ON THE EQUIVALENCE OF THE K-FUNCTIONAL AND MODULI OF CONTINUITY AND SOME APPLICATIONS

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1. Introduction.

In many papers on approximation theory the K-functional introduced by J. Peetre has turned out to be a simple and efficient tool for the description of smoothness properties of functions. On the other hand, in classical approximation theory smoothness properties of functions are usually stated in terms of moduli of continuity. The close relationship between these two quantities, and in fact their equivalence, is well known for functions defined on the n-dimensional Euclidean space R^n (see e.g. [5] for the one-dimensional case). Here we establish this equivalence for Lipschitz-graph domains (LG domains) Ω of R^n (see their definition below).

To give more details let D^β denote as usual the differential operator $(\partial/\partial x_1)^{\beta_1} \dots (\partial/\partial x_n)^{\beta_n}$ with multiindex $\beta = (\beta_1,\dots,\beta_n)$, $|\beta| = \sum_{i=1}^{n} \beta_i$, and introduce for $1 \le p \le \infty$, $r = 0,1,2,\dots$ the seminorms

$$(1.1) \qquad |f|_{p,r} = \sup_{|\beta|=r} \|D^\beta f\|_p$$

for functions $f \in C^r(\Omega)$, the space of r-times continuously differentiable functions on the domain Ω. Here $\|\cdot\|_p$ denotes the usual norm for the Lebesgue spaces $L_p(\Omega)$. We then define for $r=1,2,\dots$

$$(1.2) \qquad \begin{aligned} H_p^0(\Omega) &= L_p, \quad 1 \le p < \infty; \quad H_\infty^0(\Omega) = C(\Omega) \\ H_p^r(\Omega) &= W_p^r(\Omega), \quad 1 \le p < \infty; \quad H_\infty^r(\Omega) = C^r(\Omega) \end{aligned}$$

and the K-functional $K_r: (0,\infty)\times H_p^0(\Omega) \longrightarrow [0,\infty)$

$$(1.3) \qquad K_r(t,f) = \inf \{\|f-g\|_p + t|g|_{p,r} : g \in H_p^r(\Omega)\}$$

Here $W_p^r(\Omega)$ is the Sobolev space of functions $f \in L^p(\Omega)$ with derivatives $D^\beta f$ (in the distributional sense) belonging to $L^p(\Omega)$, $|\beta| \leq r$. We remark that the infimum could also be extended over the smaller subspace $\mathcal{D}$ of infinitely often differentiable functions on Ω with compact support. This is because $\mathcal{D}$ is dense in W_p^r, $1 \leq p < \infty$, with respect to the norm $\sup_{0 \leq j \leq r} |\cdot|_{p,r}$, at least, if Ω is a LG domain (see [1, p.54]). For our purposes however, it is convenient to work with definition (1.3).

In order to introduce the other quantity of interest here we set

$$(1.4) \qquad \Omega(h) = \{x \in \Omega : x + th \in \Omega \text{ for } 0 \leq t \leq 1\}, \quad h \in R^n$$

Then we can define for functions f on Ω

$$(1.5) \qquad \Delta^r(h) f(x) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x + kh)$$

on the subset $\Omega(rh)$. The r-th modulus of smoothness then is a function $\omega_r : (0,\infty) \times H_p^0(\Omega) \longrightarrow [0,\infty)$ given by

$$(1.6) \qquad \begin{aligned} \omega_r(t,f) &= \sup_{0<|h|\leq t} \| \chi_{\Omega(rh)} \Delta^r(h) f \|_p \qquad (r=1,2,\ldots) \\ \omega_0(t,f) &= \| f \|_p \qquad (r=0) \end{aligned}$$

Our main result then reads

<u>Theorem 1</u>: If Ω is a Lipschitz-graph domain then there are positive constants M_1, M_2 depending only on Ω, r and p such that on $(0,1) \times H_p^0(\Omega)$, $1 \leq p \leq \infty$

$$(1.7) \qquad M_1\, \omega_r(t,f) \leq K_r(t^r,f) \leq M_2\, \omega_r(t,f) \qquad (r=1,2,\ldots)$$

We remark that the easy part of this theorem is the left-side inequality in (1.7). If we allow a further term $t^r \| f \|_p$ (which for some purposes is irrelevant), the right-hand side inequality would be an easy consequence of a suitable extension theorem for functions in $C^r(\Omega)$ or $W_p^r(\Omega)$, $1 \leq p < \infty$, and the corresponding inequality for the whole R^n. However, in order to establish the sharper inequality (1.7) by this kind of argument we would need a refined extension theorem preserving moduli of continuity. As a matter of fact we shall obtain such an extension theorem as an application of (1.7).

Another application will be concerned with the classical Marchaud-inequalities for moduli of continuity(For simplicity we did not carry out our approach for the technically more involved case of directional moduli of continuity where similar results have been established by Besov [2]). Further applications give a refinement of estimates in [3] of linear functionals defined on $L_p(\Omega)$ and theorems for the L_p-approximation by algebraic polynomials.

2. Comparison of the K-functional and moduli of continuity.

We begin by listing some properties of the moduli of continuity:

i) for fixed $f \in H_p^0(\Omega)$ the quantity $\omega_r(t,f)$ is a nondecreasing function of t satisfying

(2.1) $$\omega_r(\lambda t,f) \le (1+\lambda)^r\, \omega_r(t,f) \quad , \lambda > 0 ;$$

ii) for fixed $t > 0$ the triangle inequality

(2.2) $$\omega_r(t,f_1+f_2) \le \omega_r(t,f_1) + \omega_r(t,f_2), \quad f_1,f_2 \in H_p^0(\Omega)$$

holds;

iii) for $f \in H_p^0(\Omega)$, $t > 0$ there holds

(2.3) $$\omega_r(t,f) \le 2^j\, \omega_{r-j}(t,f), \quad 0 \le j \le r ;$$

iv) for $f \in H_p^j(\Omega)$, $1 \le j \le r$, and $t > 0$ there holds

(2.4) $$\omega_r(t,f) \le n^{j/2}\, t^j \sup_{|\beta|=j} \omega_{r-j}(t,D^\beta f) ;$$

v) there exists a constant $M = M(j,r,\Omega)$ such that for $f \in H_p^0(\Omega)$, $t>0$, $1 \le j \le r-1$ and $0 < t \le 1$

(2.5) $$\omega_j(t,f) \le M \left[t^j \|f\| + t^j \int_t^1 s^{-1-j}\, \omega_r(s,f)\, ds\right] ;$$

vi) the condition $\int_0^1 s^{-1-j}\, \omega_r(s,f)\, ds < \infty$ implies *)

*) In the following constants often will be denoted simply by M. Their value may vary from line to line.

$$(2.6)\qquad |f|_{p,j} \le M\,[\|f\|_p + \int_0^1 s^{-1-j}\,\omega_r(s,f)\,ds]\ ,$$

$$(2.7)\qquad \omega_{r-j}(t,D^\beta f) \le M \int_0^t s^{-1-j}\,\omega_r(s,f)\,ds\ ,\qquad |\beta| = j.$$

Concerning a proof of (2.1) - (2.4) we refer to [8],[11]. The inequalities in v), vi) give a sort of converse to iii), iv) and are known as Marchaud-type inequalities. A proof of them will be given in Section 3.

We now establish Theorem 1 and proceed by a series of lemmas. First the left-hand side inequality of (1.7) can be proved for general open domains $G \subset R^n$.

Lemma 1: Let Ω be an open set in R^n. For all $0 < t < \infty$ and $f \in \overset{O}{H}_p(\Omega)$, $1 \le p \le \infty$, there holds $(r=1,2,\ldots)$

$$\omega_r(t,f) \le \max\,(2^r,\, n^{r/2})\; K_r(t^r,f)$$

Proof: This is a simple standard argument. One splits up $f = (f-g) + g$ and applies (2.2) - (2.4) to obtain

$$\begin{aligned}\omega_r(t,f) &\le \omega_r(t,f-g) + \omega_r(t,g)\\ &\le 2^r\|f-g\|_p + n^{r/2}\,t^r|g|_{p,r}\\ &\le \max\,(2^r,n^{r/2})\;[\|f-g\|_p + t^r|g|_{p,r}]\end{aligned}$$

Since g was arbitrary this immediately implies (1.7) in view of definition (1.3).

To prove the other part of (1.7) we confine ourselves to $0 < t \le 1$ and LG-domains Ω. In order to describe this property we need some notations. Let $\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i$ denote the inner product for vectors $x,y \in R^n$ with associated norm $\sqrt{\langle x,x\rangle} = |x|$ and denote by $S_\rho(x)$, $C(q,\gamma,\rho)$ the open ball $\{\, y\in R^n\colon |y-x| < \rho \,\}$ and the cone $\{\, z\in R^n\colon \langle z,q\rangle/|z| > \cos\gamma,\ |z| < \rho\}$, respectively, where x,q are vectors $\in R^n$ with $|q| = 1$ and ρ,γ real numbers with $\rho>0$, $\cos\gamma > 0$. Then we make the following (cf. [1,p.66])

Definition: An open subset of R^n will be called a LG domain if

i) there exists a finite number of open sets U_i, $1 \le i \le s$ and a $\delta > 0$ such that the boundary $\partial\Omega$ of Ω is covered by the open sets $U_i^{\delta} = \{x \in U_i : S_\delta(x) \subset U_i\} \ne \emptyset$, $1 \le i \le s$;

ii) there are open cones $C_i \equiv C_i(q_i, \gamma_i, \rho_i)$ such that for each $x \in \Omega \cap U_i$ the set $x + C_i$ is in Ω, $1 \le i \le s$.

We first need a special version of a result of J.H.B. Kemperman (written communication).

Lemma 2: Let $\mathfrak{h}$ be a vector space over R, $L(\mathfrak{h})$ the set of linear transformations on $\mathfrak{h}$ and $E: R^n \longrightarrow L(\mathfrak{h})$ a mapping such that $E(h_1+h_2) = E(h_1)E(h_2)$ for all $h_1, h_2 \in R^n$ and $\Delta(h) = E(h) - E(0)$ for $h \in \mathbb{R}^n$. If D^* denotes the set of subsets of $\{1,\dots,r\}$, $D \in D^*$ and $|D|$ is the number of elements in D, then for $h_1,\dots,h_r \in R^n$ one has the identity

$$\prod_{k=1}^{r} \Delta(h_k) = \sum_{D\in D^*} (-1)^{|D|} E\Big(\sum_{j\notin D} h_j\Big) \Delta^r \Big(\sum_{j\in D} j^{-1} h_j\Big) \tag{2.8}$$

Proof: (J.H. Kemperman)
For arbitrary $k \in \{0,1,\dots,r\}$ there holds

$$\prod_{j=1}^{r} \Delta((j-k)h_j) = \prod_{j=1}^{r} \big[E((j-k)h) - E(0)\big]$$

$$= \sum_{D\in D^*} (-1)^{r-|D|} E\Big(\sum_{j\in D} jh_j\Big) E\Big(-\sum_{j\in D} kh_j\Big)$$

The left side does not vanish only for $k = 0$. Therefore, after multiplying both sides by $(-1)^{r-k}\binom{r}{k}$ and summing over k, we obtain

$$(-1)^r \prod_{j=1}^{r} \Delta(jh_j) = \sum_{D\in D^*} (-1)^{r-|D|} E\Big(\sum_{j\in D} jh_j\Big) \Delta^r\Big(-\sum_{j\in D} h_j\Big)$$

Next we substitute h_j by $-h_j$ and derive from this by the identity

$$\Delta(-jh_j) = -E(-jh_j)\,\Delta(jh_j)$$

the formula

$$E(-h_1-2h_2-\dots-rh_r)\prod_{j=1}^{r}\Delta(jh_j)=\sum_{D\varepsilon D^*}(-1)^{r-|D|}E(-\sum_{j\varepsilon D}jh_j)\Delta^r(\sum_{j\varepsilon D}h_j)$$

Now multiplying both sides by $E(+h_1+2h_2+\dots+rh_r)$ and then substituting h_j by h_j/j one gets (2.8).

There are similar identities with other terms on the right-hand side possible. This one here has the advantage that the vectors $\sum_{j\notin D}h_j+k\sum_{j\varepsilon D}j^{-1}h_j$ of R^n lie in a cone $C(q,\gamma,\rho)$ for $o\le k\le r$ and all $D\ \varepsilon\ D^*$ if $h_1,\dots h_r\ \varepsilon\ C(q,\gamma,\rho)$ and are small enough.

The next lemma considers what can be done for a proof of Theorem 1 with the use of Steklov-means. This is well known in the case of R^1 (see e.g. [5]). Here we work with "simple" domains Ω so that only this kind of argument is needed. In particular the case $\Omega = R^n$ is included.

Lemma 3: Let the open set $V\subset\Omega$ have a cone property in the sense that there is a cone $C(q,\gamma,\rho)$ such that $x+C(q,\gamma,\rho)\subset\Omega$ whenever $x\ \varepsilon\ V$. Then there exists a constant $M>o$, depending only on r, the data of $C(q,\gamma,\rho)$, and n such that for all $1\le p\le\infty$, all $f\ \varepsilon\ H_p^o(\Omega)$ and $0<t\le 1$

$$K_r(t^r,f)_V \le M\,\omega_r(t,f)$$

Here $K_r(t,f)_V$ denotes the K-functional of the restriction of f to V.

Proof: Suppose first that V is such that the cone $C=C(q,\gamma,\rho)$ contains the set $\{x\varepsilon R^n;\ 0<|x|\le r^2(1+\log r)\sqrt{n}\ \text{for}\ 0\le x_i,\ 1\le i\le n\}$ and $f\varepsilon H_p^o(\Omega)$. If e_i, $1\le i\le n$, denotes the unit vector in direction of the i-th coordinate axis and $0<s\le 1$ then for $x\ \varepsilon\ V$ we introduce the Steklov-means $g_s(x)$

$$(2.9)\qquad g_s(x) = -\sum_{k=1}^{r}(-1)^{r-k}\binom{r}{k}\int_0^1\dots\int_0^1 f(x+k\cdot\sum_{i=1}^{n}\sum_{j=1}^{r}s\,\sigma_{ij}e_i)\prod_{i=1}^{n}\prod_{j=1}^{r}d\sigma_{ij}.$$

Setting $E(h)f(x) = f(x+h)$ one obtains

$$\|f-g_s\|_{L_p(V)} \le \int_0^1\dots\int_0^1\|\Delta^r(s\sum_{i=1}^{n}\sum_{j=1}^{r}\sigma_{ij}e_i)f\|_{L_p(V)}\prod_{i=1}^{n}\prod_{j=1}^{r}d\sigma_{ij}$$

$$\leq \omega_r(r\sqrt{n}\, s,f) \quad .$$

On the other hand for $|\beta| = r$ and $x \in V$ by a familiar formula (cf. Butzer-Berens [5,p. 12]) a.e. in x

$$D^\beta g_s(x) = -s^{-r} \sum_{k=1}^{r} (-1)^{r-k} \binom{r}{k} \int_0^1 \cdots \int_0^1 E(ks \sum_{i=1}^{n} \sum_{j=1}^{r-\beta_i} \sigma_{ij} e_i) \cdot \prod_{i=1}^{n} \Delta^{\beta_i}(kse_i) f(x) \prod_{i=1}^{n} \prod_{j=1}^{r-\beta_i} d\sigma_{ij}$$

so that

$$\|D^\beta g_s\|_{L_p(V)} \leq s^{-r} \sum_{k=1}^{r} \binom{r}{k} \int_0^1 \cdots \int_0^1 \|E(ks \sum_{i=1}^{n} \sum_{j=1}^{r-\beta_i} \sigma_{ij} e_i) \prod_{i=1}^{n} \Delta^{\beta_i}(kse_i) f\|_{L_p(V)} \prod_{i=1}^{n} \prod_{j=1}^{r-\beta_i} d\sigma_{ij}.$$

In view of $\beta_1 + \ldots + \beta_n = r$ an application of Lemma 2, relation (2.8), then yields

$$\|D^\beta g_s\|_{L_p(V)} \leq s^{-r}(2^r-1)^2 \omega_r((1+\log r)rs,\ f)_\Omega$$

Setting $s = \frac{1}{r} \min (\frac{1}{\sqrt{n}}, \frac{1}{1+\log r})\ t$ it follows that

$$\|f-g_s\|_{L_p(V)} + t^r |g_s|_{L_p(V),r} \leq \omega_r(t,f) + 2^{2r}(r \max (\sqrt{n},1+\log r))^r \cdot \omega_r(t,f)$$

for $0 < t \leq r \max (\sqrt{n}, 1+\log r)$, hence a fortiori

$$K_r(t^r,f)_V \leq (1 + (4r \max (\sqrt{n},1+\log r))^r)\ \omega_r(t,f)$$

If the cone C does not have the above properties then there exists a nonsingular linear transformation $T: R^n \longrightarrow R^n$ such that TC is a cone with these properties. By the just proved part there exists a constant $M(r,n)$ such that

$$K_r(t^r, f\circ T^{-1})_{TV} \leq M(r,n)\ \omega_r(t, f\circ T^{-1})_{T\Omega}$$

for $0 < t \leq 1$. But obviously there are constants $M' = M'(T,r)$,

$M'' = M''(T,r)$ such that

$$K_r(t^r,f)_V \le M' K_r(t^r,f\circ T^{-1})_{TV}$$

$$\omega_r(t,f\circ T^{-1})_{T\Omega} \le M'' \omega_r(t,f)_\Omega$$

for all $0 < t < \infty$. This observation completes the proof.

The next lemma is the key step and treats the case when Ω is the union of two domains of the type in Lemma 3. Then the use of the function (2.9) is not possible for the whole domain V.

<u>Lemma 4</u>: Let $\Omega_1 \subset \Omega_3$ and $\Omega_2 \subset \Omega_4$ be open subsets $\subset R^n$ with $\text{dist}(\partial\Omega_i,\partial\Omega_{i+2}) \ge \varepsilon$ for $i=1,2$. Set $V_i = \Omega_i \cap \Omega$ for a given subset of R^n and suppose that there exist $\Omega_5 \supset \{x\varepsilon\Omega_3 \colon S_\delta(x) \subset \Omega_3\}$, $\delta < \varepsilon$, and a cone $C(q,\gamma,\delta_o)$ such that $\Omega_5 \cap \Omega + C \subset \Omega_3 \cap \Omega = V_3$. Then there exists a constant $M > 0$ (independent of f and t) such that for $f \in H_p(\Omega)$, $1 \le p \le \infty$, and $0 < t \le 1$

$$K_r(t,f)_{V_1 \cup V_2} \le M \left[K_r(t,f)_{V_3} + K_r(t,f)_{V_4}\right] .$$

<u>Proof</u>: Let Ω_6 be the set $\{x\varepsilon\Omega_3 \colon S_{(\varepsilon+\delta)/2}(x) \subset \Omega_3\}$ and η a nonnegative C^∞-function on R^n, $\|\eta\|_1 = 1$, with $\text{supp}(\eta) \subset \{x\varepsilon R^n ; |x| < (\varepsilon-\delta)/2\}$. Then the function $\varphi = \eta * \chi_{\Omega_6}$, where χ_{Ω_6} is the characteristic function of Ω_6 vanishes outside of the set $\Omega_5 \supset \{x\varepsilon\Omega_3 \colon S_\delta(x) \subset \Omega_3\}$ and is equal to 1 on Ω_1. Now, take functions $g_1 \in H_p^r(V_3)$, $g_2 \in H_p^r(V_4)$ such that

$$\text{(2.10)} \qquad \begin{cases} \|f-g_1\|_{V_3} + t|g_1|_{p,r,V_3} \le 2\, K_r(t,f)_{V_3} , \\ \|f-g_2\|_{V_4} + t|g_2|_{p,r,V_4} \le 2\, K_r(t,f)_{V_4} , \end{cases}$$

and define g on $V_1 \cup V_2$ by $g = \varphi g_1 + (1-\varphi)g_2$, so that

$$\text{(2.11)} \qquad K_r(t,f)_{V_1 \cup V_2} \le \|f-g\|_{V_1 \cup V_2} + t|g|_{r,p,V_1 \cup V_2} .$$

The first term is estimated by $(V_5 = \Omega_5 \cap \Omega)$

$$\|f-g\|_{V_1\cup V_2} \le \|\varphi[f-g_1] + (1-\varphi)[f-g_2]\|_{V_1\cup V_2}$$

$$\le \|f-g_1\|_{(V_1\cup V_2)\cap V_5} + \|f-g_2\|_{V_2}$$

$$\le \|f-g_1\|_{V_3} + \|f-g_2\|_{V_4} \quad ,$$

and the second by

$$|g|_{r,p,V_1\cup V_2} = |g_1|_{r,p,V_1} + |g_2|_{r,p,(V_1\cup V_2)\setminus V_5} + $$

$$+ |g|_{r,p,(V_1\cup V_2)\cap(V_5\setminus V_1)} \;.$$

Here only the third term requires further consideration, for which we observe by Leibniz's rule

$$|g|_{r,p,(V_1\cup V_2)\cap(V_5\setminus V_1)} = |g_2 + \varphi(g_1-g_2)|_{r,p,V_2\cap(V_5\setminus V_1)}$$

$$\le |g_2|_{r,p,V_2} + M \sum_{j=0}^{r} |g_1-g_2|_{j,p,V_2\cap V_5}$$

Since by assumption we may assume $V_2\cap V_5 + C \subset V_4\cap V_3$ a familiar argument (cf. [1,p.76]) gives

$$\sum_{j=0}^{r} |g_1-g_2|_{j,p,V_2\cap V_5} \le M\,[\|g_1-g_2\|_{V_3\cap V_4} + |g_1-g_2|_{r,p,V_3\cap V_4}]\;\cdot$$

$$\le M\,[\|f-g_1\|_{V_3} + \|f-g_2\|_{V_4} + |g_1|_{r,p,V_3}$$

$$+ |g_2|_{r,p,V_4}] \;.$$

Summing up all these estimates we see that the right-hand side of (2.11) can be bounded by the quantities appearing in (2.10).

We are now in the position to prove the second half of Theorem 1. We state this in the following

<u>Lemma 5</u>: Let Ω be a LG-domain in R^n. Then there exists a constant M only depending on r,p,n and Ω such that for $f \in \overset{o}{H}_p(\Omega)$, $1 \le p \le \infty$, and $0 < t \le 1$

$$(2.12) \qquad K_r(t,f) \leq M\,\omega_r(t,f)$$

Proof: We may assume in the definition of a LG-domain, by taking the minima γ of the γ_i and ρ of the ρ_i, $1 \le i \le s$, that the angles γ_i and the radii ρ_i are all the same. Furthermore, by a change of scale, we may assume $\rho \geq 1$ and that the δ in the definition can be chosen $\delta = 2ks$ where the integer k satisfies $k \geq \max(2, 1+4/s)$. In view of $U_i^\tau = \{x \in U_i : S_\tau(x) \in U_i\}$ for $1 \leq \tau \leq ks$ it is obvious that $\operatorname{dist}(\partial U_i^\tau, \partial U_i^{\tau-1}) \geq 1$, $1 \leq i \leq s$ and $\operatorname{dist}(\partial \bigcup_{i=j}^{s} U_i^\tau, \partial \bigcup_{i=j}^{s} U_i^{\tau-1}) \geq 1$, $1 \leq j \leq s$. We then introduce the sets

$$\Omega_{j_1} = U_j^{ks/2-j+1}, \quad \Omega_{j5} = U_j^{ks/2-j}, \quad \Omega_{j3} = U_j^{ks/2-j-1},$$

$$\Omega_{j2} = \bigcup_{i=j+1}^{s} U_i^{ks/2-j+1} \quad \text{and} \quad \Omega_{j4} = \bigcup_{i=j+1}^{s} U_i^{ks/2-j}$$

for $j = 1, \ldots s-1$ which satisfy the assumptions of Lemma 4 since $(\Omega_{j5} \cap \Omega) + C(q_j, \gamma, 1) \subset \Omega_{j3}$. Hence it follows that $(V_{ji} = \Omega_{ji} \cap \Omega,\ 1 \leq i \leq 5)$

$$K_r(t,f)_{V_{j1} \cup V_{j2}} \leq M_j \left[K_r(t,f)_{V_{j3}} + K_r(t,f)_{V_{j4}}\right]$$

Since $V_{j3} \subset V_j \cap \Omega$ and $V_{j4} = V_{j+1,1} \cup V_{j+1,2}$ repeated application of this estimate gives

$$K_r(t,f)_{V_{11} \cup V_{12}} \leq \sum_{j=1}^{s} K_r(t,f)_{U_j \cap \Omega}\,.$$

Combining this with Lemma 3 and observing $V_{11} \cup V_{12} \supset \bigcup_{i=1}^{s} U_i^{ks/2} \cap \Omega$ we obtain

$$(2.13) \qquad K_r(t,f)_{\bigcup_{i=1}^{s} U_i^{ks/2} \cap \Omega} \leq M\,\omega_r(t,f)_\Omega\,.$$

The set $V_2 =: \bigcup_{i=1}^{s} U_i^{ks} \cap \Omega$ covers the boundary strip $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < ks\}$ of Ω (since $\delta = 2ks$). We now set $V_4 = \bigcup_{i=1}^{s} U_i^{ks/2}$, $V_i = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq ks/i\}$ for $i = 1,3$ and $V_5 = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > ks/2\}$. Then again Lemma 4 applies giving together with (2.13)

$$K_r(t,f)_\Omega = K_r(t,f)_{V_1\cup V_2} \le M\,[K_r(t,f)_{V_3} + \omega_r(t,f)_\Omega]$$

But V_3 certainly satisfies the assumption of Lemma 3 so that (2.12) follows.

3. Marchaud-type inequalities.

The essential aim of this section is to prove the Marchaud-type inequalities (2.5) - (2.7). In view of Theorem 1 we can consider equivalently this question for the corresponding K-functionals. The essential point here is the use of estimates for the seminorms (1.1) of low order in terms of those of higher order. We formulate this in the following

<u>Definition</u>: An open subset Ω of R^n has the property (L) if for all integers j,r satisfying $0 < j < r$ there exists a constant $L = L(j,r,\Omega) > 0$ such that

$$(3.1)\qquad |f|_{p,j} \le L\,[t^{-j}\|f\|_p + t^{r-j}|f|_{p,r}]$$

for $f \in H_p^r(\Omega)$, $1 \le p \le \infty$, and $0 < t \le 1$.

This property holds e.g. for LG-domains (see [1 ,p. 75-78]).

<u>Theorem 2</u>: Let $\Omega \subset R^n$ have the property (L).

a) For each $f \in H_p^o(\Omega)$, $1\le p\le\infty$, and $0<t\le 1$ there holds with $M = M(j,r,\Omega) > 0$

$$(3.2)\qquad K_j(t^j,f) \le M\,t^j\,[\|f\|_p + \int_t^1 s^{-j-1}K_r(s^r,f)\,ds]\ ;$$

b) The condition $\int_0^1 s^{-j-1}K_r(s^r,f)\,ds < \infty$, $1\le j\le r-1$, implies $f \in H_p^j(\Omega)$ and the existence of a constant $M = M(j,r,\Omega) > 0$ such that for $0<t\le 1$

$$(3.3)\qquad |f|_{p,j} \le M\,[\|f\|_p + \int_0^1 s^{-j-1}K_r(s^r,f)\,ds]\ ,$$

$$(3.4)\qquad K_{r-j}(t^{r-j},D^\beta f) \le M \int_0^t s^{-j-1}K_r(s,f)\,ds\ ,\quad |\beta| = j.$$

<u>Proof</u>: For $f \in H_p^0(\Omega)$ choose $g_t \in H_p^r(\Omega)$ such that

$$(3.5)\qquad \|f-g_t\|_p + t^r|g_t|_{p,r} \le 2\, K_r(t^r,f).$$

In view of $H_p^j(\Omega) \supset H_p^r(\Omega)$ for $1 \le j \le r-1$ we have

$$K_j(t^j,f) \le \|f-g_t\|_p + t^j|g_t|_{p,j}.$$

The use of the identity $g_t = \sum_{k=0}^{m-1} [g_{2^k t} - g_{2^{k+1}t}] - g_{2^m t}$ and the property (L) gives

$$t^j|g_t|_{p,j} \le L\, t^j \sum_{k=0}^{m-1} \Big[(2^k t)^{-j}\|g_{2^k t}-g_{2^{k+1}t}\|_p +$$

$$+ (2^k t)^{r-j}\, |g_{2^k t}-g_{2^{k+1}t}|_{p,r}\Big]$$

$$+ Lt^j \Big[(2^m t)^{-j}\, \|g_{2^m t}\|_p + (2^m t)^{r-j}\, |g_{2^m t}|_{p,r}\Big]$$

Subtracting and adding f one obtains in view of (3.5) by the triangle inequality

$$\|f - g_t\|_p + t^j|g_t|_{p,j}$$

$$\le\ M \Big[2^{-mj}\|f\|_p + t^j \sum_{k=0}^{m} (2^k t)^{-j} K_r((2^k t)^r,f)\Big]$$

$$\le\ M \Big[2^{-mj}\|f\|_p + t^j \int_{t/2}^{2^m t} \frac{K_r(s^r,f)}{s^{j+1}}\, ds\Big]$$

Choosing m so that $\frac{1}{2} < 2^m t \le 1$ one deduces

$$K_j(t^j,f) \le M\, t^j \Big[\|f\|_p + \int_{t/2}^{1} \frac{K_r(s^r,f)}{s^{j+1}}\, ds\Big]$$

Now it is easily seen that $\int_{t/2}^{1} \frac{K_r(s^r,f)}{s^{j+1}}\, ds \le 2^j \int_{t}^{1} \frac{K_r(s^r,f)}{s^{j+1}}\, ds$

which gives (3.2).

Concerning part b) let $g_t \in H_p^r(\Omega)$, $0 < t \leq 1$, be chosen as in (3.5). By the same technique as above one obtains

$$(3.6) \qquad \sum_{k=0}^{\infty} |g_{2^{-k}t} - g_{2^{-(k+1)}t}|_{p,j} \leq 2L \sum_{k=0}^{\infty} (2^{-k}t)^{-j} K_r((2^{-(k+1)}t)^r, f)$$

The right hand side can be bounded (up to a constant) by $\int_0^t \frac{K_r(s^r,f)}{s^{1+j}} ds$ which is finite by assumption. This yields the convergence of $\sum_{k=0}^{\infty} (g_{2^{-k}t} - g_{2^{-(k+1)}t})$ in the norm of $H_p^j(\Omega)$.

Consequently $g_{2^{-k}t}$ converges to a limit element in $H_p^j(\Omega)$.

But in view of $\lim_{t \downarrow 0} K_r(t^r, f) = 0$, one has by (3.5)

$\lim_k \|g_{2^{-k}t} - f\|_p = 0$, showing that this element must be equal to f.

To obtain (3.3) one uses (3.6) for $t = 1$ giving

$$|f|_{p,j} \leq |g_1|_{p,j} + 2L \sum_{k=0}^{\infty} 2^{-kj} K_r(2^{-(k+1)r}, f) .$$

From this we deduce (3.3) by the monotonicity of $K_r(t,f)$ in t and (3.5).

In order to establish (3.4) we observe that $(|\beta| = j)$

$$\begin{aligned} K_{r-j}(t^{r-j}, D^\beta f) &\leq \|D^\beta f - D^\beta g_t\|_p + t^{r-j} |D^\beta g_t|_{p,r-j} \\ &\leq |f - g_t|_{p,j} + t^{r-j} |g_t|_{p,r} \\ &\leq \sum_{k=0}^{\infty} |g_{2^{-(k+1)}t} - g_{2^{-k}t}|_{p,j} + t^{r-j} |g_t|_{p,r}. \end{aligned}$$

Then application of (3.6) gives

$$K_{r-j}(t^{r-j}, D^\beta f) \leq M \left[\int_0^t s^{-j-1} K_r(s^r, f)\, ds + t^{-j} K_r(t^r, f) \right]$$

which immediately implies (3.4)

Now Theorem 1 shows that Theorem 2 contains nothing else than the desired Marchaud-type inequalities (2.5) - (2.7).

But Theorem 2 can be proved under more general circumstances using an approximation-theoretic approach which actually inspired the proof of Theorem 2. In order to describe the general situation let X,Y be linear spaces with $Y \subset X$ and (semi)norms $|\cdot|_X$ and $|\cdot|_Y$, respectively. Then for $f \in X$, $0 < t < \infty$, we define

$$(3.1) \qquad K(t,f;X,Y) = \inf \{|f-g|_X + t|g|_Y : g \in Y\}$$

and want to establish relations between the various K-functionals arising when X or Y is replaced by a third linear space Z. The situation above is exhibited by the choice $X = H_p^0(\Omega)$, $Y = H_p^r(\Omega)$ provided with seminorm (1.1).

Now the approximation-theoretic approach consists in assuming that there is an approximation system $\{P_n\}_{n=1}^{\infty}$ of subspaces of X satisfying

$$(3.7) \qquad P_n \subset P_{n+1}.$$

We assume that the family $\{P_n\}_{n=1}^{\infty}$ satisfies a Jackson-type inequality of order $\alpha > 0$ with respect to the subspace Y, i.e.

$$(J_Y) \qquad E_n(f;X) = \inf_{p_n \in P_n} |f-p_n|_X \le M\, 2^{-n\alpha}|f|_Y \qquad (f \in Y)$$

holds, $M > 0$ being independent of n and f. Furthermore we will assume that $\{P_n\}_{n=1}^{\infty}$ satisfies a Bernstein-type inequality of order $\alpha > 0$ with respect to Y, i.e.

$$(B_Y) \qquad P_n \subset Y, \qquad |p_n|_Y \le M\, 2^{n\alpha}|p_n|_X \qquad (p_n \in P_n)$$

holds, $M > 0$ being independent of n and f.

We will also need sharper versions of these Jackson- and Bernstein-type inequalities. Concerning this point we prove

<u>Lemma 6</u>: Suppose (J_Y) and (B_Y) hold. Then (B_Y) can be sharpened to

$$(B_Y') \qquad P_n \subset Y, \qquad |p_n|_Y \le M\, 2^{n\alpha}\, K(2^{-n\alpha},p_n;X,Y) \qquad (p_n \in P_n)$$

if and only if (J_Y) can be sharpened to

$$(J_Y') \qquad E_n(f;X,Y) = \inf_{p_n \in P_n} (|f-p_n|_X + 2^{-n\alpha}|p_n|_Y) \le M\, 2^{-n\alpha}|f|_Y \qquad (f\in Y).$$

<u>Proof</u>: Let (B_Y) and (J'_Y) be satisfied. Then for arbitrary $g\in Y$ choose $q_n \in P_n$ such that

$$|g - q_n|_X + 2^{-n\alpha}|q_n|_Y \le 2\, E_n(g;X,Y).$$

It follows by (B_Y) for fixed $p_n \in P_n$

$$\begin{aligned} |p_n|_Y &\le |p_n-q_n|_Y + |q_n|_Y \\ &\le M\, 2^{n\alpha}\, [|p_n-g|_X + |g-q_n|_X] + |q_n|_Y \\ &\le M\, 2^{n\alpha}\, [E_n(g;X,Y) + |p_n-g|_X] , \end{aligned}$$

and by (J_Y)

$$|p_n|_Y \le M\, 2^{n\alpha}\, [|p_n-g|_X + 2^{-n\alpha}\, |g|_Y],$$

whence (B'_Y) follows by taking the infimum over $g\in Y$.

Conversely let (J_Y) and (B'_Y) be satisfied. For given $f\in Y$ choose q_n such that $|f-q_n|_X \le 2\, E_n(f;X)$. Then it follows by (B'_Y)

$$\begin{aligned} 2^{-n\alpha}|q_n|_Y &\le M\, K(2^{-n\alpha},q_n;X,Y) \\ &\le M\, [K(2^{-n\alpha},q_n-f;X,Y) + K(2^{-n\alpha},f;X,Y)] \\ &\le M\, [E_n(f;X) + 2^{-n\alpha}|f|_Y] \end{aligned}$$

Combining this with (J_Y) gives

$$E_n(f;X,Y) \le 2\, E_n(f;X) + 2^{-n\alpha}|q_n|_Y \le M\, 2^{-n\alpha}|f|_Y .$$

In applications one is mainly interested in sharp Bernstein-type inequalities (B'_Y), however it is (J'_Y) which is more easily verified. In particular this is the case in the example mentioned after Theorem 2. (A similar assertion together with other examples has been considered in [9]).

We now show that existence of such inequalities implies Marchaud-type inequalities for K-functionals.

<u>Theorem 3</u>: a) Suppose Y,Z are subspaces of X such that (J_Y) and (B_Z) with order $\beta > 0$ hold. Then for each $f \in X$ and $0 < t \leq 1$ one has

$$(3.8) \qquad K(t^{\beta},f;X,Z) \leq M\, t^{\beta} \{|f|_X + \int_t^1 u^{-\beta-1} K(u^{\alpha},f;X,Y)\, du\}$$

b) Suppose X,Y,Z are such that (J'_Y), (B_Y) with order α and (B_Z) with order $\beta > 0$, $\alpha > \beta$, are satisfied. Suppose further that Z is a Banach space under norm $\|\cdot\|_Z = |\cdot|_X + |\cdot|_Z$. Then $\int_0^1 u^{-\beta-1} K(u^{-\alpha},f;X,Y)\, du < \infty$ implies $f \in Z$ and the inequality, $0 < t \leq 1$,

$$(3.9) \qquad K(t^{\alpha-\beta},f;Z,Y) \leq M \int_0^t u^{-\beta-1} K(u^{\alpha},f;X,Y)\, du.$$

<u>Proof</u>: It is sufficient to establish (3.8), (3.9) for $0 < t \leq 1/2$. For such t choose k with

$$(3.10) \qquad 2^{-k-1} < t < 2^{-k}.$$

Similarly to (3.5) choose for $n = 1,2,...$ elements $q_n = q_n(f) \in P_n$ such that

$$(3.11) \qquad |f-q_n|_X \leq 2\, E_n(f;X) .$$

Then, concerning part a), we estimate by the same (Bernstein) technique as in Theorem 2

$$\begin{aligned} K(t^{\beta},f;X,Z) &\leq 2\, E_k(f;X) + 2^{-k\beta}\, |q_k|_Z \\ &\leq 2\, E_k(f;X) + 2^{-k\beta} \Big[\sum_{j=1}^{k-1} 2^{j\beta} |q_{j+1}-q_j|_X \\ &\qquad + |q_1|_X\Big] \end{aligned}$$

$$(3.12) \qquad \leq M\, 2^{-k\beta} \{|f|_X + \sum_{j=1}^{k} 2^{j\beta}\, E_j(f;X)\} ,$$

where (3.7), (3.10), (3.11) and (B_Z) were used. A standard argument of interpolation theory yields from (J_Y)

$$(3.13) \qquad E_n(f;X) \le M\,K(2^{-n\alpha},f;X,Y).$$

Inserting this in (3.12) and using the monotonicity of the K-functional in t we then obtain (3.8).

Concerning part b), we first show $f\varepsilon Z$. Here we observe that for the q_n of (3.11)

$$|q_n-q_m|_Z \le \sum_{j=m}^{n-1} 2^{(j+1)\beta}|q_{j+1}-q_j|_X \le 2\cdot 2^{\beta} \sum_{j=m}^{n} 2^{j\beta} E_j(f;X)$$

for any n,m with $n > m$. The same estimate obviously holds for $|q_n-q_m|_X$ so that by (3.13)

$$(3.14) \qquad \|q_n-q_m\|_Z \le M \sum_{j=m}^{\infty} 2^{j\beta} K(2^{-j\alpha},f;X,Y) \le M \int_0^{2^{-m-1}} u^{-\beta-1} K(u^{\alpha},f;X,Y)\,du.$$

Thus $\{q_n\}$ is a Cauchy-sequence for $n \longrightarrow \infty$ which consequently has a limit in Z which must be equal to f in view of (3.11), (3.13) and the finiteness of the integral.

It follows

$$(3.15) \qquad K(t^{\alpha-\beta},f;Z,Y) \le |f-q_k|_Z + 2^{k(\beta-\alpha)}|q_k|_Y .$$

Here the first term can be bounded by the right-hand side of (3.9) when passing to the limit $n \longrightarrow \infty$ in (3.14). For the second term we use the fact that, according to Lemma 6 a sharpened Bernstein-type inequality (B'_Y) holds, giving

$$2^{k(\beta-\alpha)}|q_k|_Y \le M\,2^{k\beta} K(2^{-k\alpha},q_k;X,Y) \le M\,2^{k\beta}\{E_k(f;X) + K(2^{-k\alpha},f;X,Y)\} .$$

Then (3.13) and

$$2^{k\beta} K(2^{-k\alpha},f;X,Y) \le M \int_{2^{-k-2}}^{2^{-k-1}} u^{-\beta-1} K(u^{\alpha},f;X,Y)\,du$$

show that also the second term in (3.10) can be bounded by the right-hand side of (3.9).

We remark that one can show the existence of families $\{P_n\}_{n=1}^{\infty}$ for the spaces $H_p^0(\Omega)$ which satisfy (sharp) Jackson- and Bernstein-type inequalities of order r with respect to the subspace $H_p^r(\Omega)$ provided with seminorm $|\cdot|_{p,r}$. One possibility (for LG-domains Ω) is e.g. the use of the quasiinterpolants of de Boor - Fix. This would give a second proof of Theorem 1.

4. Further applications

Theorem 1 may be used to establish an extension theorem preserving moduli of continuity uniformly in t. To this end we "interpolate" an extension theorem of Stein [10,pp.181] for LG-domains Ω where the sets $\Omega \cap U_i$ (see ii) in the definition of LG-domains) have the particular form $\Omega \cap U_i = U_i \cap \Omega_i$, where Ω_i are special Lipschitz-domains in R^n of the form

$$\Omega_i = \{(x,y): y > \varphi_i(x),\ y\varepsilon R^1,\ x\varepsilon R^{n-1}\}\ ,$$

and the functions $\varphi_i: R^{n-1} \longrightarrow R^1$ satisfy a Lipschitz-condition

$$|\varphi_i(x) - \varphi_i(x')| \le M_i|x-x'| \qquad (x,x'\varepsilon R^{n-1}).$$

The extension theorem of Stein then states that for such domains Ω there exists a linear operator E mapping functions on Ω to functions on R^n with the properties

a) $E(f)_{|\Omega} = f$

b) E maps $H_p^r(\Omega)$ continuously into $H_p^r(R^n)$ for all p, $1 \le p \le \infty$, and all $r = 0,1,2,\ldots$.

(There are other such extension theorems but with more regularity conditions on the boundary, cf. [1,pp. 83]).

<u>Corollary 1</u>: Let $\Omega_1 \subset \Omega_2$ be bounded LG-domains $\subset R^n$ of the type above. There exists a bounded linear extension operator T from $H_p^0(\Omega_1)$ into $H_p^0(\Omega_2)$, $1\le p\le\infty$ *), such that for $0 < t \le 1$

$$\text{(4.1)} \qquad \omega_r(t,Tf)_{\Omega_2} \le M\,\omega_r(t,f)_{\Omega_1}$$

with a constant $M > 0$ independent of t and f.

*) In case $p=\infty$ it is necessary to consider the closure $\bar{\Omega}_1$ instead of Ω_1.

Proof: Let E be the bounded linear extension operator from above and P_r the extension to $L_p(\Omega_1)$, $1 \le p \le \infty$, of the orthogonal projection of $L_2(\Omega_1)$ into the space of polynomials of total degree r-1. We claim that the operator

$$Tf = E(f - P_r f) + P_r f$$

is the desired extension operator. It is obviously a bounded linear operator from $H_p^0(\Omega_1)$ into $H_p^0(\Omega_2)$, $1 \le p \le \infty$, in particular for $f \in H_p^0(\Omega_1)$

$$\|Tf\|_{p,\Omega_2} \le M\|f-P_r f\|_{p,\Omega_1} + \|P_r f\|_{p,\Omega_1}$$
$$\le M\|f\|_{p,\Omega_1} .$$

Furthermore for any $g \in H_p^r(\Omega_1)$ one has

$$|Tg|_{p,r,\Omega_2} \le M\|g-P_r g\|_{p,r,\Omega_1}$$

Since (cf. [1,pp. 85-86]) there exists a constant M independent of g such that

$$\|g-P_r g\|_{p,r,\Omega_1} \le M\ |g|_{p,r,\Omega_1}$$

it follows by the same argument as in Theorem 3

$$K_r(t,Tf)_{\Omega_2} \le M\ [\|f-g\|_{p,\Omega_1} + t|g|_{p,r,\Omega_1}]$$

Taking the infimum over $g \in H_p^r(\Omega_1)$ and applying Theorem 1 then yields (4.1).

Note that it is not possible to replace Ω_2 by R^n in the above theorem since then the right-hand side of (4.1) may vanish (f being a polynomial) whereas the left-hand side does not.

The next application gives a refinement of estimates in Bramble-- Hilbert [3] on linear functionals.

Corollary 2: Let F(f) be a bounded linear functional on $C(\Omega)$ which vanishes for polynomials of total degree $\le r-1$. Then, if ρ denotes the diameter of the domain $\Omega \subset R^n$, one has with constant

$M > 0$ independent of ρ and f

$$(4.2) \qquad |F(f)| \leq M\, \omega_r(\rho; f)_{C(\Omega)}$$

<u>Proof</u>: The result of [3] states that $|F(g)| \leq M\, \rho^r |g|_{r,\infty,\Omega}$ for $g \in C^r(\Omega)$. We then use the same interpolation argument as in Lemma 1 and apply Theorem 1 to obtain (4.2).

<u>Corollary 3</u>: Let Ω be an LG- domain with diameter $\rho > 0$. Then for each $f \in H^0_p(\Omega)$, $1 \leq p \leq \infty$, there exists a polynomial $q_r(f)$ of total degree $\leq r-1$ such that

$$\| f - q_r(f) \|_p \leq M\, \omega_r(f, \rho) \qquad (r = 1,2,\ldots)$$

with a constant independent of f and ρ.

<u>Proof</u>: According to [3] (see also [7,pp. 85-86]) for $g \in H^r_p(\Omega)$ there is a polynomial $q_r(g)$ such that $\| g - q_r(g) \|_p \leq M\, \rho^r |g|_{r,p,\Omega}$. We then estimate $\inf \{ \| f - q_r \|_p : q_r$ a polynomial of total degree $r-1\}$ by the same argument as in Corollary 2.

Corollary 3 states a multidimensional analogue of a theorem of Whitney (see [4]).

Finally we consider another simple application to multidimensional polynomial approximation. A one-dimensional result for the interval $I = (a,b)$ in De Vore [6] states that there exists a sequence $\{L_m\}_{m=r}^{\infty}$ of linear operators L_m mapping $H^0_p(I)$, $1 \leq p \leq \infty$, into the space of polynomials of degree $\leq m$ such that

$$(4.3) \qquad \| L_m(f) - f \|_{p,I} \leq M\, \omega_r(f, 1/m)_I \qquad (r=1,2,\ldots),$$

M being a constant independent of f and m. Using tensor products it is easy to construct linear operators $L_{\mu,n}$ mapping $H^0_p(I^n)$, $1 \leq p \leq \infty$, into the space $\mathcal{P}_\mu$ of multivariate polynomials of total degree $\leq \mu$ such that

$$\| L_{\mu,n}(f) - f \|_{p,I^n} \leq M\, \omega_r(f, 1/\mu)_{I^n}$$

for $\mu \geq \mu_o \geq \max(n,r)$, $r = 1,2,\ldots$.

Now the application of the extension operator in Corollary 1 gives

Corollary 4: Let Ω be a bounded LG-domain in R^n, $1 \leq p \leq \infty$, and $r = 1,2,\ldots$. There exists a sequence $\{P_\mu\}_{\mu_o}^{\infty}, \mu_o \geq \max(n,r)$, mapping $H_p^o(\Omega) \longrightarrow \mathcal{P}_\mu$ such that with a constant M independent of μ and f

$$\|P_\mu(f) - f\|_{p,\Omega} \leq M\,\omega_r(f,1/\mu)_\Omega .$$

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Remark: After writing this paper the authors were informed by Prof. Yu. Brudnyi (Jaroslavl) that he has proved an extension theorem of the type of Corollary 1 for general rearrangement invariant Banach spaces in "On the Theorem of Extension for one Family of Functional Spaces" (Issledovanijy po lineirym operateram i teorii functii. UI - Leningrad 1976). This also would provide another proof of our theorem 1."