INVALIDITY OF THE ELEMENTS OF BEST APPROXIMATION AND A THEOREM OF GLAESER

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In the theory of approximation of functions it is well known that for the best approximation of polynomials of an even function it is sufficient to restrict oneself to even polynomials. We show that this fact is typical: Under sufficiently general conditions, the invariance of the approximable element relative to a continuous compact group of transformations implies the same invariance also for the best approximating elements. Then, making use of the obtained result, we present a new method for proving statements of the type of Glaeser's theorem [1].

1. Assume that a Banach space B with the norm $\|\cdot\|_B$ and a compact topological group G are connected in the following manner: to each element $s \in G$ there is associated a continuous linear operator $T_s : B \to B$, $\|T_s\| = 1$ and, moreover, $T_e = I$, $T_{st} - T_s T_t$ ($s \in G$, $t \in G$), where e is the identity element of G and I is the identity operator. We denote by B^G the set of elements $f \in B$ which are invariant relative to the group of operators T_s (i.e., they satisfy the condition $T_s f = f\ \forall s \in G$). We have the following theorem.

THEOREM 1. Let $\mathfrak{M}$ be a closed subspace of B satisfying the conditions: 1) $T_s\varphi \in \mathfrak{M}\ \ \forall \varphi \in \mathfrak{M},\ \forall s \in G$; 2) $\forall \varphi \in \mathfrak{M}$ the function $T_s\varphi$ is continuous with respect to s as a function from G into $\mathfrak{M}$.

Then, if $f \in B^G$, the best approximation of f by the subspace $\mathfrak{M}$ in the norm of B is realized on the elements from $\mathfrak{M}^G$, i.e.,

$$E(f, \mathfrak{M}) \overset{\text{def}}{=} \inf_{\varphi \in \mathfrak{M}} \|f - \varphi\|_B = \inf_{\varphi \in \mathfrak{M}^G} \|f - \varphi\|_B.$$

Proof. For an arbitrary $\varepsilon > 0$ we find an element $\varphi \in \mathfrak{M}$ such that $\|f - \varphi\|_B \leqslant E(f, \mathfrak{M}) + \varepsilon$.

Because of the compactness of the group G, there exists a normalized Haar measure $\mu(s)$ on it (see, e.g., [2]). Then conditions 1) and 2) ensure the existence of the integral $\varphi_1 = \int_G T_s \varphi d\mu(s)$ (see [2]), and from the closedness of $\mathfrak{M}$ it follows that $\varphi_1 \in \mathfrak{M}$. Now, by virtue of the invariance of the measure μ relative to translations, we obtain that $\forall t \in G$ we have

$$T_t\varphi_1 = \int_G T_t T_s \varphi d\mu(s) = \int_G T_s \varphi d\mu(s) = \varphi_1$$

(the possibility of introducing the operator T_s under the integral sign is proved, e.g., in [2]). Finally, making use of the invariance of f relative to the group of operators T_s, we obtain

$$\|f - \varphi_1\|_B = \left\| \int_G T_s(f - \varphi) d\mu(s) \right\|_B \leqslant \int_G \|T_s(f - \varphi)\|_B d\mu(s) \leqslant \|f - \varphi\|_B \leqslant E(f, \mathfrak{M}) + \varepsilon.$$

The theorem is proved.

2. Theorem 1 can be strengthened in some special cases.

Let G_m be a compact continuous group of transformations of the m-dimensional Euclidean space R^m; let K be a compactum in R^m such that $\forall x \in K\ \forall s \in G_m\ \ sx \in K$ where sx is the image of x under the transformation s; let C(K) be the Banach space of functions continuous on K with the norm $\|f\|_{C(K)} = \max_K |f|$; let $P_{n,m}^{(\beta)}$ be the space of the algebraic polynomials $P(x) = \sum_{(\alpha,\beta) \leqslant n} c_\alpha x^\alpha$ of m variables and of β-weighted degree n, where $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m)$ are multiindices, $(x, y) = \sum_{i=1}^m x_i y_i$, $x, y \in R^m$, $x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_m^{\alpha_m}$; for $\beta = (1, \ldots, 1)$, the corresponding space of polynomials will be denoted by $P_{n,m}$. For $f \in C(K)$ we set

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$$E^{(\beta)}_{n,m}(f;K)=\inf_{P\in P^{(\beta)}_{n,m}}\|f-P\|_{C(K)};\quad E_{n,m}(f;K)=\inf_{P\in P_{n,m}}\|f-P\|_{C(K)}.$$

We denote by $\sigma_1(x),\ldots,\sigma_k(x)$ the minimal system of homogeneous generators of the algebra $P^{G_m}_{n,m}$ and we set $\sigma=(\sigma_1,\ldots,\sigma_k)$; $R^m\to R^k$; in addition, let $\sigma(E)$ denote the image of the set $E\subset R^m$ under the mapping σ. We have the following theorem.

THEOREM 2. If $\sigma_i\in P_{\beta_i,m}$, $\beta_i\geq 1$, $1\leq i\leq k$, then for an arbitrary function $f\in C(K)^{G_m}$ there exists a function $g=g_f\in C(\sigma(K))$ such that $f(x)=g(\sigma(x))$ $\forall x\in K$ and we have the equalities

$$E_{n,m}(f;K)=\inf_{P\in P^{G_m}_{n,m}}\|f-P\|_{C(K)}=E^{(\beta)}_{n,k}(g,\sigma(K)). \tag{1}$$

Proof. Let $P_r\in P_{r,m}$ be polynomial of best approximation of degree r for f in the metric of C(K), r = 0, 1, ... By virtue of Theorem 1 [for $T_sf(x)=f(sx)$] we have $P_r\in P^{G_m}_{r,m}$ and $P_r=Q_r(\sigma_1,\ldots,\sigma_k)$, where $Q_r\in P^{(\beta)}_{r,k}$, r = 0, 1, ... Consequently, for $r\to\infty$, the polynomials Q_r converge uniformly on $\sigma(K)$ to some function $g\in C(\sigma(K))$ and, moreover, Q_n is the polynomial of best approximation of β-weighted degree n for g in the metric of $C(\sigma(K))$. From here there follows the validity of Theorem 2.

3. We give some examples of the application of Theorem 2. Let $B^m_r=\left\{x\in R^m:|x|^2_m=\sum_{i=1}^m x_i^2\leqslant r^2\right\}$ be an m-dimensional ball.

1) G_m is the group of rotations around zero, $\sigma_1(x)=|x|^2$. The polynomial of best approximation of degree n for $f\in C(B^m_r)^{G_m}$ will be a polynomial of the form $T(|x|^2)$, where $T\in P_{[n/2],1}$ is the best polynomial for g_f on $[-r, r]$.

2) G_m is the group of rotations around the vector $\gamma\in B^m_1$, $\sigma_1(x)=(\gamma,x)$. The polynomial of best approximation of degree n for $f\in C(B^m_r)^{G_m}$ has the form $T((\gamma,x))$, where $T\in P_{n,1}$ is the polynomial of best approximation for g_f on $[-r|\gamma|, r|\gamma|]$.

3) $G_m=\Gamma_m$ is the group of permutations of the set $(1,\ldots,m)$, σ_i, $1\leq i\leq m$, are the elementary symmetric polynomials. The polynomial of best approximation of degree n for $f\in C(B^m_r)^{G_m}$ will be a polynomial of the form $T(\sigma_1,\ldots,\sigma_m)$, where $T\in P^{(\beta)}_{n,m}$ is the best polynomial for g_f on $\sigma(B^m_r)$, $\beta=(1,\ldots,m)$.

4) $G_m=D^m$ is the Cartesian product of m groups $D=\{-1,1\}$, $\sigma_i(x)=x_i^2$, $1\leq i\leq m$. The polynomial of best approximation of degree n for $f\in C(B^m_r)^{G_m}$ has the form $T(x_1^2,\ldots,x_m^2)$, where $T\in P_{[n/2],m}$ is the best polynomial for g_f on the simplex $\left\{x\in R^m:\sum_{i=1}^m x_i\leqslant r^2,\ x_i\geqslant 0,\ 1\leqslant i\leqslant m\right\}$.

4. We make use of Theorem 2 for the investigation of the properties of functions which are invariant relative to the group G_m and which belong to the Gevrey class $G_\lambda(R^m)$, $\lambda\geq 1$ (regarding definitions, see, e.g., [3]). We have the following theorem.

THEOREM 3. For an arbitrary function $f\in G_\lambda(R^m)^{G_m}$, there exists a function $g\in G_\lambda(\Omega^0)$,* where $\Omega=\sigma(R^m)$ is such that $f(x)=g(\sigma(x))$ $\forall x\in R^m$.

For the class C^∞, statements of the global type, similar to Theorem 3, have been obtained by other methods by a series of authors. In [1] (see also [4]) the case $G_m=\Gamma_m$ is considered; in [5] (see also [6]) the case $G_m=D^m$ has been studied; in [7] and [8] one has obtained similar results in more general situations.

In order to prove Theorem 3 we make use of a result from [3].

LEMMA. Let A be an open set in R^m and let f be continuous on A. In order that $f\in G_\lambda(A)$, it is necessary and sufficient that for any compactum $K\in A$ there should exist constants $L<\infty$ and $a\in(0,1)$ such that for all natural numbers n the inequality $E_{n,m}(f;K)\leqslant La^{n^{\frac{1}{\lambda}}}$ should hold.

Proof of Theorem 3. By virtue of Theorem 2, for any natural number l there exists a function $g_l\in C(\Omega_l)$, where $\Omega_l=\sigma(B^m_l)$, such that $f(x)=g_l(\sigma(x))$. From here there follows the existence of a function g continuous on Ω and such that $f(x)=g(\sigma(x))$ $\forall x\in R^m$. It remains to prove that $g\in G_\lambda(\Omega^0)$. Since $f\in G_\lambda(R^m)$, making use of the lemma and of relation (1), we obtain $(n=1,2,\ldots;\ l=1,2,\ldots)$

*Ω^0 is the interior of the set Ω.

$$E_{n,k}(g,\Omega_l)\leqslant E_{n,k}^{(\beta)}(g;\Omega_l)=E_{n,m}(f;B_l^m)\leqslant La^{n^{\frac{1}{\lambda}}}. \qquad (2)$$

By virtue of the relation $\lim_{x\to\infty}|\sigma(x)|_k=\infty$ we have that for any compactum $K\subset\Omega^0$ there exists l_0 for which $K\subset\Omega_{l_0}$. Consequently, from (2) and from the lemma there follows that $g\in G_\lambda(\Omega^0)$. The theorem is proved.

Remark. If instead of the lemma one makes use of the constructive character of the class C^∞ (see [3]), then the indicated proof can be applied also to the class C^∞.

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A PROPERTY OF A CERTAIN CLASS OF LAPLACE-TYPE TRANSFORMS

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In [1] one has proved a property of the Cesaro summation methods, called the (c)-property. In the present paper we prove a similar property for a certain class of Laplace type transforms. It can be applied to obtain Tauberian theorems and also for the investigation of the properties of the Voronoi methods.

1. Assume that there is given a function $s(p)$ $(0\le p<\infty)$ and that for all $y>0$ there exists its transform

$$\Phi(y)=\frac{1}{F(y)}\int_0^\infty e^{h(p)-\frac{p}{y}}s(p)\,dp,$$

where $h(p)$ satisfies the following conditions: 1) $0<h'(p)\to 0$ $(p\to\infty)$; 2) $h''(\bar{p})<0$; $-h''(p)$ does not increase; $-p^2h''(p)$ does not decrease; 3) $\forall_{0<\delta<1}\int_0^\infty e^{\delta p^2h''(p)}dp<\infty$; and $F(y)=\int_0^\infty e^{h(p)-\frac{p}{y}}dp$.

THEOREM 1. Let

$$\forall_{0<\delta<1}\; s(p)=O(e^{-\delta p^2h''(p)}). \qquad (1)$$

If $\Phi(y)\to s$ $(y\to\infty)$ and if the closed convex set G is a (c)-set for the function $s(p)$, then $s\in G$. If the point at infinity is a (c)-point of the function $s(p)$, then $\overline{\lim_{y\to\infty}}|\Phi(y)|=\infty$.*

Proof. From the properties of function $h(p)$ it follows that: a) $\lim h''(y)/h''(x)=1$ $(1<y/x\to 1)$, $x\to\infty$; b) there exists a function $\psi(p)\to\infty$ $(p\to\infty)$ such that $-p^2h''(p)/\psi^2(p)\to\infty$ $(p\to\infty)$; c) to each sufficiently large

* For the definitions of a (c)-set and of the point at infinity being a (c)-point, see [1, 2].

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