norms $\{\|\cdot\|_{F,p}\}_{p=1}^{\infty}$ in F such that, for every $f \in F$,

$$\sup_p \|f\|_{F,p} = \|f\|_F, \tag{2.2}$$

and for every sequence $g_n \in B$, $n = 1, 2, \ldots$, with $\sup_n \|g_n\|_F < \infty$, there exist a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ and an element $g^* \in B$ satisfying the relation

$$\lim_{k\to\infty} \|g^* - g_{n_k}\|_{F,p} = 0, \qquad p = 1, 2, \ldots. \tag{2.3}$$

Definition 2.2. We say that a subspace B of a normed space F satisfies the GCC* if, for every sequence $g_n \in B$, $n = 1, 2, \ldots$, with $\sup_n \|g_n\|_F < \infty$, there exist a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ and an element $g^* \in B$ such that for all $f \in F$, the following inequality holds:

$$\|f - g^*\|_F \le \liminf_{k\to\infty} \|f - g_{n_k}\|_F. \tag{2.4}$$

Remark 2.3. It is well known that B satisfies the GCC with $\|f\|_{F,p} := \|f\|_F$, $p = 1, 2, \ldots$, $f \in F$, if and only if B is a finite-dimensional space [37, Theorem 1.22]. Examples of subspaces of infinite dimension, satisfying the GCC, will be discussed in Remark 2.5.

The following statement shows that the GCC* is a weaker condition than the GCC.

Proposition 2.1. *If a subspace B of a normed space F satisfies the GCC, then B satisfies the GCC*.*

Proof. Let B satisfy the GCC and let $\{g_n\}_{n=1}^{\infty}$ be a bounded sequence in B. Then, by Definition 2.1, there exist a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ and $g^* \in B$ such that (2.3) holds. Consequently, taking account of (2.2), we obtain, for every $f \in F$,

$$\|f - g^*\|_F = \sup_p \|f - g^*\|_{F,p} \le \sup_p \liminf_{k\to\infty} \|f - g_{n_k}\|_{F,p} \le \liminf_{k\to\infty} \|f - g_{n_k}\|_F.$$

Therefore, the GCC* is satisfied with g^* and $\{g_{n_k}\}_{k=1}^{\infty}$ from the GCC. ∎

Next we prove that both conditions guarantee the existence of an element of best approximation to any $f \in F$.

Proposition 2.2. *If B satisfies either the GCC or GCC*, then for every $f \in F$ there exists $g^* \in B$ satisfying $E(f, B, F) = \|f - g^*\|_F$.*

Proof. Due to Proposition 2.1, it suffices to prove the proposition in the case when B satisfies the GCC*. Let $f \in F$ and let $g_n \in B$ satisfy the inequality

$$\|f - g_n\|_F < E(f, B, F) + n^{-1}, \qquad n = 1, 2, \ldots.$$

Then $\sup_n \|g_n\|_F \le 2\|f\|_F + 1$ and, by the GCC*, there exist $g^* \in B$ and a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ such that (2.4) holds. This implies that $\|f - g^*\|_F \le E(f, B, F)$. Therefore, g^* is an element of best approximation to f. ■

Further, we discuss special cases of Theorem 2.1 for Banach function spaces.

Groups, Spaces, Subspaces, and Sets. All our examples are special cases of the following situation.

Let F be a Banach space of functions defined on a subset Ω_m of the m-dimensional Euclidean space $\mathbf{R}^m$ and let $G = G_m$ be a compact group of transformations $s : \Omega_m \to \Omega_m$ with the identity transformation e. In the capacity of T_s, we consider either $T_s\psi(x) = \psi(sx)$ or $T_s\psi(x) = h(s)\psi(sx)$, $\psi \in F$, where sx is the image of $x \in \Omega_m$ and $h : G_m \to \{-1, 1\}$ is a continuous homomorphism from G_m to the group $\{-1, 1\}$ (see Remark 2.2). In the case $T_s\psi(x) = \psi(sx)$ and $f \in F^{G_m}$, we say that f *is invariant under* G_m.

We assume that the norm in F is invariant under G_m, that is, for all $f \in F$ and all $s \in G_m$, the equality $\|f(s\cdot)\|_F = \|f\|_F$ holds, which implies $\|T_s\|_{F\to F} = 1$, $s \in G_m$. In particular, F is invariant under G_m when $F = F(\Omega_m)$ is a Banach rearrangement-invariant space of measurable functions on a closed set $\Omega_m \subseteq \mathbf{R}^m$ [13, Sec. 2.2], [26, Sec. 2.4] and G_m is a group of measure-preserving transformations such as linear transformations $sx = A(s)x + x^*(s)$, where $A(s)$ is an $m \times m$ matrix with $|\det A(s)| = 1$ and $x^*(s) \in \mathbf{R}^m$, $s \in G_m$.

Examples of such spaces include the space $C(\Omega_m)$ of all continuous functions on Ω_m with the finite norm $\|f\|_{C(\Omega_m)} := \sup_{\Omega_m} |f|$; the space $L_p(\Omega_m)$, $1 \le p < \infty$, of all measurable on Ω_m functions with the finite norm $\|f\|_{L_p(\Omega_m)} := (\int_{\Omega_m} |f|^p\,dx)^{1/p}$; and also Lorentz spaces $L_\psi(\Omega_m)$, Marcinkiewicz spaces $M_\psi(\Omega_m)$, and Orlicz spaces $O_\psi(\Omega_m)$ [13, Sec. 2.2], [26, Sec. 2.5]. In the univariate case, we use notation $C[a, b] := C([a, b])$, $L_p[a, b] := L_p([a, b])$.

Let $\mathbf{Z}^m$ be a set of all vectors $\alpha = (\alpha_1, \ldots, \alpha_m)$ with integral coordinates, $|\alpha| := \sum_{j=1}^m \alpha_j$, $\mathbf{Z}_+^m = \{\alpha \in \mathbf{Z}^m : \alpha_j \ge 0, 1 \le j \le m\}$, $\mathbf{T}^m$ the m-dimensional torus, and let V be a convex centrally symmetric (with respect to the origin) body in $\mathbf{R}^m$.

In our examples and applications, we discuss the following traditional subspaces of algebraic and trigonometric polynomials and entire functions of exponential type in classic approximation theory. Let $\mathcal{P}_{n,m}$ be the set of all algebraic polynomials $P(x) = \sum_{\alpha\in\mathbf{Z}_+^m,\, |\alpha|\le n} c_\alpha x_1^{\alpha_1} \ldots x_m^{\alpha_m}$ in m variables with real coefficients of degree at most n and $\mathcal{T}_V$ the set of all real-valued trigonometric polynomials $T(x) = \sum_{\alpha\in V\cap\mathbf{Z}^m} c_\alpha \exp(-i \sum_{j=1}^m \alpha_j x_j)$ of m variables with their spectra in V.

Finally, let B_V be the set of all real-valued entire functions g of exponential type in m variables satisfying the inequality

$$|g(z)| \leq C(\varepsilon, g)\ \exp\left((1+\varepsilon)\sup_{t\in V}|(t,z)|\right)$$

for all $z = (z_1, \dots, z_m) \in \mathbf{C}^m$ and any $\varepsilon > 0$ (see [39, Sec. 3.4]); here, $(t, z) := \sum_{j=1}^{m} t_j z_j$. In particular, for $m = 1$, V is a closed interval $[-\sigma, \sigma]$ and we set $B_\sigma := B_{[-\sigma,\sigma]}$, $\sigma > 0$. It is clear that $\mathcal{T}_V \subseteq B_V$.

Throughout the paper we shall use the following groups G_m and sets Ω_m.

Let $D(m)$ be the group of all rotations (about the origin) of $\mathbf{R}^m$. We identify $D(m)$ with the group of all $m \times m$ orthogonal matrices which is isomorphic to $D(m)$ since $s \in D(m)$ if and only if $sx = A(s)x$, where $A(s)$ is an $m \times m$ orthogonal matrix with $|\det A(s)| = 1$.

Let $D(m, a)$ be the group of all rotations (or $m \times m$ orthogonal matrices) s satisfying the condition $sa = a$, where $a \neq 0$ is a fixed vector from $\mathbf{R}^m$. For example, if $a = (\cos\gamma, \sin\gamma)$, then $D(2, a) = \{I, A_\gamma\}$, where I is the 2×2 identity matrix and $A_\gamma := \begin{bmatrix} \cos 2\gamma & \sin 2\gamma \\ \sin 2\gamma & -\cos 2\gamma \end{bmatrix}$.

Next, let $V_m(\rho) := \{x \in \mathbf{R}^m : |x| \leq \rho\}$ be the ball of radius ρ, $V_m := V_m(1)$ the unit ball in $\mathbf{R}^m$, and $S^{m-1} = \{x \in \mathbf{R}^m : |x| = 1\}$ the unit $(m-1)$-dimensional sphere in $\mathbf{R}^m$, $m \geq 2$. Finally, note that χ_Ω denotes the characteristic function of a set Ω.

Remark 2.4. Throughout the paper polynomial approximation on $\Omega_m \subset \mathbf{R}^m$ means approximation by the restrictions of polynomials to Ω_m.

An Invariance Theorem in Banach Function Spaces. In the case of Banach function spaces, condition (2) in Theorem 2.1 can be replaced by a simpler condition. Let Ω_m, G_m, and T_s be defined as in the previous subsection. Then the following invariance theorem for a Banach function space F holds.

Theorem 2.2. *Let G_m be a compact group of transformations on $\Omega_m \subseteq \mathbf{R}^m$ and let F be a Banach function space with the norm invariant under G_m. In addition, let a closed subspace B of F and G_m satisfy the conditions*:

(1) *for all $g \in B$ and all $s \in G$, $g(s\cdot) \in B$*;
(2′) *for every $g \in B$ and each $x \in \Omega_m$, the linear functional $g(sx) : G_m \to \mathbf{R}^1$ is continuous in $s \in G_m$* ;
(3) $\|T_s\|_{F\to F} = 1$, $s \in G$; *and*
(4) *B satisfies the GCC* (*see Definition* 2.1).

If $f \in F^{G_m}$, then there exists a function $g^* \in B$ of best approximation to f which is invariant under $\{T_s\}_{s\in G_m}$.

Proof. Let T_s be defined by $T_s\psi(x) = \psi(sx)$ or $T_s\psi(x) = h(s)\psi(sx)$, where $\psi \in F$, $s \in G_m$, $x \in \Omega_m$, and $h : G_m \to \{-1, 1\}$ is a continuous homomorphism from G_m to the group $\{-1, 1\}$. Next, by condition (4) and Proposition 2.2, there exists an element

$g_0 \in B$ of best approximation to f. Further, by condition (2′), $T_s g_0(x)$ is continuous as a function from G_m to $\mathbf{R}^1$ for each $x \in \Omega_m$, that is, $T_s g_0 \in C(G_m)$. Since G_m is a compact group, there exists the only Haar measure $\mu(s)$ on G_m with $\mu(G_m) = 1$ such that the integral $g^*(x) := \int_{G_m} T_s g_0(x)\, d\mu(s)$ exists [37, Theorem 5.14]. In addition, $g^* \in B$ by [37, Theorem 3.27], and the proof of the relations $g^* \in B \cap F^{G_m}$ and $E(f, B, F) = \|f - g^*\|_F$ is similar to that of Theorem 2.1. ■

The following example shows that in some cases condition (2′) in Theorem 2.2 is less restrictive than condition (2) in Theorem 2.1.

Example 2.1. Let $m = 2$, $F = C(\mathbf{R}^2)$, $\Omega_2 = \mathbf{R}^2$, $B = B_Q \cap C(\mathbf{R}^2)$, where $Q := \{x = (x_1, x_2) \in \mathbf{R}^2 : |x_1| \le 1, |x_2| \le 1\}$ is the unit square, and let G_2 be the group of all proper rotations s of $\mathbf{R}^2$ about the origin through angle $\gamma \in [0, 2\pi)$, that is, $sx = A_\gamma x$ with

$$A_\gamma = \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix}, \qquad \gamma \in [0, 2\pi).$$

For any entire function $g \in B_Q \cap C(\mathbf{R}^2)$ of exponential type, the function

$$g(sx) = g(x_1 \cos\gamma - x_2 \sin\gamma, x_1 \sin\gamma + x_2 \cos\gamma)$$

is continuous in $\gamma \in [0, 2\pi)$ for each fixed $x \in \mathbf{R}^2$. Thus condition (2′) is satisfied. Next we show that $g_1(sx)$ is not continuous in γ as a function from G_2 to $B_Q \cap C(\mathbf{R}^2)$, where $g_1(x) := \sin x_1 \sin x_2 \in B_Q \cap C(\mathbf{R}^2)$.

Let us set

$$u_p := 2(5p + 1), \qquad \gamma(p) := \arccos \frac{u_p^2 - 1}{u_p^2 + 1} = \arcsin \frac{2u_p}{u_p^2 + 1},$$

$$0 \le \gamma(p) \le \pi/2, \quad p = 0, 1, \ldots.$$

Then $u_p^2 + 1$ is divisible by 5 and we set $k_p := (u_p^2 + 1)/5$. Further, setting $x_1(p) = x_2(p) := k_p \pi$ and $x(p) := (x_1(p), x_2(p))$, we have

$$\begin{aligned} &|g_1(x(p)) - g_1(A_{\gamma(p)} x(p))| \\ &\quad = |\sin[\cos(\gamma(p))x_1(p) - \sin(\gamma(p))x_2(p)] \\ &\qquad \times \sin[\sin((\gamma(p))x_1(p) + \cos(\gamma(p))x_2(p)]| \\ &\quad = \left|\sin \frac{\pi(u_p^2 - 2u_p - 1)}{5} \sin \frac{\pi(u_p^2 + 2u_p - 1)}{5}\right| \\ &\quad = \sin(\pi/5)\sin(2\pi/5) > 0.5. \end{aligned}$$

Thus $\lim_{p\to\infty} \gamma(p) = 0$, while, for each $p = 0, 1, \ldots$,

$$\sup_{x \in \mathbf{R}^2} |g_1(x) - g_1(A_{\gamma(p)} x)| \ge |g_1(x(p)) - g_1(A_{\gamma(p)} x(p))| > 0.5.$$

Therefore, condition (2) is not satisfied.

Examples of Special Invariance Theorems. Here, we consider two examples of special invariance theorems for rearrangement-invariant spaces, groups of linear transformations, and subspaces of algebraic polynomials and entire functions of exponential type.

Example 2.2. Let Ω_m be a compact set in $\mathbf{R}^m$, $F(\Omega_m)$ a Banach rearrangement-invariant space, and $B = \mathcal{P}_{n,m}$. It is easy to see that $\mathcal{P}_{n,m}$ is invariant under any compact group G_m of linear transformations $sx = A(s)x + x^*(s) : \Omega_m \to \Omega_m$, where $A(s)$ is an $m \times m$ matrix with $|\det A(s)| = 1$, $x^* \in \mathbf{R}^m$. Thus condition (1) of Theorem 2.2 is satisfied. Since polynomials from $\mathcal{P}_{n,m}$ and linear transformations s are continuous functions, condition ($2'$) of Theorem 2.2 is satisfied as well. Further, if T_s is defined by $T_s\psi(x) = \psi(sx)$, where $\psi \in F$, $s \in G_m$, $x \in \Omega_m$, then condition (3) of Theorem 2.2 is satisfied since $F = F(\Omega_m)$ is a Banach rearrangement-invariant space. Furthermore, $\mathcal{P}_{n,m}$ is a subspace of finite dimension of any Banach rearrangement-invariant space $F(\Omega_m)$ so B satisfies the GCC (see Remark 2.2). Thus, condition (4) of Theorem 2.2 is satisfied. Therefore, Theorem 2.2 implies the following result: for every $f \in F(\Omega_m)$ invariant under G_m there exists a polynomial $P^* \in \mathcal{P}_{n,m}$ of best approximation to f invariant under G_m.

Example 2.3. Let $\Omega_m = \mathbf{R}^m$ and let G_m be a compact group of linear transformations $sx = A(s)x + x^*(s) : \mathbf{R}^m \to \mathbf{R}^m$, where $A(s)$ is an $m \times m$ matrix with $|\det A(s)| = 1$ and $x^*(s) \in \mathbf{R}^m$, $s \in G_m$. Let V be a convex centrally symmetric (with respect to the origin) body in $\mathbf{R}^m$ satisfying the following.

Transpose Condition (TC). $(A(s))^T x \in V$ for any $s \in G_m$ and any $x \in V$, where $(A(s))^T$ is the transpose of $A(s)$.

We assume that $F(\mathbf{R}^m)$ is a Banach rearrangement-invariant space satisfying the following.

Uniform Extension Condition (UEC). For any increasing sequence of measurable sets $\{M_n\}_{n=1}^{\infty}$ with $\bigcup_{n=1}^{\infty} M_n = \mathbf{R}^m$ and, for every $f \in F(\mathbf{R}^m)$,

$$\lim_{n\to\infty} \|f\|_{F(M_n)} = \|f\|_{F(\mathbf{R}^m)}. \tag{2.5}$$

Then the following result holds.

Corollary 2.1. *For every $f \in F(\mathbf{R}^m)$ invariant under G_m there exists a function $g^* \in B_V$ of best approximation to f invariant under G_m.*

To prove the corollary, we need some properties of entire functions of exponential type.

Lemma 2.1. *Let $F(\mathbf{R}^m)$ be a Banach rearrangement-invariant space. Then the following statements hold*:

(a) *If $g \in B_V \cap F(\mathbf{R}^m)$, then $\|g\|_{C(\mathbf{R}^m)} \le C\|g\|_{F(\mathbf{R}^m)}$, where C is independent of g.*
(b) *Let $g_n \in B_V$ satisfy the inequalities $\|g_n\|_{C(\mathbf{R}^m)} \le 1$, $n = 1, 2, \dots$. Then there exist a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ and a function $g^* \in B_V \cap C(\mathbf{R}^m)$ such that*

$$\lim_{k\to\infty} g_{n_k}(x) = g^*(x) \tag{2.6}$$

uniformly on any compact in $\mathbf{R}^m$.
(c) *The set $B_V \cap F(\mathbf{R}^m)$ is closed in $F(\mathbf{R}^m)$.*

Proof. Statements (a) and (b) are proved in [21, Theorem 5.2] and [15, Lemma 2], respectively. Next, let $\lim_{n\to\infty} \|g_n - f_0\|_{F(\mathbf{R}^m)} = 0$, where $g_n \in B_V \cap F(\mathbf{R}^m)$, $n = 1, 2, \dots$, and $f_0 \in F(\mathbf{R}^m)$. Then using statements (a) and (b), we can find a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ and a function $g^* \in B_V \cap C(\mathbf{R}^m)$ such that (2.6) holds uniformly on any compact $K \subset \mathbf{R}^m$. Then we have

$$\|(f_0 - g^*)\chi_K\|_{F(\mathbf{R}^m)} \le \lim_{k\to\infty} (\|f_0 - g_{n_k}\|_{F(\mathbf{R}^m)} + C\|g^* - g_{n_k}\|_{C(K)}) = 0,$$

where C depends only on K and $F(\mathbf{R}^m)$. Hence $f_0(x) = g^*(x)$ a.e. on K and since K is an arbitrary compact, we obtain that $f_0(x) = g^*(x)$ a.e. on $\mathbf{R}^m$. This proves statement (c) of the lemma. ■

Proof of Corollary 2.1. We shall show that all conditions of Theorem 2.2 are satisfied. We first note that $B \cap F(\mathbf{R}^m)$ is a closed subspace of $F(\mathbf{R}^m)$, by Lemma 2.1(c). Next, if $sx = A(s)x + x^*(s)$, then for every $g \in B_V \cap F(\mathbf{R}^m)$, the function $g(s\cdot)$ is an entire function and, for each $z \in \mathbf{C}^m$ and any $\varepsilon > 0$,

$$\begin{aligned}
|g(sz)| = |g(A(s)z + x^*(s))| &\le C(\varepsilon, g, V, x^*(s)) \exp\left[(1+\varepsilon)\sup_{t\in V} |(t, A(s)z)|\right] \\
&= C(\varepsilon, g, V, x^*(s)) \exp\left[(1+\varepsilon)\sup_{t\in V} |((A(s))^T t, z)|\right] \\
&\le C(\varepsilon, g, V, x^*(s)) \exp\left[(1+\varepsilon)\sup_{t\in V} |(t, z)|\right],
\end{aligned}$$

by the TC. In addition, $\|g(s\cdot)\|_{F(\mathbf{R}^m)} = \|g\|_{F(\mathbf{R}^m)}$. Thus $g(s\cdot) \in B_V \cap F(\mathbf{R}^m)$ and conditions (1) and (3) of Theorem 2.2 are satisfied for $B = B_V \cap F(\mathbf{R}^m)$. Moreover, it is easy to see that condition (2′) of this theorem is satisfied as well.

Further, we introduce a sequence of semi-norms on $F(\mathbf{R}^m)$ and study their properties. Let us set

$$\|f\|_{F(\mathbf{R}^m),p} := \|f\chi_{V_m(p)}\|_{F(\mathbf{R}^m)} = \|f\|_{F(V_m(p))}, \qquad f \in F(\mathbf{R}^m)), \quad p = 1, 2, \dots.$$

Then the UEC implies

$$\sup_p \|f\|_{F(\mathbf{R}^m),p} = \|f\|_{F(\mathbf{R}^m)}. \tag{2.7}$$

Next, for every sequence $g_n \in B_V \cap F(\mathbf{R}^m)$, $n = 1, 2 \ldots$, with $\sup_n \|g_n\|_{F(\mathbf{R}^m)} < \infty$, we have by statements (a) and (b) of Lemma 2.1 that there exist a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ and a function $g^* \in B_V \cap C(\mathbf{R}^m)$ such that (2.6) holds uniformly on any compact in $\mathbf{R}^m$. Moreover, by Lemma 2.1(a),

$$(2.8) \qquad \lim_{k\to\infty} \|g^* - g_{n_k}\|_{F(\mathbf{R}^m),p} \le C \lim_{k\to\infty} \|g^* - g_{n_k}\|_{C(V_m(p))} = 0, \qquad p = 1, 2, \ldots,$$

where C depends only on m, p, and $F(\mathbf{R}^m)$.

Relations (2.7) and (2.8) show that the subspace $B_V \cap F(\mathbf{R}^m)$ satisfies the GCC (see Definition 2.1) so condition (4) of Theorem 2.2 is satisfied. Therefore all conditions of Theorem 2.2 are satisfied and Corollary 2.1 follows from Theorem 2.2.

Remark 2.5. If a Banach rearrangement-invariant space $F(\mathbf{R}^m)$ satisfies either of the following conditions, then the UEC is satisfied.

Fatou Condition [26, Sec. 2.0.3]. For any sequence of functions $\{f_n\}_{n=1}^{\infty}$ satisfying $\sup_n \|f_n\|_{F(\mathbf{R}^m)} < \infty$, the convergence $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. implies that $f \in F(\mathbf{R}^m)$ and $\|f\|_{F(\mathbf{R}^m)} \le \liminf_{n\to\infty} \|f_n\|_{F(\mathbf{R}^m)}$.

Regularity Condition [26, Sec. 2.0.3]. For any decreasing sequence of measurable sets $\{M_n\}_{n=1}^{\infty}$ with $\bigcap_{n=1}^{\infty} M_n = \emptyset$, and for every $f \in F(\mathbf{R}^m)$, $\lim_{n\to\infty} \|f\|_{F(M_n)} = 0$.

Indeed, let $\{M_n\}_{n=1}^{\infty}$ be an increasing sequence of measurable sets, $\bigcup_{n=1}^{\infty} M_n = \mathbf{R}^m$. If $F(\mathbf{R}^m)$ satisfies the Regularity Condition, then

$$(2.9) \qquad \begin{aligned} \|f\|_{F(\mathbf{R}^m)} \ge \limsup_{n\to\infty} \|f\|_{F(M_n)} &\ge \liminf_{n\to\infty} \|f\|_{F(M_n)} \\ &\ge \lim_{n\to\infty} (\|f\|_{F(\mathbf{R}^m)} - \|f\|_{F(\mathbf{R}^m - M_n)}) \\ &= \|f\|_{F(\mathbf{R}^m)}. \end{aligned}$$

If $F(\mathbf{R}^m)$ satisfies the Fatou Condition, then using the relation $\lim_{n\to\infty} f(x)\chi_{M_n}(x) = f(x)$ a.e. on $\mathbf{R}^m$, we obtain

$$(2.10) \qquad \|f\|_{F(\mathbf{R}^m)} \ge \limsup_{n\to\infty} \|f\|_{F(M_n)} \ge \liminf_{n\to\infty} \|f\|_{F(M_n)} \ge \|f\|_{F(\mathbf{R}^m)}.$$

Thus (2.9) and (2.10) imply (2.5).

In particular, the spaces $L_p(\mathbf{R}^m)$, $1 \le p < \infty$, satisfy the Regularity Condition, therefore they satisfy the UEC. The space $C(\mathbf{R}^m)$ does not satisfy both conditions. Nevertheless, it is easy to see that $C(\mathbf{R}^m)$ satisfies the UEC.

In addition, note that subspaces $B_V \cap F(\mathbf{R}^m)$ of $F(\mathbf{R}^m)$ provide examples of subspaces of infinite dimension satisfying the GCC.

3. An Extremal Problem for Trigonometric Polynomials

Though invariance theorems are mostly applied to problems in multivariate approximation theory, the following example shows that they can be efficient in univariate approximation as well.

We consider a problem of finding a trigonometric polynomial

$$Q(x) = a_0 + \sum_{l=1}^{n} (a_l \cos lx + b_l \sin lx)$$

with minimal $\|Q\|_F$ provided that either $a_k = 1$ or $b_k = 1$ where k, $0 \le k \le n$, is a fixed integer. Here, F is a Banach space of 2π-periodic univariate functions that contains all trigonometric polynomials Q. We also assume that F is a shift- and symmetry-invariant space, that is, if $f(x) \in F$, then $f(x+\gamma) \in F$, $f(-x) \in F$, and $\|f(x+\gamma)\|_F = \|f(-x)\|_F = \|f\|_F$ for all $f \in F$ and each $\gamma \in [0, 2\pi)$.

In the case $F = C(\mathbf{T}^1)$, the problem was posed by Bernstein [5, pp. 29–31] who obtained some estimates and asymptotics. A general approach to this problem for $F = C(\mathbf{T}^1)$ (even in more general settings) was developed by Rogosinski [36] who, in particular, reduced it to the case $k = 1$. The complete constructive solution for the uniform norm was found by van der Corput and Visser [11] for $n/3 < k \le n$ and by Mulholland [30] for $0 \le k \le n/3$. Rahman [32], Taikov [42], and, more recently, Ash and the author [3] discussed similar problems in L_p-metrics, $1 \le p < \infty$; see [33, Secs. 16.1, 16.2] for more generalizations and discussions. The proofs of the mentioned results are chiefly based on the standard criteria for such extremal problems. In this section we solve the problem for general F and $n/3 < k \le n$, by using Theorem 2.2. In addition, we use Theorem 2.2 to reduce the $L_p(\mathbf{T}^1)$-problem to the case $k = 1$.

Theorem 3.1. *Set* $N := [(n+k)/2k]$. *Then the following statements hold*:

(a) *for* $1 \le k \le n$,

$$\min_{\{Q: a_k = 1\}} \|Q\|_F = \min_{\{Q: b_k = 1\}} \|Q\|_F = \inf_{c_{2j-1},\, 2 \le j \le N} \left\| \cos kx - \sum_{j=2}^{N} c_{2j-1} \cos(k(2j-1)x) \right\|_F ;$$

(b) *for* $n/3 < k \le n$,

$$\min_{\{Q: a_k = 1\}} \|Q\|_F = \min_{\{Q: b_k = 1\}} \|Q\|_F = \|\cos kx\|_F;$$

(c) *in particular, for* $F = L_p(\mathbf{T}^1)$, $1 \le p \le \infty$,

$$\min_{\{Q: a_k = 1\}} \|Q\|_{L_p(\mathbf{T}^1)} = \min_{\{Q: b_k = 1\}} \|Q\|_{L_p(\mathbf{T}^1)} = \begin{cases} \inf_{c_{2j-1},\, 2 \le j \le N} \|\cos x - \sum_{j=2}^{N} c_{2j-1} \cos((2j-1)x)\|_{L_p(\mathbf{T}^1)}, & 1 \le k \le n/3, \\ \|\cos x\|_{L_p(\mathbf{T}^1)} = \left(\dfrac{\pi \Gamma(p+1)}{2^{p-1} \Gamma(p/2+1)^2} \right)^{1/p}, & n/3 < k \le n. \end{cases}$$

Proof. Statement (c) is a direct consequence of statements (a) and (b). To prove statement (a), we first establish the following equalities for $1 \le k \le n$:

$$\min_{\{Q: a_k = 1\}} \|Q\|_F = \min_{\{Q: b_k = 1\}} \|Q\|_F = E(\cos kx, B, F), \tag{3.1}$$

where B is the linear set of all trigonometric polynomials of the form $g(x) = \sum_{l=0,\, l\neq k}^{n}(a_l \cos lx + b_l \sin lx)$. Indeed, let $Q^*(x) = a_0^* + \sum_{l=1}^{n}(a_l^* \cos lx + b_l^* \sin lx)$ be an extremal polynomial satisfying $\|Q^*\|_F = \min_{\{Q: a_k=1\}} \|Q\|_F$. Then

$$\min_{\{Q:a_k=1\}} \|Q\|_F = \|Q^*(x - \pi/(2k))\|_F \geq \min_{\{Q:b_k=1\}} \|Q\|_F.$$

The inequality $\min_{\{Q:a_k=1\}} \|Q\|_F \leq \min_{\{Q:b_k=1\}} \|Q\|_F$ can be proved similarly. Next, for some $\gamma \in [0, 2\pi)$ and some $Q_1 \in B$,

$$\|Q^*\|_F = \|\sqrt{1 + b_k^{*2}} \cos(kx - \gamma) - Q_1(x)\|_F \geq E(\cos kx, B, F),$$

while the inequality $\|Q^*\|_F \leq E(\cos kx, B, F)$ is trivial. Thus (3.1) holds.

Further, the function $f(x) := \cos kx$ is even, $2\pi/k$-periodic, and satisfies the condition $\cos(k(2^{-p}\pi - x)) = -\cos kx$, where $k = 2^p k_1$ and k_1 is odd. Thus f is invariant under the following three finite groups of operators: $T_s^{(i)} : F \to F$, $1 \leq i \leq 3$, where

$$\begin{aligned}
&T_s^{(1)}\psi(x) := \psi(sx), && s \in G_1^{(1)} := \{e, A_1\};\\
&T_s^{(2)}\psi(x) := \psi(sx), && s \in G_1^{(2)} := \{e, A_2, \ldots, A_2^{k-1}\};\\
&T_s^{(3)}\psi(x) := -\psi(sx), && s \in G_1^{(3)} := \{e, A_3\}, \qquad \psi \in F.
\end{aligned}$$

Here, e is the identity transformation and transformations A_i, $1 \leq i \leq 3$, are given by

$$A_1 x := -x, \qquad A_2 x := x + 2\pi/k, \qquad A_3 x := 2^{-p}\pi - x, \qquad x \in \mathbf{T}^1.$$

It is easy to see that all conditions of Theorem 2.2 are satisfied since B is a finite-dimensional subspace of continuous functions in F and, in addition, B is invariant under the transformations A_i with $\|T_s^{(i)}\|_{F\to F} = 1$, $1 \leq i \leq 3$. Therefore, by Theorem 2.2, there exists an even and $2\pi/k$-periodic polynomial $g^*(x) = c_0 + \sum_{l=2}^{[n/k]} c_l \cos(klx) \in B$ of best approximation to $\cos kx$ in the metric of F. In addition, this polynomial satisfies the condition $g^*(2^{-p}\pi - x) = -g^*(x)$, that is,

$$2c_0 + \sum_{l=2}^{[n/k]} c_l((-1)^l + 1)\cos(klx) = 0, \qquad x \in \mathbf{T}^1.$$

Hence $c_{2j} = 0$, $0 \leq j \leq [n/k]/2$, so $g^*(x) = \sum_{j=2}^{N} c_{2j-1} \cos(k(2j-1)x)$. This establishes statement (a). In particular, $N < 2$ if $n/3 < k \leq n$. Therefore, $g^*(x) = 0$ for all $x \in [0, 2\pi)$ if $n/3 < k \leq n$. This yields statement (b). ■

4. Polynomial Approximation of Radial Functions and the Bernstein Constant

An Invariance Theorem. Many radial function such as the Poisson kernel associated with the upper half-plane, the Gauss–Weierstrass kernel, the Bessel–Macdonald kernel, and the fundamental function for the iterated Laplace operator, play an important role in multivariate analysis. The following result shows that polynomials of best approximation to radial functions are radial as well.

Theorem 4.1. *Let $F(V_m)$ be a Banach rearrangement-invariant space of functions on the unit ball V_m and let $f(x) = \varphi(|x|^2) \in F(V_m)$ be a radial function on V_m, where $\varphi : (0,1) \to \mathbf{R}^1$ is a function of a single variable. Then there exists a polynomial $P^* \in \mathcal{P}_{n,m}$ of best approximation to f of the form $P^*(x) = P_1(|x|^2)$, where $P_1 \in \mathcal{P}_{[n/2],1}$.*

The theorem follows immediately from Example 2.2 and the following well-known characterization of polynomials invariant under $D(m)$ [39, Lemma 4.2.11].

Proposition 4.1. *If $P \in \mathcal{P}_{n,m}$ is invariant under the rotation group $D(m)$, then $P(x) = P_1(|x|^2)$, where $P_1 \in \mathcal{P}_{[n/2],1}$.*

In particular, Theorem 4.1 holds for $F(V_m) = L_p(V_m)$, $1 \le p \le \infty$, where $L_\infty(V_m) := C(V_m)$. In this case, Theorem 4.1 is well known to approximation analysts.

The Bernstein Constant in $L_1(V_2)$. We shall apply Theorem 4.1 to the multivariate Bernstein constant $B_{\lambda,p,m}$ defined by

$$B_{\lambda,p,m} := \lim_{n\to\infty} n^{\lambda+m/p} E(f_{\lambda,m}, \mathcal{P}_{n,m}, L_p(V_m)), \tag{4.1}$$

if the limit in (4.1) exists. Here, and in the sequel, $f_{\lambda,m}(x) := |x|^\lambda$, $x \in \mathbf{R}^m$.

In his celebrated papers [4], [6], Bernstein proved that the limit in (4.1) exists in the case $m = 1$, $p = \infty$, and $\lambda > 0$. This result and Theorem 4.1 immediately imply a multivariate version of the Bernstein asymptotic:

$$\lim_{n\to\infty} n^\lambda E(f_{\lambda,m}, \mathcal{P}_{n,m}, C(V_m)) = B_{\lambda,\infty,1}, \qquad \lambda > 0, \tag{4.2}$$

i.e., $B_{\lambda,\infty,m} = B_{\lambda,\infty,1}$, $m \ge 1$. Indeed, by Theorem 4.1,

$$\begin{aligned} E(f_{\lambda,m}, \mathcal{P}_{n,m}, C(V_m)) &= \inf_{P_1\in\mathcal{P}_{[n/2],1}} \max_{x\in V_m} |(|x|^2)^{\lambda/2} - P_1(|x|^2)| \\ &= \inf_{P_1\in\mathcal{P}_{[n/2],1}} \max_{t\in[0,1]} |t^{\lambda/2} - P_1(t)| \\ &= E(f_{\lambda,1}, \mathcal{P}_{2[n/2],1}, C[-1,1]). \end{aligned} \tag{4.3}$$

Thus (4.3) and the Bernstein asymptotic imply (4.2).

Nikolskii [31] established an integral representation for $B_{\lambda,1,1}$, $\lambda > -1$, and showed that, for odd λ,

$$B_{\lambda,1,1} = (8/\pi)|\sin(\pi\lambda/2)|\Gamma(\lambda+1)\sum_{k=0}^{\infty}(-1)^k(2k+1)^{-\lambda-2}. \tag{4.4}$$

Bernstein [8] later noted that (4.4) holds for all $\lambda > -1$. More information on the Bernstein constants $B_{\lambda,p,1}$ can be found in recent papers by Lubinsky [27] and the author [22].

In the following theorem, we compute $E(f_{\lambda,2}, \mathcal{P}_{n,2}, L_1(V_2))$ and $B_{\lambda,1,2}$.

Theorem 4.2.

(a) *For* $-2 < \lambda < 2[n/2], \lambda \neq 0, 2, \ldots,$

$$E(f_{\lambda,2}, \mathcal{P}_{n,2}, L_1(V_2)) = 4|\sin(\pi\lambda/2)| \int_0^\infty y^{\lambda+1} \log\left(\frac{(y+\sqrt{y^2+1})^{2([n/2]+2)}+1}{(y+\sqrt{y^2+1})^{2([n/2]+2)}-1}\right) dy. \tag{4.5}$$

(b) *For* $\lambda > -2, \lambda \neq 0, 2, \ldots,$

$$B_{\lambda,1,2} = \lim_{n\to\infty} n^{\lambda+2} E(f_{\lambda,2}, \mathcal{P}_{n,2}, L_1(V_2)) = 8|\sin(\pi\lambda/2)|\Gamma(\lambda+2)\sum_{k=0}^{\infty}(2k+1)^{-\lambda-3}. \tag{4.6}$$

To prove this result, we need a Markov-type theorem on polynomial $L_1[0,1]$-approximation with the weight t.

Proposition 4.2. *Let* $\varphi : (0,1) \to \mathbf{R}^1$ *satisfy the condition* $\varphi^{(k+1)}(t) \neq 0$, $t \in (0,1)$. *If* $\Phi(t) := \varphi(t^2) \in L_{1,t}[0,1]$ *with* $\|h\|_{L_{1,t}[0,1]} := \int_0^1 |h(t)|t\,dt$, *then*

$$E(\Phi, \mathcal{P}^*_{2k,1}, L_{1,t}[0,1]) = \left|\int_0^1 t\Phi(t)\,\mathrm{sign}\,U_{2k+3}(t)\,dt\right|, \tag{4.7}$$

where $U_N(t) := \sin((N+1)\arccos t)/\sqrt{1-t^2} \in \mathcal{P}_{N,1}$ *is the Chebyshev polynomial of the second kind and* $\mathcal{P}^*_{2k,1}$ *is the set of all even polynomials from* $\mathcal{P}_{2k,1}$.

Proof. We first note that by the substitution $t^2 = (1+x)/2$, we have

$$E(\Phi, \mathcal{P}^*_{2k,1}, L_{1,t}[0,1]) = (\tfrac{1}{4})E(\varphi((1+\cdot)/2), \mathcal{P}_{k,1}, L_1[-1,1]). \tag{4.8}$$

Since $(d/dx)^{k+1}[\varphi((1+x)/2)] \neq 0$ for $x \in (-1,1)$, we obtain by the classical Markov theorem [43, Sec. 2.8.11], [1, Sec. 51],

$$E(\varphi((1+\cdot)/2), \mathcal{P}_{k,1}, L_1[-1,1]) = \left|\int_{-1}^1 \varphi((1+x)/2)\,\mathrm{sign}\,U_{k+1}(x)\,dx\right| = 4\left|\int_0^1 t\Phi(t)\,\mathrm{sign}\,U_{k+1}(2t^2-1)\,dt\right|. \tag{4.9}$$

Taking account of the identity $U_n(2t^2-1) = U_{2n+1}(t)/(2t)$, we arrive at (4.7) from (4.8) and (4.9). ∎

Combining Theorem 4.1 and Proposition 4.2, we obtain the following bivariate version of Markov's theorem for radial functions.

Proposition 4.3. *Let $\varphi : (0,1) \to \mathbf{R}^1$ satisfy the condition $\varphi^{([n/2]+1)}(t) \neq 0, t \in (0,1)$. If $f(x) = \varphi(|x|^2) \in L_1(V_2)$, then*

$$E(f, \mathcal{P}_{n,2}, L_1(V_2)) = 2\pi \left| \int_0^1 t\varphi(t^2) \operatorname{sign} U_{2[n/2]+3}(t)\, dt \right|.$$

Proof of Theorem 4.2. Setting $\varphi(t) = t^{\lambda/2}$ we have, by Proposition 4.3,

$$E(f_{\lambda,2}, \mathcal{P}_{n,2}, L_1(V_2)) = 2\pi \left| \int_0^1 t^{\lambda+1} \operatorname{sign} U_{2[n/2]+3}(t)\, dt \right|. \tag{4.10}$$

To compute the integral in the right-hand side of (4.10), we shall use the following relation:

$$\begin{aligned} &\left| \int_{-1}^1 (1-x)^\mu \operatorname{sign} U_{N+1}(x)\, dx \right| \\ &\quad = \frac{2|\sin(\pi\mu/2)|}{\pi} \int_1^\infty (u-1)^\mu \log\left(\frac{(u+\sqrt{u^2-1})^{N+2}+1}{(u+\sqrt{u^2-1})^{N+2}-1} \right) du, \\ &\quad N > \operatorname{Re}\mu > -1. \end{aligned} \tag{4.11}$$

To prove (4.11), we first note that the Fourier expansions

$$\begin{aligned} \operatorname{sign}\sin(N+2)t &= \frac{4}{\pi} \sum_{k=0}^\infty \frac{\sin(2k+1)(N+2)t}{2k+1}, \\ \frac{\sin t}{u-\cos t} &= 2\sum_{k=1}^\infty (u-\sqrt{u^2-1})^k \sin kt, \qquad u > 1, \end{aligned}$$

and the Parseval identity imply

$$\begin{aligned} \int_{-1}^1 \frac{\operatorname{sign} U_{N+1}(x)}{u-x}\, dx &= \int_0^\pi \frac{\sin t \operatorname{sign}\sin(N+2)t}{u-\cos t}\, dt \\ &= 2\log\left(\frac{(u+\sqrt{u^2-1})^{N+2}+1}{(u+\sqrt{u^2-1})^{N+2}-1} \right), \qquad u > 1. \end{aligned} \tag{4.12}$$

Further, for a complex number μ with $-1 < \operatorname{Re}\mu < 0$ and for $x \in (-1,1)$, $u > 1$, we have

$$(1-x)^\mu = -\frac{\sin(\mu\pi)}{\pi} \int_1^\infty \frac{(u-1)^\mu}{u-x} du. \tag{4.13}$$

Combining (4.12) with (4.13) and using the Fubini theorem we obtain, for $-1 < \operatorname{Re}\mu < 0$,

$$\begin{aligned} &\int_{-1}^1 (1-x)^\mu \operatorname{sign} U_{N+1}(x)\, dx \\ &\quad = -\frac{2\sin(\mu\pi)}{\pi} \int_1^\infty (u-1)^\mu \log\left(\frac{(u+\sqrt{u^2-1})^{N+2}+1}{(u+\sqrt{u^2-1})^{N+2}-1} \right) du. \end{aligned} \tag{4.14}$$

Since both expressions in the left- and right-hand sides of (4.14) are analytic functions in μ for $N > \operatorname{Re}\mu > -1$, the uniqueness of the analytic extension implies that identity (4.14) holds for any complex μ with $\operatorname{Re}\mu \in (-1, N)$. Thus (4.11) follows.

Next, making the substitution $x = 1 - 2t^2$ we have, for $\mu \in (-1, N)$,

$$\left|\int_{-1}^{1}(1-x)^{\mu}\operatorname{sign}U_{N+1}(x)\,dx\right| = 2^{\mu+2}\left|\int_{0}^{1}t^{2\mu+1}\operatorname{sign}U_{2N+3}(t)\,dt\right|. \tag{4.15}$$

Finally, setting $\mu = \lambda/2$, $N = [n/2]$, we obtain, from (4.10), (4.11), and (4.15) for $-2 < \lambda < 2[n/2]$,

$$\begin{aligned}E(f_{\lambda,2}, \mathcal{P}_{n,2}, L_1(V_2)) &= 2^{-1-\lambda/2}\pi\left|\int_{-1}^{1}(1-x)^{\lambda/2}\operatorname{sign}U_{[n/2]+1}(x)\,dx\right|\\ &= 2^{-\lambda/2}|\sin(\pi\lambda/2)|\int_{1}^{\infty}(u-1)^{\lambda/2}\log\left(\frac{(u+\sqrt{u^2-1})^{[n/2]+2}+1}{(u+\sqrt{u^2-1})^{[n/2]+2}-1}\right)du\\ &= 4|\sin(\pi\lambda/2)|\int_{0}^{\infty}y^{\lambda+1}\log\left(\frac{(y+\sqrt{y^2+1})^{2([n/2]+2)}+1}{(y+\sqrt{y^2+1})^{2([n/2]+2)}-1}\right)dy.\end{aligned}$$

Thus (4.5) follows. Note that (4.5) holds trivially for $\lambda = 0, 2, \ldots, 2[n/2]-2$, as well.

(b) We first find the asymptotical behavior of

$$\gamma_M := \int_{0}^{\infty}y^{\lambda+1}\log\left(\frac{(y+\sqrt{y^2+1})^{M}+1}{(y+\sqrt{y^2+1})^{M}-1}\right)dy$$

as $M \to \infty$. By the substitution $v = y + \sqrt{y^2+1}$ we have, for $M > \lambda + 3$,

$$\begin{aligned}\gamma_M &= 2^{-\lambda-2}\int_{1}^{\infty}v^{-\lambda-3}(v^2+1)(v^2-1)^{\lambda+1}\log\left(1+\frac{2}{v^M-1}\right)dv\\ &= 2^{-\lambda-2}\left(\int_{1}^{1+M^{-2/3}}+\int_{1+M^{-2/3}}^{\infty}\right) = 2^{-\lambda-2}(I_1(M)+I_2(M)).\end{aligned} \tag{4.16}$$

To estimate $I_1(M)$ and $I_2(M)$, we need the following elementary inequalities for $y \in [0, M^{1/3}]$, $M \geq 1$,

$$y \geq (1+y/M)^M \geq e^{y-y^2/(2M)} \geq e^{y-1/(2M^{1/3})} \geq (1-1/(2M^{1/1}))e^{y}. \tag{4.17}$$

Then using (4.17) we have, for $M \geq |\lambda+1|+2$,

$$\begin{aligned}I_2(M) &\leq 2\int_{1+M^{-2/3}}^{\infty}\frac{v^{-\lambda-3}(v^2+1)(v^2-1)^{\lambda+1}}{v^M-1}dv\\ &\leq C_1(\lambda)M^{2|\lambda+1|/3}\max_{v\geq 1+M^{-2/3}}\frac{v^{|\lambda+1|+2}}{v^M-1}\\ &\leq C_2(\lambda)M^{2|\lambda+1|/3}((1+M^{-2/3})^M-1)^{-1}\\ &\leq C_3(\lambda)M^{2|\lambda+1|/3}\exp(-M^{1/3}).\end{aligned} \tag{4.18}$$

Next, by the substitution $v = 1 + y/M$, we obtain

$$(4.19)\qquad I_1(M) \;=\; M^{-1}\int_0^{M^{1/3}} \frac{\begin{array}{c}((1+y/M)^2+1)((1+y/M)^2-1)^{\lambda+1}\\ \times\log(1+2/((1+y/M)^M-1))\end{array}}{(1+y/M)^{\lambda+3}}\,dy.$$

It is easy to see that uniformly, for $y \in [0, M^{1/3}]$,

$$(4.20)\qquad (1+y/M)^{-\lambda-3}((1+y/M)^2+1) = 2(1+O(M^{-2/3})),$$

$$(4.21)\qquad ((1+y/M)^2-1)^{\lambda+1} = (2/M)^{\lambda+1}y^{\lambda+1}(1+O(M^{-2/3})).$$

In addition, the following asymptotic follows from (4.17):

$$(4.22)\qquad (1+y/M)^M = e^y(1+O(M^{-1/3}))$$

uniformly for $y \in [0, M^{1/3}]$ as $M \to \infty$. Thus, it follows from (4.19)–(4.22) that

$$I_1(M) = \frac{2^{\lambda+2}}{M^{\lambda+2}}\int_0^{M^{1/3}} y^{\lambda+1}\log\left(\frac{e^y+1}{e^y-1}\right)(1+\Psi_M(y))\,dy,$$

where $\sup_{y\in[0,M^{1/3}]}|\Psi_M(y)| = o(1)$ as $M \to \infty$. Hence,

$$\begin{aligned}(4.23)\qquad I_1(M) \;&=\; (1+o(1))\left(\frac{2}{M}\right)^{\lambda+2}\int_0^{M^{1/3}} y^{\lambda+1}\log\left(\frac{e^y+1}{e^y-1}\right)dy\\ &=\; (1+o(1))\left(\frac{2}{M}\right)^{\lambda+2}\int_0^{\infty} y^{\lambda+1}\log\left(\frac{e^y+1}{e^y-1}\right)dy, \qquad M\to\infty.\end{aligned}$$

Then taking account of the formulas

$$\begin{aligned}(4.24)\qquad \int_0^{\infty} y^{\lambda+1}\log\left(\frac{e^y+1}{e^y-1}\right)dy \;&=\; 2\sum_{k=0}^{\infty}(2k+1)^{-1}\int_0^{\infty} y^{\lambda+1}e^{-(2k+1)y}\,dy\\ &=\; 2\Gamma(\lambda+2)\sum_{k=0}^{\infty}(2k+1)^{-\lambda-3},\end{aligned}$$

we obtain, from (4.16), (4.18), (4.23), and (4.24),

$$(4.25)\qquad \lim_{M\to\infty} M^{\lambda+2}\gamma_M = 2\Gamma(\lambda+2)\sum_{k=0}^{\infty}(2k+1)^{-\lambda-3}.$$

Finally, setting $M := 2([n/2]+2)$, we arrive at (4.6) from (4.5) and (4.25). ■

Remark 4.1. The method of the proof of Theorem 4.2 can also be applied to other bivariate radial functions such as $|x|^\lambda \log|x|$, $\lambda > -1$, and $(1+|x|^2)^{-1}$.

Remark 4.2. We believe that computation of the constants $E(f_{\lambda,m}, \mathcal{P}_{n,m}, L_1(V_m))$ and $B_{\lambda,1,m}$ for $m > 2$ is a difficult problem since the corresponding Markov-type theorems with the weight $|t|^{m-1}$, $m > 2$, do not give explicit expressions for these constants.

5. Polynomial Approximation on the Unit Ball and the Unit Sphere and a Braess Problem

Invariance Theorems. We first prove a more general version of Theorem 4.1 and then discuss polynomial approximation on the $(m-1)$-dimensional sphere S^{m-1} in $\mathbf{R}^m$, $m \geq 2$. Throughout this section we shall use the notation $(x, a) := \sum_{i=1}^m x_i a_i$ for $x \in \mathbf{R}^m$ and $a \in \mathbf{R}^m$.

Theorem 5.1. *Let $F(V_m)$ be a Banach rearrangement-invariant space of functions on the unit ball V_m and let $f(x) = \varphi(|x|^2, (x, a)) \in F(V_m)$, where $\varphi : [0, 1] \times [-1, 1] \to \mathbf{R}^1$ is a function of two variables and $a \in \mathbf{R}^m$, $a \neq 0$, is a fixed vector. Then there exists a polynomial $P^* \in \mathcal{P}_{n,m}$ of best approximation to f of the form $P^*(x) = P_2(|x|^2, (x, a))$, where P_2 is a polynomial of two variables.*

To prove the theorem, we need the following representation for polynomials invariant under the compact group $D(m, a)$, $a \neq 0$, of all rotations with a fixed point a.

Proposition 5.1. *If $P \in \mathcal{P}_{n,m}$ is invariant under $D(m, a)$, $a \neq 0$, then $P(x) = P_2(|x|^2, (x, a))$, where P_2 is a polynomial of two variables.*

Proof. We first assume that $a = ce_1 := c(1, 0, \dots, 0)$, where $c \in \mathbf{R}^1$, $c \neq 0$. Then any rotation about the x_1-axis belongs to $D(m, ce_1)$, that is, if $sx = (x_1, x_2', \dots, x_m')$, where the transformation $(x_2, \dots, x_m) \to (x_2', \dots, x_m')$ is a rotation about the origin of the $(m-1)$-dimensional subspace $\{x \in \mathbf{R}^m : x_1 = 0\}$ of $\mathbf{R}^m$, then $s \in D(m, ce_1)$.

Next we note that polynomials x_1^k, $k = 0, 1, \dots,$ are invariant under $D(m, ce_1)$. In addition, since P is invariant under $D(m, ce_1)$, we have from the representation $P(x) = \sum_{k=0}^n x_1^{n-k} P_k(x_2, \dots, x_m)$ that polynomials $P_k \in \mathcal{P}_{k,m}$, $0 \leq k \leq n$, are invariant under $D(m, ce_1)$. Hence, taking into account that P_k are independent of x_1, we conclude that $P_k(\rho x') = P_k(x')$, $0 \leq k \leq n$, for any $x' = (x_2, \dots, x_m)$ and for every rotation ρ of the subspace $\{x \in \mathbf{R}^m : x_1 = 0\}$ of $\mathbf{R}^m$. Then, by Proposition 4.1,

$$P_k(x_2, \dots, x_m) = \sum_{j=0}^{[k/2]} b_{j,k}(x_2^2 + \cdots + x_m^2)^j, \qquad 0 \leq k \leq n.$$

This implies

$$P(x) = \sum_{0 \leq k+2l \leq n} c_{k,l} x_1^k (x_2^2 + \cdots + x_m^2)^l = \sum_{0 \leq k+2l \leq n} d_{k,l} x_1^k |x|^{2l}.$$

Therefore, the proposition holds for $P_2(u, v) := \sum_{0 \leq k+2l \leq n} c^{-k} d_{k,l} u^k v^l$ if $a = ce_1$ and $x_1 = c^{-1}(x_1, ce_1)$.

We now assume that a is a fixed vector from $\mathbf{R}^m$ and $P \in \mathcal{P}_{n,m}$ invariant under $D(m, a)$. Let $\tau \in D(m)$ be the rotation such that $\tau(|a|e_1) = a$. Then the polynomial $Q(x) := P(\tau x)$ of degree at most n is invariant under $D(m, |a|e_1)$. Indeed, for any $s \in D(m, |a|e_1)$, there exists the only rotation $s^* := \tau s \tau^{-1} \in D(m, a)$ such that $s = \tau^{-1} s^* \tau$. Since, for all $x \in \mathbf{R}^m$,

$$Q(sx) = P(s^* \tau x) = P(\tau x) = Q(x),$$

we conclude that Q is invariant under $D(m, |a|e_1)$. Then using the validity of Proposition 5.1 for $a = ce_1$, we obtain that

$$Q(x) = P(\tau x) = \sum_{0\le k+2l\le n} d'_{k,l} x_1^k |x|^{2l}.$$

Hence, taking account of the identities

$$(5.1) \qquad x_1 = (x, e_1); \qquad (\tau^{-1}x, e_1) = (x, \tau e_1) = |a|^{-1}(x, a); \qquad |\tau^{-1}x| = |x|,$$

we have $P(x) = \sum_{0\le k+2l\le n} d''_{k,l}(x, a)^k |x|^{2l}$. Thus the proposition is valid for $P_2(u, v) := \sum_{0\le k+2l\le n} d''_{k,l} u^k v^l$. ■

Proof of Theorem 5.1. Since $(sx, sa) = (x, a)$ for all $s \in D(m, a)$, the function $f(x) = \varphi(|x|^2, (x, a))$ is invariant under $G_m = D(m, a)$ on V_m. Therefore, $f \in F(V_m)^{G_m}$, where $T_s\psi(x) = \psi(sx)$, $\psi \in F(V_m)$, $s \in G_m$, $x \in V_m$. Next, $|\det s| = 1$ for all $s \in G_m$, and we have from Example 2.2 that there exists a polynomial $P^* \in \mathcal{P}_{n,m}$ of best approximation to f whose restriction to V_m is invariant under $D(m, a)$. Since polynomials $P^*(sx)$ and $P^*(x)$ coincide on the unit ball, we conclude that $P^*(sx) = P^*(x)$ for every $s \in D(m, a)$ and all $x \in \mathbf{R}^m$. Then the representation $P^*(x) = P_2(|x|^2, (x, a))$ follows from Proposition 5.1. ■

Approximation on the unit sphere S^{m-1} is a popular topic in multivariate approximation. The Poisson kernel for the unit ball, the spherical Bernoulli function, and some other kernels [17] have the form of $f(x) = \varphi((x, a))$, where $x \in S^{m-1}$ and $a \in S^{m-1}$. The following analogue of Theorem 5.1 for S^{m-1} shows that there exists a polynomial of best approximation to f of the same form.

Theorem 5.2. *Let $F(S^{m-1})$ be a Banach rearrangement-invariant space of functions on S^{m-1} and let $f(x) = \varphi((x, a)) \in F(S^{m-1})$, where $\varphi : [-1, 1] \to \mathbf{R}^1$ is a function of a single variable and $a \in S^{m-1}$ is a fixed vector. Then there exists a polynomial $P^* \in \mathcal{P}_{n,m}$ of best approximation to f of the form $P^*(x) = P_1((x, a))$ for $x \in S^{m-1}$, where $P_1 \in \mathcal{P}_{n,1}$.*

To prove the theorem, we need a spherical analogue of Proposition 5.1.

Proposition 5.2. *If the restriction $P_{S^{m-1}}$ of a polynomial $P \in \mathcal{P}_{n,m}$ to S^{m-1} is invariant under $D(m, a)$, $a \in S^{m-1}$, then $P(x) = P_1((x, a))$, where $P_1 \in \mathcal{P}_{n,1}$ and $x \in S^{m-1}$.*

It seems plausible that this proposition is a corollary of Proposition 5.1. However, we could not prove it. That is why we give the direct proof of the statement.

Proof of Proposition 5.2. We first assume that $a = e_1 = (1, 0, \dots, 0)$. Since any rotation about the x_1-axis belongs to $D(m, e_1)$, $P_{S^{m-1}}$ is constant on any $(m-2)$-dimensional sphere $S_{m-2,\lambda} := S_{m-1} \cap \Pi_\lambda$ where $\Pi_\lambda := \{x \in \mathbf{R}^m : x_1 = \lambda\}$, $-1 \le \lambda \le 1$ (for $m = 2$, $S_{m-2,\lambda}$ consists of two points on S^1 symmetric about the x_1-axis). Therefore, $P_{S^{m-1}}$ depends only on x_1, and $P(x_1, x_2, \dots, x_m) = P(x_1, \sqrt{1 - x_1^2}, 0, \dots, 0)$

for $x \in S^{m-1}$. Since the transformation $(x_1, x_2, x_3, \dots, x_m) \to (x_1, -x_2, x_3, \dots, x_m)$ belongs to $D(m, e_1)$, $P_{S^{m-1}}$ is even in x_2. Hence $P(x_1, \sqrt{1-x_1^2}, 0, \dots, 0) = P_1(x_1)$, $x \in S^{m-1}$, where $P_1 \in \mathcal{P}_{n,1}$. Therefore, Proposition 5.2 is established for $a = e_1$.

Let now $a \in S^{m-1}$ be a fixed point and let $\tau \in D(m)$ be the rotation such that $\tau e_1 = a$. Then the function $P_{S^{m-1}}(\tau x)$ is invariant under $D(m, e_1)$ (see the proof of Proposition 5.1). Therefore, $P(\tau x) = P_1(x_1)$ for $P_1 \in \mathcal{P}_{n,1}$, $x \in S^{m-1}$, and it follows from (5.1) that $P(x) = P_1((x, a))$, $x \in S^{m-1}$. This proves the proposition. ∎

Proof of Theorem 5.2. The function f is invariant under $G_m = D(m, a)$ on S^{m-1}. Therefore $f \in F(S^{m-1})^{G_m}$, where $T_s\psi(x) = \psi(sx)$, $\psi \in F(S^{m-1})$, $s \in G_m$, $x \in S^{m-1}$. Since $|\det s| = 1$ for all $s \in G_m$, Example 2.2 implies that there exists a polynomial $P^* \in \mathcal{P}_{n,m}$ of best approximation to f whose restriction to S^{m-1} is invariant under $D(m, a)$. Applying Proposition 5.2 to $P = P^*$, we obtain that $P^*(x) = P_1((x, a))$, $x \in S^{m-1}$. ∎

Remark 5.1. A weaker version of Theorem 5.2 and its applications in the case $F(S^{m-1}) = L_p(S^{m-1})$, $1 \le p \le \infty$, was discussed in [17].

A Braess Problem. We apply Theorem 5.2 to the following problem posed by Braess [10]. Let $a \in \mathbf{R}^m$ be a fixed vector with $r := |a| \ge 0$ and let

$$f_{\lambda,a,m}(x) := |x-a|^{-\lambda}, \qquad f_{\log,a,m}(x) := \log|x-a|, \qquad x \in \mathbf{R}^m.$$

In the univariate case, the asymptotic behavior of the error of best polynomial approximation of these functions has been studied by Bernstein. In particular, he proved for $r > 1$ the following asymptotics as $n \to \infty$ [5, Sec. 2.5, eqs. (3.2) and (44bis)], Akhiezer [1, p. 325]:

$$E(f_{\lambda,a,1}, \mathcal{P}_{n,1}, C[-1,1]) = \frac{n^{\lambda-1}(r-\sqrt{r^2-1})^n(1+o(1))}{\Gamma(\lambda)(r^2-1)^{\lambda/2+1/2}}, \qquad \lambda \in \mathbf{R}^1, \tag{5.2}$$

$$E(f_{\log,a,1}, \mathcal{P}_{n,1}, C[-1,1]) = \frac{n^{-1}(r-\sqrt{r^2-1})^n(1+o(1))}{(r^2-1)^{1/2}}, \tag{5.3}$$

and extended (5.2) to $0 \le r \le 1$ and $\lambda < 0$, $\lambda \ne -2, -4, \dots$, [6], [7],

$$\begin{aligned} E(f_{\lambda,a,1}, \mathcal{P}_{n,1}, C[-1,1]) &= (1-r^2)^{|\lambda|/2} E(f_{\lambda,0,1}, \mathcal{P}_{n,1}, C[-1,1])(1+o(1)) \\ &= n^{-|\lambda|}(1-r^2)^{|\lambda|/2} B_{|\lambda|,\infty,1}(1+o(1)), \\ &\qquad 0 \le r < 1, \end{aligned} \tag{5.4}$$

$$\begin{aligned} E(f_{\lambda,a,1}, \mathcal{P}_{n,1}, C[-1,1]) &= 2^{|\lambda|} E(f_{2\lambda,0,1}, \mathcal{P}_{2n,1}, C[-1,1])(1+o(1)) \\ &= n^{-2|\lambda|} 2^{-|\lambda|} B_{2|\lambda|,\infty,1}(1+o(1)), \qquad r = 1, \end{aligned} \tag{5.5}$$

where $B_{\mu,\infty,1}$ is the Bernstein constant defined in (4.1). Note that extensions of (5.4) to more general sets were obtained by Vasiliev [46] and Totik [44].

Recently, Braess [10] investigated the behavior of $E(f_{\lambda,a,2}, \mathcal{P}_{n,2}, C(V_2))$ for $\lambda > 0$ and $r > 1$ in connection with some numerical problems of elliptic equations. Using the ingenious Newman trick of transition from a univariate complex approximation to a bivariate real one, he proved the following estimates for $\lambda > 0$ and $r > 1$:

$$C_1(r,\lambda)n^{-1}r^{-n} \le E(f_{\lambda,a,2}, \mathcal{P}_{n,2}, C(V_2)) \le C_2(r,\lambda)n^{\lambda+1}r^{-n}, \tag{5.6}$$

where the lower estimate was established under the conditions $r \ge 3$ or $0 < \lambda < 2$. In addition, he mentioned that the similar estimates were valid for $E(f_{\log,a,2}, \mathcal{P}_{n,2}, C(V_2))$. In his paper Braess states that it is an open problem whether the lower bound in (5.6) also holds if r gets closer to 1 and if λ is large.

Here we show that better estimates than (5.6) are valid for all $r > 1$ and all $\lambda \in \mathbf{R}^1$, $\lambda \ne 0, -2, \dots$. Moreover, we extend them to m-variate approximation and to the functions $f_{\lambda,a,m}$ and $f_{\log,a,m}$, $m \ge 2$. In addition, we establish precise estimates for $E(f_{\lambda,a,m}, \mathcal{P}_{n,m}, C(V_m))$ in the case $0 \le r \le 1$, $\lambda < 0$, $\lambda \ne -2, -4, \dots$, which for $r = 1$, $m \ge 2$, are surprisingly different compared with (5.5).

Our proof is based on orthogonal expansions, invariance of a polynomial of best approximation on the unit sphere, and relations (5.2)–(5.5).

Theorem 5.3.

(a) *For any $a \in \mathbf{R}^m$ with $|a| = r > 1$ and any $\lambda \in \mathbf{R}^1$, $\lambda \ne 0, -2, \dots$, the following estimates hold*:

$$C_3(r,\lambda)n^{\lambda/2-1}r^{-n} \le E(f_{\lambda,a,m}, \mathcal{P}_{n,m}, C(V_m)) \le C_4(r,\lambda)n^{\mu-1}r^{-n}, \tag{5.7}$$

$$C_5(r)n^{-1}r^{-n} \le E(f_{\log,a,m}, \mathcal{P}_{n,m}, C(V_m)) \le C_6(r)n^{-1}r^{-n}, \tag{5.8}$$

where

$$\mu := \begin{cases} \lambda, & \lambda > 0, \\ \lambda/2, & \lambda < 0. \end{cases} \tag{5.9}$$

In addition,

$$\lim_{n\to\infty} (E(f_{\lambda,a,m}, \mathcal{P}_{n,m}, C(V_m)))^{1/n} = \lim_{n\to\infty} (E(f_{\log,a,m}, \mathcal{P}_{n,m}, C(V_m)))^{1/n} = r^{-1}. \tag{5.10}$$

(b) *For any $a \in \mathbf{R}^m$ with $|a| = r \in (0, 1]$ and any $\lambda < 0$, $\lambda \ne -2, -4, \dots$,*

$$C_7(r,\lambda)n^{-|\lambda|} \le E(f_{\lambda,a,m}, \mathcal{P}_{n,m}, C(V_m)) \le C_8(r,\lambda)n^{-|\lambda|}. \tag{5.11}$$

In addition,

$$\lim_{n\to\infty} n^{|\lambda|} E(f_{\lambda,0,m}, \mathcal{P}_{n,m}, C(V_m)) = B_{|\lambda|,\infty,1}. \tag{5.12}$$

Proof. We first note that (5.10) immediately follows from (5.7) and (5.8) while (5.12) is a direct consequence of (4.2). Next, we prove upper and lower estimates in (5.7), (5.8), and (5.11).

Upper Estimates. We first assume that $r > 1$. Using generating function relations for the Gegenbauer polynomials $C_k^{(\tau)}$ and the Chebyshev polynomials T_k of the first kind (see [40, eqs. (4.7.23) and (4.7.25)]) we have, for $x \in V_m$ and $\lambda \in \mathbf{R}^1$, $\lambda \neq 0, -2, \dots,$

$$|x-a|^{-\lambda} = r^{-\lambda} \sum_{k=0}^{\infty} C_k^{(\lambda/2)} \left(\frac{(x,a)}{r|x|} \right) \left(\frac{|x|}{r} \right)^k, \tag{5.13}$$

$$-\log|x-a| = -\log r + 2 \sum_{k=1}^{\infty} k^{-1} T_k \left(\frac{(x,a)}{r|x|} \right) \left(\frac{|x|}{r} \right)^k. \tag{5.14}$$

Next, it is easy to see that if a polynomial $P_k \in \mathcal{P}_{k,1}$ is even for an even k and odd for an odd k, then the function $P_k((x,a)/r|x|)(|x|/r)^k$ belongs to $\mathcal{P}_{k,m}$, $k = 0, 1, \dots$. Therefore, the nth partial sums of the series in (5.13) and (5.14) are polynomials from $\mathcal{P}_{n,m}$.

Then, taking account of the estimate [40, Sec. 7.33(1)],

$$\|C_k^{(\lambda/2)}\|_{C[-1,1]} \le C(\lambda) k^{\mu-1}, \qquad k = 0, 1, \dots,$$

where μ is defined by (5.9) we get, from (5.13),

$$\begin{aligned} E(f_{\lambda,a,m}, \mathcal{P}_{n,m}, C(V_m)) &\le r^{-\lambda} \max_{|x|\le 1} \left| \sum_{k=n+1}^{\infty} C_k^{(\lambda/2)} \left(\frac{(x,a)}{r|x|} \right) \left(\frac{|x|}{r} \right)^k \right| \\ &\le C(r,\lambda) \sum_{k=n+1}^{\infty} k^{\mu-1} r^{-k} \le C_4(r,\lambda) n^{\mu-1} r^{-n}. \end{aligned}$$

In addition, since $\|T_k\|_{C[-1,1]} = 1$ we have, from (5.14),

$$\begin{aligned} E(f_{\log,a,m}, \mathcal{P}_{n,m}, C(V_m)) &\le 2 \max_{|x|\le 1} \left| \sum_{k=n+1}^{\infty} k^{-1} T_k \left(\frac{(x,a)}{r|x|} \right) \left(\frac{|x|}{r} \right)^k \right| \\ &\le 2 \sum_{k=n+1}^{\infty} k^{-1} r^{-k} \le C_6(r,\lambda) n^{-1} r^{-n}. \end{aligned}$$

Thus the upper bounds in (5.7) and (5.8) are established.

Further, let $0 < r \le 1$, $\lambda < 0$, $\lambda \neq -2, -4, \dots$. Then, by (5.12),

$$\begin{aligned} E(f_{\lambda,a,m}, \mathcal{P}_{n,m}, C(V_m)) &= E(f_{\lambda,0,m}, \mathcal{P}_{n,m}, C(V_m - a)) \\ &\le (1+r)^{|\lambda|} E(f_{\lambda,0,m}, \mathcal{P}_{n,m}, C(V_m)) \\ &\le C_8(r,\lambda) n^{-|\lambda|}. \end{aligned}$$

This establishes the upper estimate in (5.11).

Lower Estimates. We first note that the restriction of any polynomial from $\mathcal{P}_{n,m}$ to the line $L := \{x \in \mathbf{R}^m : x = ta/r,\ t \in \mathbf{R}^1\}$, passing through the origin and a, is a polynomial in a single variable $t \in \mathbf{R}^1$ of degree at most n, and the restriction of $f_{\lambda,a,m}(x)$ to L is $f_{\lambda,r,1}(t)$. Hence,

$$E(f_{\lambda,a,m}, \mathcal{P}_{n,m}, C(V_m)) \ge E(f_{\lambda,r,1}, \mathcal{P}_{n,1}, C[-1,1]). \tag{5.15}$$

Therefore, the lower estimate in (5.11), in the case $0 < r < 1$, $\lambda < 0$, $\lambda \neq -2, -4, \ldots$, follows from (5.15) and (5.4).

All other lower estimates are based on the following result.

Proposition 5.3. *For any* $a \in \mathbf{R}^m$ *with* $|a| = r \geq 1$ *and every* $\varphi \in C[(r-1)^2, (r+1)^2]$,

$$E(\varphi(|x-a|^2), \mathcal{P}_{n,m}, C(V_m)) \geq E(\varphi, \mathcal{P}_{n,1}, C[(r-1)^2, (r+1)^2]). \tag{5.16}$$

Proof. The restriction of $\varphi(|x-a|^2)$ to the unit sphere S^{m-1} is the function $\psi(x) = \varphi(1 - 2r(x, b) + r^2)$, where $x \in S^{m-1}$ and $b = a/|a| \in S^{m-1}$. Then, by Theorem 5.2, there exists a polynomial $P_1 \in \mathcal{P}_{n,1}$ such that

$$\begin{aligned} E(\psi, \mathcal{P}_{n,m}, C(S^{m-1})) &= \max_{x \in S^{m-1}} |\varphi(1 - 2r(x, b) + r^2) - P_1((x, b))| \\ &= \max_{y \in [(r-1)^2, (r+1)^2]} \left| \varphi(y) - P_1\left(\frac{1 + r^2 - y}{2r}\right) \right| \\ &\geq E(\varphi, \mathcal{P}_{n,1}, C[(r-1)^2, (r+1)^2]). \end{aligned} \tag{5.17}$$

Now (5.16) follows from (5.17) and a trivial inequality

$$E(\varphi(|x-a|^2), \mathcal{P}_{n,m}, C(V_m)) \geq E(\psi, \mathcal{P}_{n,m}, C(S^{m-1})).$$

This proves the proposition. ■

Now we are in a position to complete the proof of Theorem 5.3. First let $r > 1$. Then, by Proposition 5.3,

$$\begin{aligned} E(f_{\lambda,a,m}, \mathcal{P}_{n,m}, C(V_m)) &\geq E(y^{-\lambda/2}, \mathcal{P}_{n,1}, C[(r-1)^2, (r+1)^2]) \\ &= (2r)^{-\lambda/2} E\left(\left(\frac{1 + r^2}{2r} - t\right)^{-\lambda/2}, \mathcal{P}_{n,1}, C[-1, 1]\right), \\ &\qquad \lambda \in \mathbf{R}^1, \end{aligned} \tag{5.18}$$

$$\begin{aligned} E(f_{\log,a,m}, \mathcal{P}_{n,m}, C(V_m)) &\geq (\tfrac{1}{2}) E(\log y, \mathcal{P}_{n,1}, C[(r-1)^2, (r+1)^2]) \\ &= (\tfrac{1}{2}) E\left(\log\left(\frac{1 + r^2}{2r} - t\right), \mathcal{P}_{n,1}, C[-1, 1]\right). \end{aligned} \tag{5.19}$$

Thus the lower estimates in (5.7) and (5.8) for $r > 1$ and $\lambda \in \mathbf{R}^1$, $\lambda \neq 0, -2, \ldots$, follow from relations (5.2), (5.18) and (5.3), (5.19), respectively.

Next, let $r = 1$ and $\lambda < 0$, $\lambda \neq -2, -4, \ldots$. Then, by Proposition 5.3,

$$\begin{aligned} E(f_{\lambda,a,m}, \mathcal{P}_{n,m}, C(V_m)) &\geq 2^{-\lambda} E(y^{-\lambda/2}, \mathcal{P}_{n,1}, C[0, 4]) = E(y^{-\lambda/2}, \mathcal{P}_{n,1}, C[0, 1]) \\ &= E(|t|^{-\lambda}, \mathcal{P}_{2n,1}, C[-1, 1]) \geq C_7(1, \lambda) n^{-|\lambda|}. \end{aligned}$$

This establishes the lower estimate in (5.11) for $r = 1$. Therefore, the proof of Theorem 5.3 is completed. ■

Remark 5.2. Note that for $a \neq 0$, $f_{\lambda,a,m}$ is a function of $|x|^2$ and (x, a). Then, by Theorem 5.1, there is a polynomial $P^* \in \mathcal{P}_{n,m}$ of best approximation to $f_{\lambda,a,m}$ in $C(V_m)$ of the form $P^*(x) = P_2(|x|^2, (x, a))$, where P_2 is a bivariate polynomial. Therefore, for $r \neq 0$,

$$E(f_{\lambda,a,m}, \mathcal{P}_{n,m}, C(V_m)) = E(\psi_{\lambda,r}, \mathcal{P}_{n,2}, C([0, 1] \times [-1, 1])) = E(f_{\lambda,a^*,2}, \mathcal{P}_{n,2}, C(V_2)),$$

where $a^* := (0, r) \in \mathbf{R}^2$ and $\psi_{\lambda,r}(u, v) := (u^2 - 2rv + r^2)^{-\lambda/2}$. This shows that the m-variate problem can be reduced to the bivariate Braess one. Nevertheless, the problem of estimating $E(\psi_{\lambda,r}, \mathcal{P}_{n,2}, C([0, 1] \times [-1, 1]))$ appears to be difficult, and we use a different approach to the proof of Theorem 5.3.

Remark 5.3. Theorem 5.3 establishes the exact order of decay of $E(f_{\lambda,a,m}, \mathcal{P}_{n,m}, C(V_m))$ for $\lambda < 0$,

$$E(f_{\lambda,a,m}, \mathcal{P}_{n,m}, C(V_m)) \sim \begin{cases} n^{\lambda/2-1} r^{-n}, & r > 1, \\ n^{-|\lambda|}, & 0 \leq r \leq 1. \end{cases}$$

In addition,

$$E(f_{\log,a,m}, \mathcal{P}_{n,m}, C(V_m)) \sim n^{-1} r^{-n}, \qquad r > 1.$$

The problem, whether multivariate analogues of asymptotics (5.2), (5.3), and (5.4) are valid, is open.

6. Approximation of Radial Functions by Entire Functions of Exponential Type

An Invariance Theorem. The following result shows that entire functions of best approximation to radial functions are radial as well.

Theorem 6.1. *Let $F(\mathbf{R}^m)$ be a Banach rearrangement-invariant space of functions on $\mathbf{R}^m$, satisfying the UEC from Example* 2.3. *If $f(x) = \varphi(|x|) \in F(\mathbf{R}^m)$, where $\varphi : (0, \infty) \to \mathbf{R}^1$ is a function of a single variable, then there exists an entire function $g^* \in B_{V_m(\sigma)} \cap F(\mathbf{R}^m)$ of best approximation to f of the form $g^*(x) = g_1(|x|)$, where $g_1 \in B_\sigma$ is an even function of a single variable.*

The proof is based on Example 2.3 and on the following analogue of Proposition 4.1.

Proposition 6.1. *If the restriction of $g \in B_{V_m(\sigma)}$ to $\mathbf{R}^m$ is invariant under the rotation group $D(m)$, then $g^*(x) = g_1(|x|)$, where $g_1 \in B_\sigma$ is an even function of a single variable.*

Proof. Writing $g(x) = \sum_{k=0}^{\infty} P_k(x)$, where $P_k \in \mathcal{P}_{k,m}$ is a k-homogeneous component of the Taylor expansion for g, $k = 0, 1 \ldots$, we have that for any $\varepsilon > 0$, every $s \in D(m)$, and each $x \in \mathbf{R}^m$, the following identities hold:

$$\sum_{k=0}^{\infty} \varepsilon^k P_k(x) = g(\varepsilon x) = g(\varepsilon(sx)) = \sum_{k=0}^{\infty} \varepsilon^k P_k(sx).$$

Hence $P_k(sx) = P_k(x)$, $k = 0, 1, \ldots$, for every $s \in D(m)$, and each $x \in \mathbf{R}^m$. Therefore, by Proposition 4.1, $P_k(x) = 0$ for an odd k and $P_k(x) = c_k|x|^k$ for an even k, $k = 0, 1, \ldots$, $x \in \mathbf{R}^m$. This implies that $g(x) = \sum_{p=0}^{\infty} c_{2p}|x|^{2p}$. It remains to show that the function $g_1(u) := \sum_{p=0}^{\infty} c_{2p}u^{2p}$ belongs to B_σ. Indeed, since $g \in B_{V_m(\sigma)}$ we have, for any $\varepsilon > 0$ and any $z = (z_1, \ldots, z_m) \in \mathbf{C}^m$,

$$|g(z)| = \left|\sum_{p=0}^{\infty} c_{2p}\left(\sum_{j=1}^{m} z_j^2\right)^p\right| \le C(\varepsilon, g)\exp\left((1+\varepsilon)\sup_{t\in V_m(\sigma)}\left|\sum_{j=1}^{m} t_j z_j\right|\right)$$

$$= C(\varepsilon, g)\exp\left((1+\varepsilon)\sigma\left(\sum_{j=1}^{m}|z_j|^2\right)^{1/2}\right).$$

Hence, setting $z_2 = \cdots = z_m = 0$ we have, for any $z_1 \in \mathbf{C}^1$,

$$|g_1(z_1)| = \left|\sum_{p=0}^{\infty} c_{2p}z_1^{2p}\right| \le C(\varepsilon, g)\exp((1+\varepsilon)\sigma|z_1|).$$

Thus $g_1 \in B_\sigma$. ■

Proof of Theorem 6.1. The function f is invariant under the compact group $G_m = D(m)$. Therefore, $f \in F(\mathbf{R}^m)^{G_m}$, where $T_s\psi(x) = \psi(sx)$, $\psi \in F(\mathbf{R}^m)$, $s \in G_m$, $x \in \mathbf{R}^m$. Next, $V = V_m(\sigma)$ satisfies the Transpose Condition from Example 2.3 since for any orthogonal matrix $s \in D(m)$ and every $x \in V_m(\sigma)$, $s^T x = s^{-1}x \in V_m(\sigma)$. Thus, all the conditions of Example 2.3 are satisfied. Then there exists an entire function $g^* \in B_{V_m(\sigma)} \cap F(\mathbf{R}^m)$ of best approximation to f whose restriction to $\mathbf{R}^m$ is invariant under $D(m)$. Finally, applying Proposition 6.1 to $g = g^*$, we establish Theorem 6.1. ■

Remark 6.1. Theorem 6.1 holds for $F(\mathbf{R}^m) = C(\mathbf{R}^m)$ and $F(\mathbf{R}^m) = L_p(\mathbf{R}^m)$, $1 \le p < \infty$, since these spaces satisfy the UEC (see Remark 2.5).

Approximation of $|x|^\lambda$ in $L_1(\mathbf{R}^2)$. We shall apply Theorem 6.1 to approximation of $f_\lambda(x) := |x|^\lambda$, $x \in \mathbf{R}^2$, by functions from $B_{V_2(\sigma)}$ in the metric of $L_1(\mathbf{R}^2)$.

Theorem 6.2. *For* $\lambda > 0$, $\lambda \ne 2, 4, \ldots$,

$$E(f_\lambda, B_{V_2(\sigma)}, L_1(\mathbf{R}^2)) = 8|\sin(\pi\lambda/2)|\Gamma(\lambda+2)\sigma^{-\lambda-2}\sum_{k=0}^{\infty}(2k+1)^{-\lambda-3}. \tag{6.1}$$

Proof. *Step* 1. Since $f_\lambda \notin L_1(\mathbf{R}^2)$, we first prove that for every $\varepsilon > 0$ there exists a radial function $g(x) = g_\varepsilon(|x|) \in B_{V_2(\varepsilon)}$, where $g_\varepsilon \in B_\varepsilon$ is even, such that the following inequalities hold:

$$|g_\varepsilon(|x|)| \le C(1+|x|^2)^N, \qquad x \in \mathbf{R}^2, \tag{6.2}$$

$$||x|^\lambda - g_\varepsilon(|x|)| \le C(1+|x|^2)^{-2}, \qquad x \in \mathbf{R}^2, \tag{6.3}$$

where $N \geq 0$ is an integer. To prove it, we introduce the function $(0 < a < b)$,

$$\psi_{a,b}(x) := \begin{cases} 0, & |x| \leq a, \\ c\int_a^{|x|} \exp(-(u-a)^{-2}(b-u)^{-2})\,du, & a < |x| < b, \\ 1, & |x| \geq b, \end{cases}$$

where $c := (\int_a^b \exp(-(u-a)^{-2}(b-u)^{-2})\,du)^{-1}$. Then $\psi_{\varepsilon/2,\varepsilon}$ is a radial infinitely differentiable function on $\mathbf{R}^2$ and $\psi_{\varepsilon/2,\varepsilon}(x) = 0$ for $|x| \leq \varepsilon/2$ and $\psi_{\varepsilon/2,\varepsilon}(x) = 1$ for $|x| \geq \varepsilon$.

Next we note that the Fourier transform of the tempered distribution f_λ for $m = 2$ and $\lambda > 0$ is

$$\mathcal{F}(f_\lambda)(y) = \pi 2^{\lambda+2}(\Gamma(\lambda/2+1)/\Gamma(-\lambda/2))|y|^{-\lambda-2}, \qquad y \in \mathbf{R}^2,$$

(see [25, Sec. 2.3.3]). Then the function $h_\lambda := \psi_{\varepsilon/2,\varepsilon}\mathcal{F}(f_\lambda)$ satisfies the conditions

$$\frac{\partial^{l_1+l_2}h_\lambda(x)}{\partial x_1^{l_1}\partial x_2^{l_2}} \in L_1(\mathbf{R}^2), \qquad 0 \leq l_1 + l_2 \leq 4.$$

Therefore, the inverse Fourier transform $\mathcal{F}^{-1}(h_\lambda)$ satisfies the inequality

(6.4) $$|\mathcal{F}^{-1}(h_\lambda)(x)| \leq C(1+|x|^2)^{-2}, \qquad x \in \mathbf{R}^2.$$

Further, the function $H := \mathcal{F}(f_\lambda) - h_\lambda$ is a tempered distribution with the support in $V_2(\varepsilon)$. Then by the generalized Paley–Wiener theorem [37, Theorem 7.23], the function $g := \mathcal{F}^{-1}(H)$ belongs to $B_{V_2(\varepsilon)}$ and has polynomial growth on $\mathbf{R}^2$, that is,

(6.5) $$|g(x)| \leq C(1+|x|^2)^N$$

for some integer $N \geq 0$. Moreover, g is invariant under $D(2)$. To prove this statement, we use the following fact [25, Sec. 2.3.1]: if a tempered distribution f of m variables is invariant under $D(m)$, that is, f satisfies the condition $f(sx) = f(x)$ for all $s \in D(m)$, then $\mathcal{F}(f)$ and $\mathcal{F}^{-1}(f)$ are invariant under $D(m)$. Hence, the tempered distributions $\mathcal{F}(f_\lambda)$, h_λ, and H are invariant under $D(2)$, consequently, $g = \mathcal{F}^{-1}(H)$ is invariant under $D(2)$ as well.

Next, by Proposition 6.1, $g(x) = g_\varepsilon(|x|)$, where $g_\varepsilon \in B_\varepsilon$ is an even function. Since $\mathcal{F}^{-1}(h_\lambda)(x) = f_\lambda(x) - g_\varepsilon(|x|)$, inequalities (6.2) and (6.3) follow from (6.5) and (6.4), respectively.

In particular, (6.3) implies that

(6.6) $$\|f_\lambda - g_\varepsilon(|\cdot|)\|_{L_1(\mathbf{R}^2)} < \infty$$

and, in addition,

(6.7) $$|t(|t|^\lambda - g_\varepsilon(t))| \leq C(1+|t|^2)^{-1}, \qquad t \in \mathbf{R}^1.$$

Step 2. Since, by (6.6), $f_\lambda(x) - g_\varepsilon(|x|) \in L_1(\mathbf{R}^2)$, we can apply Theorem 6.1 to $f(x) = f_\lambda(x) - g_\varepsilon(|x|)$ and $F(\mathbf{R}^2) = L_1(\mathbf{R}^2)$. Then there exists an even function

$g_\sigma \in B_\sigma$ such that, for $\sigma > \varepsilon$,

$$(6.8) \qquad \begin{aligned} E(f_\lambda, B_{V_2(\sigma)}, L_1(\mathbf{R}^2)) &= E(f_\lambda - g_\varepsilon(|\cdot|), B_{V_2(\sigma)}, L_1(\mathbf{R}^2)) \\ &= \|f_\lambda - g_\varepsilon(|\cdot|) - g_\sigma(|\cdot|)\|_{L_1(\mathbf{R}^2)} \\ &= \pi E(|t|^\lambda - g_\varepsilon(t), B_\sigma, L_{1,t}(\mathbf{R}^1)), \end{aligned}$$

where $\|h\|_{L_{1,t}(\mathbf{R}^1)} := \int_{\mathbf{R}^1} |th(t)|\, dt$.

This shows that a bivariate approximation problem is reduced to a univariate approximation problem in a weighted L_1-space. To solve the latter one, we first establish the following equality:

$$(6.9) \qquad E(|t|^\lambda - g_\varepsilon(t), B_\sigma, L_{1,t}(\mathbf{R}^1)) = E(t(|t|^\lambda - g_\varepsilon(t)), B_\sigma, L_1(\mathbf{R}^1)),$$

where $t(|t|^\lambda - g_\varepsilon(t)) \in L_1(\mathbf{R}^1)$, by (6.7). Indeed, the inequality

$$(6.10) \qquad E(|t|^\lambda - g_\varepsilon(t), B_\sigma, L_{1,t}(\mathbf{R}^1)) \geq E(t(|t|^\lambda - g_\varepsilon(t)), B_\sigma, L_1(\mathbf{R}^1))$$

is trivial. Next, let $G_\sigma \in B_\sigma$ satisfy the equality

$$E(t(|t|^\lambda - g_\varepsilon(t)), B_\sigma, L_1(\mathbf{R}^1)) = \|t(|t|^\lambda - g_\varepsilon(t)) - G_\sigma\|_{L_1(\mathbf{R}^1)}.$$

Without loss of generality we can assume that G_σ is an odd function. Then $G_\sigma^*(t) := G_\sigma(t)/t \in B_\sigma$ and we have

$$(6.11) \qquad \begin{aligned} E(t(|t|^\lambda - g_\varepsilon(t)), B_\sigma, L_1(\mathbf{R}^1)) &= \|\, |t|^\lambda - g_\varepsilon(t) - G_\sigma^*\|_{L_{1,t}(\mathbf{R}^1)} \\ &\geq E(|t|^\lambda - g_\varepsilon(t), B_\sigma, L_{1,t}(\mathbf{R}^1)). \end{aligned}$$

Thus (6.10) and (6.11) imply (6.9).

Step 3. Next we find the Fourier sin-transform of the function $t|t|^\lambda - tg_\varepsilon(t)$, which is integrable on $\mathbf{R}^1$ by (6.7). Namely, we prove that, for $|y| > \varepsilon$,

$$(6.12) \qquad \begin{aligned} \Lambda(y) &:= \int_{\mathbf{R}^1} (t|t|^\lambda - tg_\varepsilon(t)) \sin ty\, dt \\ &= -2\sin(\pi\lambda/2)\Gamma(\lambda+2)|y|^{-\lambda-2} \operatorname{sign} y. \end{aligned}$$

Let $S(\mathbf{R}^1)$ be the Schwartz class of all rapidly decreasing functions on $\mathbf{R}^1$. Then denoting the right-hand side of (6.12) by $\varphi(y)$ we have, for any $h \in S(\mathbf{R}^1)$ with its support outside $[-\varepsilon, \varepsilon]$,

$$(6.13) \qquad \int_{\mathbf{R}^1} (t|t|^\lambda - tg_\varepsilon(t))\mathcal{F}(h)(t)\, dt = \int_{\mathbf{R}^1} t|t|^\lambda \mathcal{F}(h)(t)\, dt - \int_{\mathbf{R}^1} tg_\varepsilon(t)\mathcal{F}(h)(t)\, dt,$$

where the second integral in the right-hand side of (6.13) exists by (6.2). Next, again using the generalized Paley–Wiener theorem [37, Theorem 7.23], we have

$$(6.14) \qquad \int_{\mathbf{R}^1} tg_\varepsilon(t)\mathcal{F}(h)(t)\, dt = 0.$$

Further, it is known [25, eq. (2.3.13)] that

$$\int_{\mathbf{R}^1} t|t|^{\lambda}\mathcal{F}(h)(t)\,dt = \int_{\mathbf{R}^1} \varphi(t)h(t)\,dt = \int_{|t|>\varepsilon} \varphi(t)h(t)\,dt. \tag{6.15}$$

Thus (6.13), (6.14), and (6.15) imply the equality

$$\int_{\mathbf{R}^1} (t|t|^{\lambda} - tg_{\varepsilon}(t))\mathcal{F}(h)(t)\,dt = \int_{|t|>\varepsilon} \varphi(t)h(t)\,dt. \tag{6.16}$$

Finally, choosing h in (6.16) as a peak delta-like function from $S(\mathbf{R}^1)$, supported in the interval $[y-\delta, y+\delta]$ with $0 < \delta < |y| - \varepsilon$ and letting $\delta \to 0$, we arrive at (6.12).

Step 4. Finally, we prove (6.1). It follows from (6.7) and (6.12) that the function $\psi(t) := -\text{sign}(\sin(\pi\lambda/2))(t|t|^{\lambda} - tg_{\varepsilon}(t))$ satisfies the following conditions of the Sz.-Nagy criterion [41], [1, Sec. 88] for approximation in $L_1(\mathbf{R}^1)$ by entire functions of exponential type: ψ is an odd function, satisfying $|\psi(t)| \le C(1+t^2)^{-1}$ by (6.7), and the following inequalities hold for its sin-transform $\Lambda_1(y) := -\text{sign}(\sin(\pi\lambda/2))\Lambda(y)$, where Λ is defined in (6.12),

$$\Lambda_1(y) > 0, \qquad \Lambda_1'(y) \le 0, \qquad \Lambda_1''(y) \ge 0, \qquad y > \varepsilon.$$

Then, for $\sigma > \varepsilon$,

$$\begin{aligned} E(t(|t|^{\lambda} - g_{\varepsilon}(t)), B_{\sigma}, L_1(\mathbf{R}^1)) &= \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\Lambda_1((2k+1)\sigma)}{2k+1} \\ &= (8/\pi)|\sin(\pi\lambda/2)|\Gamma(\lambda+2)\sigma^{-\lambda-2} \\ &\quad\times \sum_{k=0}^{\infty}(2k+1)^{-\lambda-3}. \end{aligned} \tag{6.17}$$

Thus, (6.1) follows from (6.8), (6.9), and (6.17). ∎

Remark 6.2. Comparing (4.6) and (6.1), we see that the following relation holds:

$$\lim_{n\to\infty} n^{\lambda+2}E(|x|^{\lambda}, \mathcal{P}_{n,2}, L_1(V_2)) = E(|x|^{\lambda}, B_{V_2(1)}, L_1(\mathbf{R}^2)), \qquad \lambda > 0. \tag{6.18}$$

Actually, (6.18) is a special case of more general relations (the so-called limit theorems) of the form

$$\lim_{n\to\infty} E(f, \mathcal{P}_{n,m}, L_p(nV_m)) = E(f, B_{V_m(1)}, L_p(\mathbf{R}^m)), \qquad 1 \le p < \infty, \tag{6.19}$$

which are valid for all measurable functions f of polynomial growth on $\mathbf{R}^m$ provided that the right-hand side of (6.19) is finite (see [16], [20]). Note that a version of (6.19) for the uniform metric holds as well [16], [20].

Remark 6.3. A general approach to Markov-type theorems in $L_1(\mathbf{R}^m)$, $m \ge 2$, for radial integrable functions was developed in [18], [19]. Note that the Sz.-Nagy criterion cannot be applied for $m > 2$, and computation of $E(|x|^{\lambda}, \mathcal{P}_{n,m}, L_1(V_m))$ appears to be a difficult problem for $m > 2$.

7. Approximation in Higher Dimensions

Here we discuss approximation in higher dimensions when a function $f(x)$, $x \in \Omega_m$, depends only on variables $x_1, \ldots, x_k$, $1 \le k < m$, $m \ge 2$. In this section we shall use the following notation: for a vector $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m)$ with positive coordinates we set $\Pi_k(\boldsymbol{\sigma}) := \{x \in \mathbf{R}^k : |x_j| \le \sigma_j, 1 \le j \le k\}$ to be a rectangular parallelepiped in $\mathbf{R}^k$, $1 \le k \le m$.

Theorem 7.1. *If $f \in C(\mathbf{R}^m)$ depends only on variables $x_1, \ldots, x_k$, $1 \le k < m$, $m \ge 2$, then there exists an entire function $g^* \in B_{\Pi_m(\boldsymbol{\sigma})} \cap C(\mathbf{R}^m)$ such that it depends only on $x_1, \ldots, x_k$ and*

$$(7.1)\qquad E(f, B_{\Pi_m(\boldsymbol{\sigma})} \cap C(\mathbf{R}^m), C(\mathbf{R}^m)) = E(f, B_{\Pi_k(\boldsymbol{\sigma})} \cap C(\mathbf{R}^k), C(\mathbf{R}^k)) = \|f - g^*\|_{C(\mathbf{R}^k)}.$$

Note that a function $f \in C(\mathbf{R}^m)$ depends only on variables $x_1, \ldots, x_k$, $1 \le k < m$, $m \ge 2$, if and only if f is invariant under the group G_m of all shift transformations s of the form $sx = x + (0, \ldots, 0, \tau_{k+1}, \ldots, \tau_m)$, where $\tau_j \in \mathbf{R}^1$, $k+1 \le j \le m$. Therefore, $f \in C(\mathbf{R}^m)^{G_m}$ for $T_s\psi(x) = \psi(sx)$, $\psi \in C(\mathbf{R}^m)$, $s \in G_m$, $x \in \mathbf{R}^m$. However, we cannot use invariance theorems from Section 2 because G_m is a locally compact group. That is why we give the direct proof of the statement.

Proof of Theorem 7.1. We first note that there exists a function $g_0 \in B_{\Pi_m(\boldsymbol{\sigma})} \cap C(\mathbf{R}^m)$ of best approximation to f. This follows from Proposition 2.2 and from the fact that $B_{\Pi_m(\boldsymbol{\sigma})} \cap C(\mathbf{R}^m)$ satisfies the GCC (see the proof of Corollary 2.1 and Remark 2.5).

Next, let us consider a sequence of functions

$$g_n(x) := (2M_n)^{k-m} \int_{-M_n}^{M_n} \cdots \int_{-M_n}^{M_n} g_0(x+\tau)\,d\tau_{k+1} \ldots d\tau_m, \qquad n = 1, 2, \ldots,$$

where $\{M_n\}_{n=1}^{\infty}$ is an increasing sequence of positive numbers with $\lim_{n\to\infty} M_n = \infty$, and $\tau \in \mathbf{R}^m$ is an arbitrary vector of the form $\tau = (0, \ldots, 0, \tau_{k+1}, \ldots, \tau_m)$.

Then $\sup_n \|g_n\|_{C(\mathbf{R}^m)} \le \|g_0\|_{C(\mathbf{R}^m)}$ and $g_n \in B_{\Pi_m(\boldsymbol{\sigma})}$, $n = 1, 2, \ldots$. Next, using Lemma 2.1(b), we can assume without loss of generality that there exists $g^* \in B_{\Pi_m(\boldsymbol{\sigma})} \cap C(\mathbf{R}^m)$ such that

$$(7.2)\qquad \lim_{n\to\infty} g_n(x) = g^*(x)$$

uniformly on any compact in $\mathbf{R}^m$. Further, by (7.2), for an arbitrary vector $y = (0, \ldots, 0, y_{k+1}, \ldots, y_m) \in \mathbf{R}^m$ and for any compact $K \subset \mathbf{R}^m$ we have

$$\begin{aligned}\max_{x\in K} |g^*(x+y) - g^*(x)| &= \lim_{n\to\infty} \max_{x\in K} |g_n(x+y) - g_n(x)| \\ &\le \|g_0\|_{C(\mathbf{R}^m)} (2M_n)^{k-m} \operatorname{Vol}_{m-k}((Q_{M_n} \setminus (Q_{M_n} + y')) \cup ((Q_{M_n} + y') \setminus Q_{M_n})) \\ &= 0.\end{aligned}$$

Here, $y' := (y_{k+1}, \ldots, y_m) \in \mathbf{R}^{m-k}$ and $Q_a := \{x \in \mathbf{R}^{m-k} : |x_i| \le a, k+1 \le i \le m\}$ is a cube in $\mathbf{R}^{m-k}$. Therefore, g^* depends only on $x_1, \ldots, x_k$.

Furthermore, it follows from (7.2) that, for any compact $K \in \mathbf{R}^m$,

$$\begin{aligned}\max_{x\in K}|f(x)-g^*(x)| &= \lim_{n\to\infty}\max_{x\in K}|f(x)-g_n(x)| \\ &\le \max_{x\in K}|f(x)-g_0(x)| = E(f, B_{\Pi_m(\boldsymbol{\sigma})}\cap C(\mathbf{R}^m), C(\mathbf{R}^m)).\end{aligned}$$

Therefore, $g^* \in B_{\Pi_m(\boldsymbol{\sigma})}\cap C(\mathbf{R}^m)$ is a function of best approximation to f. Since, by the definition, $g^* \in B_{\Pi_m(\boldsymbol{\sigma})}$ if and only if, for any $z \in \mathbf{C}^m$ and any $\varepsilon > 0$,

$$|g^*(z_1,\dots,z_m)| \le C(\varepsilon, g^*)\exp\left((1+\varepsilon)\sum_{j=1}^m \sigma_j|z_j|\right),$$

we have

$$|g^*(z_1,\dots,z_m)| = |g^*(z_1,\dots,z_k,0,\dots,0)| \le C(\varepsilon, g^*)\exp\left((1+\varepsilon)\sum_{j=1}^k \sigma_j|z_j|\right).$$

Hence the restriction $g^*_{\mathbf{R}^k}$ of g^* to $\mathbf{R}^k$ belongs to $B_{\Pi_k(\boldsymbol{\sigma})}\cap C(\mathbf{R}^k)$. In addition, it is easy to see that $g^*_{\mathbf{R}^k} \in B_{\Pi_k(\boldsymbol{\sigma})}\cap C(\mathbf{R}^k)$ is the function of best approximation to $f_{\mathbf{R}^k}$ in $C(\mathbf{R}^k)$. Consequently,

$$\begin{aligned}E(f, B_{\Pi_k(\boldsymbol{\sigma})}\cap C(\mathbf{R}^k), C(\mathbf{R}^k)) &= \|f-g^*\|_{C(\mathbf{R}^k)} = \|f-g^*\|_{C(\mathbf{R}^m)} \\ &= E(f, B_{\Pi_m(\boldsymbol{\sigma})}\cap C(\mathbf{R}^m), C(\mathbf{R}^m)).\end{aligned}$$

This yields (7.1). ■

Remark 7.1. A similar invariance result for approximation of continuous functions on $\mathbf{T}^m$ by trigonometric polynomials from $\mathcal{T}_{\Pi_m(\boldsymbol{n})}$ follows from Theorem 2.2. A nonperiodic analogue of this result for polynomial approximation on the m-dimensional unit cube follows directly from the periodic case by the standard substitution $x_i = \cos t_i$, $1 \le i \le m$.

Acknowledgment. The author thanks Professor Franz Peherstorfer for providing references [30], [32], and [36].

References

[1] N. I. Akhiezer (1965): Lectures on the Theory of Approximation, 2nd ed. Moscow: Nauka (Russian).

[2] N. N. Andreev, V. A. Yudin (2001): *Polynomials of least deviation from zero, and Chebyshev-type cubature formulas*. Proc. Steklov Inst. Math., **232**(1):39–51.

[3] J. M. Ash, M. Ganzburg (1999): *An extremal problem for trigonometric polynomials*. Proc. Amer. Math. Soc., **127**:211–216.

[4] S. Bernstein (1913): *Sur les meilleure approximation de $|x|$ par des polynômes des degrés donnés*. Acta Math., **37**:1–57.

[5] S. N. Bernstein (1937): Extremal Properties of Polynomials and the Best Approximation of Continuous Functions of a Single Real Variable. Moscow: State United Scientific and Technical Publishing House (Russian).

[6] S. BERNSTEIN (1938): *Sur les meilleure approximation de $|x|^p$, par des polynômes de degrés trés élevés*. Bull. Acad. Sci. USSR Sér. Math., **2**:181–190.
[7] S. N. BERNSTEIN (1938): *On the best approximation of $|x-c|^p$*. Dokl. Akad. Nauk. SSSR, **18**:379–384 (Russian).
[8] S. N. BERNSTEIN (1954): *On the best approximation of $|x-c|^p$*. In: Collected Works, Vol. II. Moscow: Akad. Nauk SSSR, pp. 273–280 (Russian).
[9] D. BRAESS (1986): Nonlinear Approximation Theory. Series in Computational Mathematics, Vol. 7. Berlin: Springer-Verlag.
[10] D. BRAESS (2001): *Note on the approximation of powers of the distance in two-dimensional domains*. Constr. Approx., **17**:147–151.
[11] J. G. VAN DER CORPUT, C. VISSER (1946): *Inequalities concerning polynomials and trigonometric polynomials*. Indag. Math., **8**:238–247.
[12] N. DANFORD, J. T. SCHWARTZ (1988): Linear Operators, Part I: General Theory. New York: Wiley.
[13] R. A. DEVORE, G. G. LORENTZ (1993): Constructive Approximation. New York: Springer-Verlag.
[14] N. V. EFIMOV, S. B. STEČKIN (1961): *Approximative compactness and Chebshev sets*. Dokl. Akad. Nauk SSSR, **140**:522–524 (Russian).
[15] M. I. GANZBURG (1981): *Multidimensional Jackson theorems*. Siberian Math. J., **22**:223–231.
[16] M. I. GANZBURG (1983): *Multidimensional limit theorems of the theory of best polynomial approximations*. Siberian Math. J., **23**:316–331.
[17] M. I. GANZBURG (1987): *Exact constants for best polynomial approximation on an $(m-1)$-dimensional sphere*. Soviet Math. (Iz. VUZ), **31**(12):44–52.
[18] M. I. GANZBURG (1990): *A Markov-type theorem for approximation in integral metric with power weight*. Soviet Math. (Iz. VUZ), **34**(1):9–18.
[19] M. I. GANZBURG (1992): *Markov's type theorems for approximation of radial functions in the integral metric*. Russian Math. (Iz. VUZ), **36**(2):7–16.
[20] M. I. GANZBURG (2000): *Limit theorems and best constants of approximation theory*. In: Handbook on Analytic-Computational Methods in Applied Mathematics. Boca Raton, FL: CRC Press, pp. 507–569.
[21] M. I. GANZBURG (2001): *Polynomial inequalities on measurable sets and their applications*. Constr. Approx., **17**:275–306.
[22] M. I. GANZBURG (2002): *The Bernstein constant and polynomial interpolation at the Chebyshev nodes*. J. Approx. Theory, **119**:193–213.
[23] M. I. GANZBURG, S. A. PICHUGOV (1982): *Invariance of the elements of best approximation and a theorem of Glaeser*. Ukrainian Math. J., **33**(5):508–510.
[24] M. I. GANZBURG, D. S. LUBINSKY (to appear): *Best approximating entire functions to $|x|^\alpha$ in L_2*.
[25] I. M. GELFAND, G. E. SHILOV (1964): Generalized Functions, Vol. I: Properties and Operators. New York: Academic Press.
[26] S. G. KREIN, YU. I. PETUNIN, E. M. SEMENOV (1982): Interpolation of Linear Operators. Providence, RI: American Mathematical Society.
[27] D. S. LUBINSKY (2007): *On the Bernstein constants of polynomial approximation*. Constr. Approx., **25**:303–366.
[28] G. MEINARDUS (1963): *Invarianz bei linearen Approximationen*. Arch. Rational Mech. Anal., **14**:301–303.
[29] G. MEINARDUS (1967): Approximation of Functions: Theory and Numerical Methods. New York: Springer-Verlag.
[30] H. P. MULHOLLAND (1956): *On two extremum problems for polynomials on the unit circle*. J. London Math. Soc., **31**:191–199.
[31] S. M. NIKOLSKII (1947): *On the best mean approximation by polynomials of the function $|x-c|^s$*. Izv. Akad. Nauk SSSR, Ser. Mat., **11**:139–180 (Russian).
[32] Q. I. RAHMAN (1963): *Inequalities concerning polynomials and trigonometric polynomials*. J. Math. Anal. Appl., **6**:303–324.
[33] Q. I. RAHMAN, G. SCHMEISSER (2002): Analytic Theory of Polynomials. Oxford: Clarendon Press.
[34] R. A. RAITSIN (1968): *S. N. Bernstein's limit theorem for best approximations in the mean*, *and some of its applications*. Izv. Vyssh. Uchebn. Zaved. Mat., **12**(10):81–86 (Russian).
[35] R. A. RAITSIN (1969): *The best mean square approximation*, *by polynomials and by entire functions of finite degree of functions which have algebraic singular point*. Izv. Vyssh. Uchebn. Zaved. Mat., **13**(4):59–61 (Russian).

[36] W. W. ROGOSINSKI (1954): *Extremum problems for polynomials and trigonometric polynomials*. J. London Math. Soc., **29**:259–275.
[37] W. RUDIN (1973): Functional Analysis. New York: McGraw-Hill.
[38] A. SMOLUK (1979): *Niezmiennicze aproksymacje optymalne* (*Invariant optimal approximation*). Ekonometria, **152**(174):79–91.
[39] E. M. STEIN, G. WEISS (1971): Introduction to Fourier Analysis on Euclidean Spaces. Princeton, NJ: Princeton University Press.
[40] G. SZEGŐ (1975): Orthogonal Polynomials. Amer. Math. Soc. Colloq. Publ., Vol. 23. Providence, RI: American Mathematical Society.
[41] B. SZ.-NAGY (1939): *Über gewisse Extremalfragen bei transformierten trigonometrischen Entwicklungen. II. Nichtperiodischer Fall*. Berichte Acad. d. Wiss., Leipzig, **91**.
[42] L. V. TAIKOV (1965): *A group of extremal problems for trigonometric polynomials*. Uspekhi Mat. Nauk, **20**(3): 205–211 (Russian).
[43] A. F. TIMAN (1963): Theory of Approximation of Functions of a Real Variable. New York: Macmillan.
[44] V. TOTIK (2002): *Approximation on compact subsets of* **R**. In: Advanced Problems in Constructive Approximation. International Series of Numerical Mathematics (M. D. Buhmann, D. H. Mache, eds.), Vol. 142. Basel: Birkhaüser Verlag, pp. 263–274.
[45] R. S. VARGA, A. J. CARPENTER (1985): *On the Bernstein conjecture in approximation theory*. Constr. Approx., **1**:333–348.
[46] R. K. VASILIEV (1998): Chebyshev Polynomials and Approximation Theory on Compact Subsets of the Real Axis. Saratov: Saratov University.
[47] Y. XU (2004): *On polynomials of least deviation from zero in several variables*. Experiment. Math., **13**:103–112.

M. Ganzburg
Department of Mathematics
Hampton University
Hampton, VA 23668
USA
michael.ganzburg@hamptonu.edu