The contradiction thus obtained shows that our assumption that the series $\sum_{k=1}^{\infty} X_k$ is B-summable almost surely is not valid. Thus the series $\sum_{k=1}^{\infty} X_k$ is almost surely C-summable but not B-summable.

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BEST APPROXIMATION OF SUMS OF ELEMENTS AND A THEOREM OF NEWMAN AND SHAPIRO

M. I. Ganzburg

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1. Let F be a normed space with norm $\|\cdot\|_F$, and let B be a subspace of F. For $f \in F$ we set

$$E(f, B, F) = \inf_{g \in B} \|f - g\|_F. \tag{1}$$

In this paper we consider the following problem: find the conditions on the elements $f_k \in F$, $1 \leq k \leq N$, $N \geq 2$, under which we have the equality

$$E\left(\sum_{k=1}^{N} f_k, B, F\right) = \sum_{k=1}^{N} E(f_k, B, F). \tag{2}$$

In this paper we obtain criteria for the equality (2) or for its integral analog to hold (Theorems 1, 2; Corollary 1). As a consequence we present the known result of Newman and Shapiro [1] on the validity of (2) in the case of the approximation of functions of m variables of the form $\sum_{k=1}^{m} \psi_k(x_k)$ by generalized polynomials in the uniform metric. It is shown that the variant of the Newman–Shapiro theorem, in the case of the approximation in the integral metric, does not hold (Corollary 3). We also show an analog of the Newman–Shapiro theorem for the approximation by entire functions of exponential type (Theorem 3). The proof of this result is based on a new limit relation for best polynomial approximations of continuous functions (Theorem 4).

2. As usual, F^* denotes the conjugate space and $B^{\perp} = \{\varphi \in F^* : \varphi(g) = 0 \ \forall g \in B\}$. Let $G_\varepsilon(f) = \{\varphi \in B^{\perp} : \|\varphi\|_{F^*} = 1, \varphi(f) \geq E(f, B, F) - \varepsilon\}$, $\varepsilon \geq 0$.

We have the following theorem.

<u>THEOREM 1.</u> In order that equality (2) be true, it is necessary and sufficient that

$$\forall \varepsilon > 0 \ \bigcap_{k=1}^{N} G_\varepsilon(f_k) \neq \varnothing$$

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<u>Proof.</u> We denote $If = \sum_{k=1}^{N} f_k$.

If (2) is satisfied, then, by virtue of the duality theorem for best approximations (see, for example, [2, p. 141]), we have $g_\varepsilon(If) \neq \phi$, $\varepsilon > 0$, and $\forall\varphi \in G_\varepsilon(If)$ we obtain

$$\sum_{k=1}^{N} \varphi(f_k) \geqslant E(If, B, F) - \varepsilon = \sum_{k=1}^{N} E(f_k, B, F) - \varepsilon \tag{3}$$

Making use of the inequality $\varphi(f_k) \leq E(f_k, B, F)$, $1 \leq k \leq N$, and (3), we have

$$\varphi(f_k) \geqslant E(f_k, B, F) - \varepsilon, \quad 1 \leqslant k \leqslant N. \tag{4}$$

Consequently, $\varphi \in \bigcap_{k=1}^{N} G_\varepsilon(f_k)$, and the necessity part of the theorem is proved.

Assume now that $\varphi \in \bigcap_{k=1}^{N} G(f_k)$. Then the equalities (4) are satisfied and we have the inequalities

$$E(If, B, F) \leqslant \sum_{k=1}^{N} E(f_k, B, F) \leqslant \sum_{k=1}^{N} \varphi(f_k) + N \cdot \varepsilon \leqslant E(If, B, F) + N\varepsilon.$$

By virtue of the arbitrariness of $\varepsilon > 0$, from here we obtain the validity of the equality (2) and the theorem is proved.

Let Q be a space with a positive measure μ and let $f_\lambda : Q \to F$ be a continuous vector-valued function. Further, let $If = \int_Q f_\lambda d\mu(\lambda)$ be an integral with the properties: $If \in F$ and $\varphi(If) = \int_Q \varphi(f_\lambda) d\mu(\lambda) \ \forall \varphi \in F^*$. Such an integral exists, for example, in the case of a compact Q [3, p. 90 of the Russian edition].

We say that F and B satisfy condition E if $\forall f \in F$ there exists an element $g_0 = g_0 f \in B$ for which the infimum is attained in (1).

The element g_0 is said to be an element of best approximation from B for f in the metric of F and it is denoted by T(f). We say that F and B satisfy condition C if for any continuous $f_\lambda : Q \to F$ there exists a function $T(f_\lambda) : Q \to B$, continuous with respect to λ.

If F and B satisfy conditions E and C, then the assertion of Theorem 1 can be generalized and refined. We have the following theorem.

<u>THEOREM 2.</u> If F and B satisfy the conditions E and C, then for the equality

$$E(If, B, F) = \int_Q E(f_\lambda, B, F) d\mu(\lambda) \tag{5}$$

to hold it is necessary and sufficient that $\bigcap_{\lambda \in Q \setminus E_0} G_0(f_\lambda) \neq \varnothing$ for some $E_0 \subset Q$, $\mu E_0 = 0$.

<u>Proof.</u> If (5) is satisfied, then, by virtue of the condition E and the criterion for an element of best approximation (see, for example, [2, p. 150]), we have $G_0(If) \neq \phi$ and $\forall\varphi \in G_0(If)$ we obtain

$$\int_Q \varphi(f_\lambda) d\mu(\lambda) = E(If, B, F) = \int_Q E(f_\lambda, B, F) d\mu(\lambda). \tag{6}$$

From (6) there follows the existence of a set $E_0 \subset Q$, ${}^3E_0 = 0$, such that

$$\varphi(f_\lambda) = E(f_\lambda, B, F) \quad \forall \lambda \in Q \setminus E_0. \tag{7}$$

Consequently, $\varphi \in \bigcap_{\lambda \in Q \setminus E_0} G_0(f_\lambda)$, and the necessity of the theorem is proved.

If $\varphi \in \bigcap_{\lambda \in Q \setminus E_0} G_0(f_\lambda)$, then the equalities (7) hold.

In addition, according to the conditions E and C, there exists a continuous $g_\lambda = Tf_\lambda$ and, consequently, the integral Ig is finite [3, p. 90 of the Russian edition] and we have

$$E(If, B, F) \leqslant \| If - Ig \|_F \leqslant \int_Q E(f_\lambda, B, F)\, d\mu(\lambda) = \varphi(If) \leqslant E(If, B, F).$$

From here we obtain the validity of equality (5) and the theorem is proved.

Taking as Q a segment of the natural series, $\{k\}_{k=1}^N$, $\mu(k) = 1$, $1 \le k \le N$ I (in this case the condition C is trivially satisfied), from Theorem 2 we obtain the following corollary.

COROLLARY 1. If F and B satisfy condition E, then for equality (2) to hold it is necessary and sufficient that $\bigcap_{k=1}^N G_0(f_k) \neq \varnothing$.

In a series of cases the following statement is useful.

LEMMA 1. If F and B satisfy condition E and equality (2) holds, then for $T\left(\sum_{k=1}^N f_k\right)$ one can take the element $\sum_{k=1}^N T(f_k)$.

Proof. We have the inequalities

$$E\left(\sum_{k=1}^N f_k, B, F\right) \leqslant \left\| \sum_{k=1}^N (f_k - T(f_k)) \right\|_F \leqslant \sum_{k=1}^N E(f_k, B, F) = E\left(\sum_{k=1}^N f_k, B, F\right)$$

and the lemma is proved.

Remark 1. The example of a Hilbert space shows that, in general, the converse statement is not true.

3. Let X_k, $1 \le k \le m$ be Hausdorff spaces; let $X_0 = X_1 \times \ldots \times X_m$ be the Cartesian product of X_k, $1 \le k \le m$; let $C(X_k)$ be the space of functions f, continuous on X_k, with the norm $\|f\|_{C(X_k)} = \sup_{y_k \in X_k} |f(y_k)|$, $0 \leqslant k \leqslant m$; let μ_k be a positive measure on X_k, $1 \le k \le m$; let $\mu_0 = \mu_1 \times \ldots \times \mu_m$ be the measure on X_0; let $L_q(\Sigma_k, X_k, \mu_k)$ be the Banach space of functions, measurable with respect to the measure μ_k and with finite norm $\|f\|_{L_q(\Sigma_k, X_k, \mu_k)} = \left(\int_{X_k} |f|^q d\mu_k\right)^{1/q}$, $1 \leqslant q \leqslant \infty$, where Σ_k is the family of subsets of X_k, measurable with respect to μ_k, $0 \le k \le m$; let B_k be an n_k-dimensional subspace of $C(X_0)$ [or of $L_1(\Sigma_k, X_k, \mu_k)$], $1 \le k \le m$.

Further, let $\{g_{i,k}\}_{i=1}^{n_k}$ be a basis of B_k, $1 \le k \le m$; let B_0 be the n-dimensional subspace of $C(X_0)$ [or of $L_1(\Sigma_0, X_0, \mu_0)$] of elements of the form

$$P(y) = P(y_1, \ldots, y_m) = \sum_{\substack{1 \leqslant i_k \leqslant n_k \\ 1 \leqslant k \leqslant m}} c_{i_1,\ldots,i_m} \prod_{k=1}^m g_{i_k,k}(y_k),$$

where $c_{i_1,\ldots,i_m}$ are real numbers and $n = \prod_{k=1}^m n_k$.

COROLLARY 2 (Newman, Shapiro [1]). Let $g_{1,k} \equiv 1$, let X_k be a compactum, $1 \le k \le m$, and let $B_0 \subset C(X_0)$. Then the following statements hold:

a) if $f_k \in C(X_k)$, $1 \le k \le m$, then for the function $f_0(y) = \sum_{k=1}^m f_k(y_k)$ we have the equality

$$E(f_0, B_0, C(X_0)) = \sum_{k=1}^m E(f_k, B_k, C(X_k));$$

b) there exists an element of best approximation from B_0 for f_0 in the metric of $C(X_0)$ of the form $T(f_0)=\sum_{k=1}^{m} T(f_k)$, where $T(f_k) \in B_k$ is the element, least deviating from f_k in $C(X_k)$, $1 \le k \le m$.

Proof. Let $\mu_k = \mu_k^+ - \mu_k^-$ be the measure, existing according to the criterion of the element of best approximation in $C(X_k)$ (see, for example, [2, p. 157]), concentrated on the set $M_k = M_k^+ \cup M_k^-$, $\operatorname{card} M_k \le n + 1$, with the properties $\mu_k \in B_k^{\perp}$, $\operatorname{var}\mu_k = 1$,

$$\int_{X_k} (f_k - T(f_k))\, d\mu_k^{\pm} = \pm \operatorname{var} \mu_k^{\pm} E(f_k, B_k, C(X_k)). \tag{8}$$

Here μ_k^+, $-\mu_k^-$ are, respectively, the positive and the negative components of the measure μ_k, $\operatorname{supp}\mu_k^{\pm} = M_k^{\pm}$, $1 \le k \le m$.

The measure $\bar{\mu}_0 = \beta(\mu_1^+ \times \ldots \times \mu_m^+ - \mu_1^- \times \ldots \times \mu_m^-)$, where $\beta > 0$ is selected from the condition $\operatorname{var}\mu_0 = 1$, is defined on X_0 and it is concentrated on the set $(M_1^+ \times \ldots \times M_m^+) \cup (M_1^- \times \ldots \times M_m^-)$. It is easy to verify condition $\bar{\mu}_0 \in B_0^{\perp}$, while from the equalities (8) there follows the relation

$$\int_{X_0} f_k(y_k)\, d\bar{\mu}_0(y) = E(f_k, B_k, C(X_k)), \quad 1 \leqslant k \leqslant m.$$

Consequently, $\bar{\mu}_0 \in \bigcap_{k=1}^{m} G_0(h_k)$, where $h_k(y) = f_k(y_k)$, $1 \le k \le m$ and, applying Theorem 1, we obtain the validity of statement a).

In order to prove statement b) it is sufficient to note that from the criterion of the element of best approximation in $C(X_0)$ there follows the equality $T(h_k) = T(f_k)$, $1 \le k \le m$, and then to apply Lemma 1. The corollary is proved.

We say that a finite-dimensional subspace $B \subset L_1(\Sigma, X, \mu)$ satisfies condition S if from $f \in B^{\perp}$, $\mu(\operatorname{supp} f) > 0$ there follows the existence of sets $E_i \in \Sigma$, $\mu(E_i) > 0$, $i = 1, 2$, such that $f > 0$ on E_1 and $f < 0$ on E_2.

COROLLARY 3. Assume that the spaces $B_k \subset L_1(\Sigma_k, X_k, \mu_k)$, $1 \le k \le m$, satisfy condition S. Then for any $f_k \in L_1(\Sigma_k, X_k, \mu_k)$, $f_0(y) = \sum_{k=1}^{m} f_k(y_k)$ the equality

$$E(f_0, B_0, L_1(\Sigma_0, X_0, \mu_0)) = \sum_{k=1}^{m} E(f_k, B_k, L_1(\Sigma_k, X_k, \mu_k)) \tag{9}$$

is satisfied if and only if there exists a natural number k_0, $1 \le k_0 \le m$, such that $f_k \in B_k$, $1 \le k \le m$, $k \ne k_0$.

Proof. We denote $h_k(y) = f_k(y_k)$, $1 \le k \le m$. The sets $G_0(h_k)$ of functionals have the form

$$G_0(h_k) = \{\operatorname{sign}_{\psi_k}(h_k - T(h_k)) : \|\psi_k\|_{L_\infty(\Sigma_0, X_0, \mu_0)} \leqslant 1\}, \quad 1 \leqslant k \leqslant m,$$

where $\operatorname{sign}_\psi h \overset{\text{def}}{=} \begin{cases} \operatorname{sign} h, & h \neq 0, \\ \psi, & h = 0. \end{cases}$

By virtue of the criterion of the element of best approximation in $L_1(\Sigma_0, X_0, \mu_0)$ [2, p. 156], we have $T(h_k) = T(f_k)$, $1 \le k \le m$.

Now we assume that (9) is satisfied. Then, by virtue of Corollary 1, there exist ψ_k^0, $\|\psi_k^0\|_{L_\infty(\Sigma_k, X_k, \mu_k)} \leqslant 1$, $1 \leqslant k \leqslant m$, such that $\forall\, y \in X_0$, with the exception of a set of μ_0-measure zero, we have

$$H_k(y) = H_j(y), \quad 1 \leqslant j, \quad k \leqslant m, \tag{10}$$

where $H_k(y) \overset{\text{def}}{=} \operatorname{sign}_{\psi_k^0}[f_k(y_k) - T(f_k)(y_k)]$. If there exist j, k, $1 \le j < k \le m$ such that $f_j \notin B_j$, $f_k \notin B_k$, then, by virtue of the criterion of the element of best approximation in $L_1(\Sigma_k, X_k, \mu_k)$, we have $H_j \in B_j^\perp$, $H_k \in B_k^\perp$.

Making use of condition S, we obtain that there exists a set E_0, $\mu_0 E_0 > 0$, on which $H_j > 0$, $H_k < 0$, contradicting equality (10). The corollary is proved.

Remark 2. Condition S is satisfied, for example, in the cases when B contains a constant or the basis B consists of nonnegative functions.

Remark 3. The following example shows that, in the case of approximation in $L_1(\Sigma_0, X_0, \mu_0)$, the equality $T(f_0) = \sum_{k=1}^{m} T(f_k)$, $m > 1$, is not always true. Let $X_k = [-1, 1]$, let μ_k be the Lebesgue measure, let $f_k(y_k) = y_k^2$, and assume that B_k consists of constants, $1 \le k \le m$. Then $T(f_k) = 4^{-1}$, $1 \le k \le m$, $T(f_0) = (2^{m-1}/\omega_m)^{2/m} \ne m/4$ for $m > 1$ (this follows from the transcendence of the number $2^{m-1}/\omega_m$, $m > 1$), where ω_m is the volume of the m-dimensional ball of unit radius.

4. Let R^m be the m-dimensional Euclidean space; $B_{\sigma,m}$, $\sigma = (\sigma_1, \ldots, \sigma_m)$ is the class of entire functions of m variables and of exponential type σ [4, p. 99]; $\mathcal{P}_{\mu n,m}$, $\mu = (\mu_1, \ldots, \mu_m)$, $\mu_i > 0$, $1 \le i \le m$, is the class of algebraic polynomials of m variables and of degree $\le \mu_i n$ with respect to the i-th variable, $1 \le i \le m$; $\Pi_\gamma = \{y \in R^m : |y_i| \le \gamma_i, 1 \le i \le m\}$, $\gamma = (\gamma_1, \ldots, \gamma_m)$, $\gamma_i > 0$, $1 \le i \le m$. We have the following theorem.

THEOREM 3. For the functions $f_0(y) = \sum_{k=1}^{m} f_k(y_k)$, where $f_k \in C(R^1)$, we have the equality

$$E(f_0, B_{\sigma,m}, C(R^m)) = \sum_{k=1}^{m} E(f_k, B_{\sigma_k,1}, C(R^1)) = \left\| f_0 - \sum_{k=1}^{m} g_k \right\|_{C(R^m)}, \tag{11}$$

where $E(f_k, B_{\sigma_k}, C(R^1)) = \|f_k - g_k\|_{C(R^1)}$, $1 \le k \le m$ (the existence of elements of best approximation is proved, for example, in [4, p. 181]).

In view of the absence of effective criteria for elements of best approximation from $B_{\sigma_k,1}$ into $C(R^1)$, one does not succeed to use Corollary 1 for the proof of the equalities (11). The proof of Theorem 3 is based on the following result, which is of interest also in its own right.

THEOREM 4. Let $f \in C(R^m)$, and let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of numbers satisfying the following conditions: a) $0 \le \lambda_n \le n$, $n = 1, 2, \ldots$; b) $\underline{\lim}_{n\to\infty} \lambda_n n^{-\delta} > 0$, where $\delta \in (1/3, 1)$; c) $\lambda_n = o(n)$, $n \to \infty$. Then for $\gamma = (\mu_1/\sigma_1, \ldots, \mu_m/\sigma_m)$ we have the equality

$$\lim_{n\to\infty} E(f, \mathcal{P}_{\mu n,m}, C(\Pi_{(n-\lambda_n)\gamma})) = E(f, B_{\sigma,m}, C(R^m)). \tag{12}$$

In the case $m = 1$, $\lambda_n = 0$, $n = 1, 2, \ldots$, equality (12) for almost all $\sigma_1 > 0$ has been proved by Bernshtein [5]. Another variant of the equality (12) has been obtained by the author [6].

5. For the proof of Theorem 4 we need the following result.

LEMMA 2. If λ_n, $n = 1, 2, \ldots$ satisfies the conditions of Theorem 4, then for $g \in B_{\sigma,m} \cap C(R^m)$ we have the relation

$$\lim_{n\to\infty} E(g, \mathcal{P}_{\mu n,m}, C(\Pi_{(n-\lambda_n)\gamma})) = 0. \tag{13}$$

Proof. For $g \in B_{\sigma,1} \cap C(R^1)$ one has [5]

$$E\left(g, \mathcal{P}_{\mu n,1}, C\left(\frac{-n+\lambda_n}{\sigma/\mu}, \frac{n-\lambda_n}{\sigma/\mu}\right)\right) \leqslant (n/\lambda_n)^{1/2} \exp(-(2/3)\mu\lambda_n^{3/2} n^{-1/2}) \|g\|_{C(R^1)}. \tag{14}$$

By virtue of condition b) we have $\lambda_n \ge C_1 n^{-\delta}$, where $\delta \in (1/3, 1)$ (all the constants C_i are independent of n and ε). Then from (14) there follows the inequality

$$E\left(g,\ \mathcal{P}_{\mu n,1},\ C\left(\frac{-n+\lambda_n}{\sigma/\mu},\ \frac{n-\lambda_n}{\sigma/\mu}\right)\right)\leqslant C_2\exp(-C_3 n^{\beta})\,\|g\|_{C(R^1)}, \tag{15}$$

where $\beta = 3\delta/2 - 1/2 > 0$. Denoting $e_\tau(t)=\exp(i\tau t)$, $\Phi_x(y)=\prod_{i=1}^{m} e_{x_i}(y_i)$, $x \in \Pi_\sigma$, $y \in \Pi_{(n-\lambda_n)\gamma}$, we have

$$E(\Phi_x,\ \mathcal{P}_{\mu n,m},\ C(\Pi_{(n-\lambda_n)\gamma}))\leqslant\left\|\Phi_x(y)-\prod_{i=1}^{m}P_i(y_i)\right\|_{C(\Pi_{(n-\lambda_n)\gamma})}\leqslant C_4\exp(-C_5 n^{\beta}). \tag{16}$$

Here $P_i \in \mathcal{P}_{\mu_i n,1}$ is the polynomial of best approximation for e_{x_i} in the metric of $C((-n+\lambda_n)\mu_i/\sigma_i,\ (n-\lambda_n)\mu_i/\sigma_i)$, $1 \le i \le m$.

Making use of the Paley–Wiener representation for $g \in B_{\sigma,m} \cap L_2(R^m)$ (see, for example, [4, p. 109]), from (16) we have

$$E(g,\ \mathcal{P}_{\mu n,m},\ C(\Pi_{(n-\lambda_n)\gamma}))\leqslant\left(\prod_{i=1}^{m}\sigma_i\right)^{1/2}\max_{x\in\Pi_\sigma}E(\Phi_x,\ \mathcal{P}_{\mu n,m},\ C(\Pi_{(n-\lambda_n)\gamma}))\leqslant C_6\exp(-C_5 n^{\beta}). \tag{17}$$

Further, $\forall\, g \in B_{\sigma,m} \cap C(R^m)$, $\forall\, \varepsilon > 0$ the function $g_1(y) = g((1-\varepsilon)y)\prod_{i=1}^{m}((\sin\varepsilon\sigma_i y_i)/(\varepsilon\sigma_i y_i))$ belongs to $B_{\sigma,m}$ and $\|g_1\|_{L_2(R^m)} \le C_7\varepsilon^{-m/2}\|g\|_{C(R^m)}$. We have further

$$\begin{aligned}\|g-g_1\|_{C(\Pi_{(n-\lambda_n)\gamma})}&\leqslant\|g((1-\varepsilon)\cdot)-g(\cdot)\|_{C(\Pi_{(n-\lambda_n)\gamma})}+\\&+m\max_{1\leqslant i\leqslant m}\|1-(\sin(\varepsilon(n-\lambda_n)\tau))/(\varepsilon(n-\lambda_n)\tau)\|_{C(-\mu_i,\mu_i)}\|g\|_{C(R^m)}=I_1+I_2\end{aligned} \tag{18}$$

From the inequality $\tau - \sin\tau^3/6$, $\tau > 0$, we obtain

$$I_2\leqslant C_8\varepsilon^2 n^2\|g\|_{C(R^m)}. \tag{19}$$

For the estimation of I_1 we make use of Bernshtein's inequality:

$$I_1\leqslant\max_{R^m}\left(\sum_{i=1}^{m}\left(\frac{\partial g}{\partial y_i}\right)^2\right)^{1/2}(n-\lambda_n)\,\varepsilon\left(\sum_{i=1}^{m}(\mu_i/\sigma_i)^2\right)^{1/2}\leqslant C_9\varepsilon\cdot n\,\|g\|_{C(R^m)}\,. \tag{20}$$

From the inequalities (17)-(20) we obtain ($0 < \varepsilon < 1$)

$$E(g,\ \mathcal{P}_{\mu n,m},\ C(\Pi_{(n-\lambda_n)\gamma}))\leqslant C_{10}(\varepsilon n+\varepsilon^2 n^2+\varepsilon^{-m/2}\exp(-C_5 n^{\beta}))\,\|g\|_{C(R^m)}. \tag{21}$$

Minimizing the right-hand side of (21) with respect to $\varepsilon \in (0, 1)$, we obtain the validity of the equality (13) $\forall\, g \in B_{\sigma,m} \cap C(R^m$.

<u>6. Proof of Theorem 4.</u> Let $f \in C(R^m)$ and $g \in B_{\sigma,m}$ be such that $E(f, B_{\sigma,m}, C(R^m)) = \|f - g\|_{C(R^m)}$. Then, by virtue of relation (13), we have

$$\begin{aligned}\varlimsup_{n\to\infty}E(f,\ \mathcal{P}_{\mu n,m},\ C(\Pi_{(n-\lambda_n)\gamma}))&\leqslant\varlimsup_{n\to\infty}E(f-g,\ \mathcal{P}_{\mu n,m},\ C(\Pi_{(n-\lambda_n)\gamma}))+\\&+\lim_{n\to\infty}E(g,\ \mathcal{P}_{\mu n,m},\ C(\Pi_{(n-\lambda_n)\gamma}))\leqslant E(f,\ B_{\sigma,m},\ C(R^m)).\end{aligned} \tag{22}$$

Assume now that $P_n(y)=\sum_{\max_i(\alpha_i/\mu_i)\leqslant n} C_\alpha y_1^{\alpha_1}\dots y_m^{\alpha_m}$, where $\alpha = (\alpha_1, \dots, \alpha_m)$ is a multiindex, is a polynomial, satisfying the equality

$$E(f,\ \mathcal{P}_{\mu n,m},\ C(\Pi_{(n-\lambda_n)\gamma}))=\|f-P_n\|_{C(\Pi_{(n-\lambda_n)\gamma})},\quad n=1,2,\dots$$

By virtue of a known inequality [7] we have

$$|C_\alpha|\leqslant 2\prod_{i=1}^{m}\sigma_i^{\alpha_i}n^{\alpha_i}(n-\lambda_n)^{-\alpha_i}(\alpha_i!)^{-1}\max_{\Pi_{(n-\lambda_n)\gamma}}|P_n|\le$$

$$\leqslant 4 \prod_{i=1}^{m} \sigma_i^{\alpha_i} n^{\alpha_i} (n-\lambda_n)^{-\alpha_i} (\alpha_i!)^{-1} \| f \|_{C(R_m)}.$$

From here for each $z = (z_1, \ldots, z_m)$ from the m-dimensional complex space $\mathbb{C}^m$ we obtain ($k = 1, 2, \ldots, n = 1, 2, \ldots$)

$$\Big| \sum_{\sum_{i=1}^{m} \alpha_i = k} C_\alpha z_1^{\alpha_1} \ldots z_m^{\alpha_m} \Big| \leqslant 4 \frac{(1+\lambda_n/(n-\lambda_n))^k \left(\sum_{i=1}^{m} \sigma_i |z_i| \right)^k}{k!} \cdot \| f \|_{C(R^m)}. \tag{23}$$

Assume now that the sequence $\{n_k\}_{k=1}^{\infty}$ is such that there exists the limit $D = \lim_{k\to\infty} E(f, \mathcal{P}_{\mu n_k, m}, C(\Pi_{(n_k-\lambda n_k)\gamma})$. By virtue of Lemma 2.4 from [6] and the inequalities (23) there follows that there exist a subsequence (without loss of generality we assume that it coincides with $\{n_k\}_{k=1}^{\infty}$) and an entire function g such that $\lim_{k\to\infty} P_{n_k} = g$ uniformly on any compactum in $\mathbb{C}^m$. In addition, from (23) and condition c) there follows that $\forall\, \varepsilon > 0$ the limit function satisfies the inequality

$$|g(z)| \leqslant 4 \| f \|_{C(R^m)} \exp\left((1+\varepsilon) \sum_{i=1}^{m} \sigma_i |z_i| \right),$$

and, consequently, $g \in B_{\sigma,m}$. Then we have

$$D = \lim_{k\to\infty} \| f - P_{n_k} \|_{C(\Pi_{(n_k-\lambda_{n_k})\gamma})} = \| f - g \|_{C(R^m)} \geqslant E(f, B_{\sigma,m}, C(R^m))$$

From here there follows the inequality

$$\varliminf_{n\to\infty} E(f, \mathcal{P}_{\mu n,m}, C(\Pi_{(n_k-\lambda_{n_k})\gamma})) \geqslant E(f, B_{\sigma,m}, C(R^m)). \tag{24}$$

From the inequalities (22), (24) we obtain the validity of Theorem 4.

7. Proof of Theorem 3. Let $\{\lambda_n\}_{n=1}^{\infty}$ be the same sequence as in Theorem 4. By virtue of Corollary 2, for $h_k(y) = f_k(y_k)$, $1 \leq k \leq m$, we have

$$E\left(\sum_{k=1}^{m} h_k, \mathcal{P}_{\mu n,m}, C(\Pi_{(n-\lambda_n)\gamma}) \right) = \sum_{k=1}^{m} E(h_k, \mathcal{P}_{\mu n,m}, C(\Pi_{(n-\lambda_n)\gamma})) = \sum_{k=1}^{m} E\left(f_k, \mathcal{P}_{\mu_k n,1}, C\left(-\frac{n-\lambda_n}{\sigma_k/\mu_k}, \frac{n-\lambda_n}{\sigma_k/\mu_k} \right)\right). \tag{25}$$

Taking the limit in (25) as $n \to \infty$ and making use of equality (12), we obtain the validity of the left relation in (11). Now the right equality from (11) follows from Lemma 1. The theorem is proved.

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