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CONVEXITY, MODULI OF SMOOTHNESS AND A JACKSON-TYPE INEQUALITY

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Abstract. For a Banach space B of functions which satisfies for some $m > 0$

$$(*) \qquad \max(\|F+G\|_B, \|F-G\|_B) \geqq (\|F\|_B^s + m\|G\|_B^s)^{1/s}, \quad \forall\, F, G \in B$$

a significant improvement for lower estimates of the moduli of smoothness $\omega^r(f,t)_B$ is achieved. As a result of these estimates, sharp Jackson inequalities which are superior to the classical Jackson type inequality are derived. Our investigation covers Banach spaces of functions on $\mathbb{R}^d$ or $\mathbb{T}^d$ for which translations are isometries or on S^{d-1} for which rotations are isometries. Results for C_0 semigroups of contractions are derived. As applications of the technique used in this paper, many new theorems are deduced. An L_p space with $1 < p < \infty$ satisfies $(*)$ where $s = \max(p, 2)$, and many Orlicz spaces are shown to satisfy $(*)$ with appropriate s.

1. Introduction

For a Banach space B of functions on $\mathbb{R}^d$ or $\mathbb{T}^d$ for which translations are continuous isometries and whose norm satisfies for some $1 < q \leqq 2$ and some $M \geqq 1$

$$(1.1) \quad \frac{1}{2}\|F+G\|_B + \frac{1}{2}\|F-G\|_B \leqq \big(\|F\|_B^q + M\|G\|_B^q\big)^{1/q}, \quad \forall\, F, G \in B,$$

the first author (see [9]) derived a sharp version of the Marchaud inequality i.e. an estimate of the r-th modulus of smoothness $\omega^r(f,t)_B$ (see (1.5) below) by an expression involving $\omega^{r+1}(f,t)_B$, which implies a sharper version of the converse inequality (see also [20]). Analogous results were achieved for

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functions on the sphere (see [13]). In the other direction, a sharp Jackson inequality and a sharp lower estimate of $\omega^r(f,t)_{L_p}$ for $1<p<\infty$ were given in [7] using a version of the Littlewood–Paley inequality. Here, we will use the following dual inequality to (1.1), given by

$$\max\left(\|F+G\|_B,\|F-G\|_B\right) \geqq \left(\|F\|_B^s + m\|G\|_B^s\right)^{1/s}, \quad \forall\, F,G\in B \tag{1.2}$$

for some $2\leqq s<\infty$ and $m>0$, to obtain the sharp Jackson inequality and the lower estimate of $\omega^r(f,t)_B$. This includes the result for L_p, $1<p<\infty$, since for $B=L_p$ when $1<p<\infty$, (1.2) is satisfied with $s=\max(2,p)$. An important portion of the paper will be dedicated to the lower estimate of $\sup\limits_{0<u\leqq t}\left\|\left(T(u)-I\right)^r f\right\|_B$, where $T(u)$ is a C_0 semigroup of contractions, and to applications of the lower estimate in approximation theory. An example of such an application is the sharp Jackson inequality for polynomial approximation on a simplex with Jacobi weights using the L_p norm where $1<p<\infty$ or some other Orlicz norm which satisfies (1.2).

The condition (1.2) depends on the particular norm of B and may not be satisfied by an equivalent norm of B. For our results we will need a norm on B which satisfies simultaneously (1.2) and the condition that $T(u)$ is a contraction on B or that translation by ξ is a contraction or an isometry on B, which also is not inherited by an equivalent norm. However, for the conclusion of our results any equivalent norm of B will do. In short, we need the condition that B possesses a norm for which $T(u)$ are contractions and which simultaneously satisfies (1.2); however, the results are valid for any equivalent norm on B.

The following theorem is perhaps typical of the results achieved in the present paper.

THEOREM 1.1. *Suppose B is a Banach space of functions on $\mathbb{R}^d$ or $\mathbb{T}^d$ with a norm satisfying* (1.2) *for some s, $2\leqq s<\infty$ and*

$$\begin{cases} \|f(\cdot+\xi)\|_B = \|f(\cdot)\|_B, & \lim\limits_{|h|\to 0}\|f(\cdot+h)-f(\cdot)\|_B = 0,\\ \|f(-\cdot)\|_B = \|f(\cdot)\|_B \end{cases} \tag{1.3}$$

for any $f\in B$ and $\xi,h\in\mathbb{R}^d$. Then for C independent of f, t and n

$$2^{-nr}\left\{\sum_{j=1}^{n} 2^{jrs}\omega^{r+1}(f,2^{-j})_B^s\right\}^{1/s} \leqq C\omega^r(f,2^{-n})_B \tag{1.4}$$

where

$$\begin{cases} \omega^r(f,t)_B = \sup_{|h| \leqq t} \|\Delta_h^r f\|_B, \quad \Delta_h f(x) = f(x+h) - f(x) \quad \text{and} \\ \Delta_h^{\ell+1} f = \Delta_h(\Delta_h^\ell f). \end{cases} \tag{1.5}$$

The inequality (1.4) is sharper than the classical $\omega^{r+1}(f,t)_B \leqq 2\omega^r(f,t)_B$ and is shown in [7, Section 10] to be optimal for L_p, $1 < p < \infty$.

Throughout this paper constants will be positive and may depend on the space (B, $C(\mathbb{R}^d)$, $L_p(\mathbb{R}^d)$ etc.) and on r but will be valid for all the elements of the space and will be independent of t, n, j and ℓ. Furthermore, unless otherwise specified, when a condition, result, or estimate is given in a theorem, definition, or remark concerning functions in some space, it applies to all the functions in that space.

2. The basic inequality

In this section we derive the basic inequality used throughout this paper.

THEOREM 2.1. *Suppose that B is a Banach space of functions, that $T : B \to B$ is a linear contraction operator, that is $\|Tf\|_B \leqq \|f\|_B$ and suppose also that* (1.2) *is satisfied with a given s, $2 \leqq s < \infty$ and $m > 0$. Then for some $m_1 > 0$*

$$\|\Delta_T^r f\|_B \geqq m_1 \Big(\sum_{j=0}^{\infty} 2^{-jrs} \| \Delta_{T^{2^j}}^{r+1} f \|_B^s \Big)^{1/s} \tag{2.1}$$

where

$$\Delta_{T^{2^\ell}} f = T^{2^\ell} f - f \quad \text{and} \quad \Delta_{T^{2^\ell}}^{k+1} f = \Delta_{T^{2^\ell}} \big(\Delta_{T^{2^\ell}}^k f \big). \tag{2.2}$$

REMARK 2.2. Examples of such Banach spaces on $\mathbb{R}^d$, $\mathbb{T}^d$ or S^{d-1} are L_p spaces for $1 < p < \infty$ where $s = \max(p, 2)$. An example of T on a space of functions on $\mathbb{R}^d$ (and $\mathbb{T}^d$) is $Tf(x) = f(x+\xi)$ with $x, \xi \in \mathbb{R}^d$. An example of T on a space of functions on S^{d-1} is $Tf(x) = f(\rho x)$ with $x \in S^{d-1}$ and $\rho \in SO(d)$ (the orthogonal matrices on $\mathbb{R}^d$ whose determinant equals 1). Also $T = T(t)$ may be a semigroup of contractions, the simplest being $T(t)f(x) = f(x+t)$ on $L_p(\mathbb{R}_+)$, but other examples important for applications will be described at length.

PROOF OF THEOREM 2.1. Let $\widetilde{T}$ be any linear contraction operator on B. We note that $\widetilde{T}^n f = \widetilde{T}(\widetilde{T}^{n-1} f)$ and follow [9] to define $F = \frac{1}{2}(\widetilde{T}^2 - I)\varphi$

and $G = -\frac{1}{2}(\widetilde{T} - I)^2\varphi$, $\varphi \in B$, so that $F + G = (\widetilde{T} - I)\varphi$ and $F - G = \widetilde{T}(\widetilde{T} - I)\varphi$. As $\widetilde{T}$ is a contraction, we have

$$\max\left(\|F + G\|_B, \|F - G\|_B\right) = \|F + G\|_B = \|(\widetilde{T} - I)\varphi\|_B$$

and by (1.2) with $\varphi = (\widetilde{T} - I)^{r-1}f$, $f \in B$, we obtain

$$\|(\widetilde{T} - I)^r f\|_B^s \geqq \frac{1}{2^s}\|(\widetilde{T}^2 - I)(\widetilde{T} - I)^{r-1}f\|_B^s + m\frac{1}{2^s}\|(\widetilde{T} - I)^{r+1}f\|_B^s. \tag{2.3}$$

Recalling that $\widetilde{T}$ is a contraction, we have

$$\|(\widetilde{T}^2 - I)^r f\|_B^s = \|(\widetilde{T} + I)^r(\widetilde{T} - I)^r f\|_B^s \leqq 2^{(r-1)s}\|(\widetilde{T} + I)(\widetilde{T} - I)^r f\|_B^s$$

$$= 2^{(r-1)s}\|(\widetilde{T}^2 - I)(\widetilde{T} - I)^{r-1}f\|_B^s,$$

which, combined with (2.3), yields

$$\|(\widetilde{T} - I)^r f\|_B^s \geqq \frac{1}{2^{rs}}\|(\widetilde{T}^2 - I)^r f\|_B^s + m\frac{1}{2^s}\|(\widetilde{T} - I)^{r+1}f\|_B^s. \tag{2.4}$$

Now we use (2.4) iteratively with $\widetilde{T} = T$, $\widetilde{T} = T^2$, $\widetilde{T} = T^4$, $\dots$, $\widetilde{T} = T^{2^\ell}$ to obtain

$$\|(T - I)^r f\|_B^s \geqq \frac{1}{2^{rs}}\|(T^2 - I)^r f\|_B^s + m\frac{1}{2^s}\|(T - I)^{r+1}f\|_B^s$$

$$\geqq \frac{1}{2^{2rs}}\|(T^4 - I)^r f\|_B^s + m\frac{1}{2^s}\left(\|(T - I)^{r+1}f\|_B^s + \frac{1}{2^{rs}}\|(T^2 - I)^{r+1}f\|_B^s\right)$$

$$\geqq \dots \geqq \frac{1}{2^{(\ell+1)rs}}\|(T^{2^{\ell+1}} - I)^r f\|_B^s + m\frac{1}{2^s}\left(\sum_{j=0}^{\ell}\frac{1}{2^{rsj}}\|(T^{2^j} - I)^{r+1}f\|_B^s\right)$$

$$\geqq \left(\frac{m}{2^s}\right)\sum_{j=0}^{\ell}\frac{1}{2^{rsj}}\|(T^{2^j} - I)^{r+1}f\|_B^s,$$

which implies (2.1) with $m_1 = \dfrac{m^{1/s}}{2}$. $\square$

The inequality (2.1), which is at the core of most of the results in this paper, is very simple, but to apply it successfully, we will need many and perhaps more sophisticated results.

3. The condition on the space

In this section we will discuss the condition (1.2), exhibit spaces for which it is valid and for what s. The condition (1.1) was shown in [9] to be equivalent to the condition

$$(3.1)\quad \eta_B(\sigma)=\sup_{\substack{\|F\|=1\\ \|G\|=\sigma}}\left(\frac{1}{2}\|F+G\|_B+\frac{1}{2}\|F-G\|_B-1\right),\quad \eta_B(\sigma)\leqq k\sigma^q,$$

which was extensively investigated, and spaces B satisfying (3.1) are described (see [18, p. 63]) as having modulus of smoothness of power type q. We note that the concept modulus of smoothness in [18] describes the smoothness of the unit ball of the Banach space B (in relation to a specific norm), and is not related to the concept with the same name (see for instance (1.5)) in approximation theory describing smoothness of a function (i.e. an element of B). We note that we found (1.1) easier to use in classical analysis and also easier to verify (see [8, p. 49]).

In the next theorem we show that (1.2) is dual to (1.1), and use that later to examine spaces that satisfy (1.2) and for what s. As a result we will show (later) that a big class of Orlicz spaces satisfies (1.2) and give examples of such spaces.

THEOREM 3.1. *Suppose B is a Banach space endowed with a norm which for some q, $1<q\leqq 2$, satisfies*

$$(3.2)\quad \frac{1}{2}\|x+y\|_B+\frac{1}{2}\|x-y\|_B\leqq\left(\|x\|_B^q+M\|y\|_B^q\right)^{1/q}\quad \text{for all}\quad x,y\in B.$$

Then the dual of B, $X=B^*$ (with the norm dual to that satisfying (3.2)) satisfies

$$(3.3)\qquad \max\left(\|\varphi+\psi\|_X,\|\varphi-\psi\|_X\right)\geqq\left(\|\varphi\|_X^s+m\|\psi\|_X^s\right)^{1/s}$$

for all $\varphi,\psi\in X=B^*$ with $s=\frac{q}{q-1}\left(\frac{1}{s}+\frac{1}{q}=1\right)$ and $m=M^{-1/(q-1)}$. Moreover, if for a given norm of X (3.3) is satisfied, then $B=X^*$ (with norm dual to that satisfying (3.3)) satisfies (3.2).

PROOF. Define the operator A on $(x,y)\in B\times B=\widetilde{B}$ by

$$A(x,y)=\left(\frac{x+y}{2},\frac{x-y}{2}\right)$$

which we consider as a transformation between $\widetilde{B}$ with the norm $\|(u,v)\|_{\widetilde{B}_1}=\|u\|_B+\|v\|_B$ and $\widetilde{B}$ with the (equivalent) norm $\|(u,v)\|_{\widetilde{B}_2}=\left(\|u\|_B^q+M\|v\|_B^q\right)^{1/q}$.

Using (3.2), we now have $\|A\|_{\widetilde{B}_2\to\widetilde{B}_1} \leqq 1$. The dual to $\widetilde{B}$, $\widetilde{B}^*$, is given by $(\varphi,\psi)(u,v)=\varphi u+\psi v$ where $\varphi,\psi\in B^*$. To calculate A^*, we write

$$(\bar\varphi,\bar\psi)A(x,y)=\frac{1}{2}(\bar\varphi,\bar\psi)(x+y,x-y)=\frac{1}{2}(\bar\varphi+\bar\psi,\bar\varphi-\bar\psi)(x,y)$$

$$=A^*(\bar\varphi,\bar\psi)(x,y).$$

Setting $\frac{\bar\varphi+\bar\psi}{2}=\varphi$, $\frac{\bar\varphi-\bar\psi}{2}=\psi$, we have $A^*(\varphi+\psi,\varphi-\psi)=(\varphi,\psi)$.

Since $\|A\|_{\widetilde{B}_2\to\widetilde{B}_1}\leqq 1$, we have $\|A^*\|_{\widetilde{B}_1^*\to\widetilde{B}_2^*}\leqq 1$. We now write

$$\big\|(\varphi,\psi)\big\|_{\widetilde{B}_1^*}=\sup_{\|u\|_B+\|v\|_B=1}\big|(\varphi u+\psi v)\big|\leqq\max\big(\|\varphi\|_{B^*},\|\psi\|_{B^*}\big),$$

and equality follows, choosing $v=0$ if $\|\varphi\|_{B^*}\geqq\|\psi\|_{B^*}$ and choosing $u=0$ otherwise.

For the norm of $\widetilde{B}_2^*$,

$$\big\|(\varphi,\psi)\big\|_{\widetilde{B}_2^*}=\sup\Big\{|\varphi u+\psi v|:\ \big(\|u\|_B^q+M\|v\|_B^q\big)^{1/q}=1\Big\}$$

$$\leqq\sup\Big\{\|\varphi\|_{B^*}\|u\|_B+\big\|M^{-1/q}\psi\big\|_{B^*}\big\|M^{1/q}v\big\|_B;\|u\|_B^q+M\|v\|_B^q=1\Big\}$$

$$\leqq\big(\|\varphi\|_{B^*}^s+M^{-s/q}\|\psi\|_{B^*}^s\big)^{1/s}=\big(\|\varphi\|_{B^*}^s+m\|\psi\|_{B^*}^s\big)^{1/s}$$

with $s=\frac{q}{q-1}$ and $m=M^{-1/(q-1)}$. To show equality, we choose $a\geqq 0$ and $b\geqq 0$ for which $a^q+Mb^q=1$ and (a^q,Mb^q) is proportional to

$$\big(\|\varphi\|_{B^*}^s,m\|\psi\|_{B^*}^s\big),$$

and then choose $\|u_n\|_B=a$, $\|v_n\|_B=b$ such that $\varphi u_n\to a\|\varphi\|_{B^*}$ and $\psi v_n\to b\|\psi\|_{B^*}$.

The second assertion can be obtained in a similar way using the operator O on $(x,y)\in X\times X=\widetilde{X}$ given by

$$O(x,y)=(x+y,x-y)$$

and endowing $\widetilde{X}$ with the norms

$$\big\|(x,y)\big\|_{\widetilde{X}_1}=\max\big(\|x\|_X,\|y\|_X\big)\quad\text{and}\quad\big\|(x,y)\big\|_{\widetilde{X}_2}=\big(\|x\|_X^s+m\|y\|_X^s\big)^{1/s},$$

and hence (2.2) is satisfied by all $x,y\in B_1=X^*$ and the B_1 norm. In the terminology of [18, p. 59] this implies that B_1 is uniformly smooth and hence (see [18, p. 61, Proposition 1.e.2(ii)]) X is uniformly convex. We note that (3.2) and the above now imply (see [18, p. 61, Proposition 1.e.3]) that both B and X are reflexive. Therefore, $B_1 = B$. $\square$

As a corollary of Theorem 3.1 we show that the condition (1.2) is satisfied by L_p spaces.

COROLLARY 3.2. *For L_p with $1 < p < \infty$*

$$\max\left(\|F+G\|_p, \|F-G\|_p\right) \geqq \left(\|F\|_p^{\max(p,2)} + m\|G\|_p^{\max(p,2)}\right)^{1/\max(p,2)} \tag{3.4}$$

for some $m > 0$.

PROOF. We recall that for L_p, $1 < p < \infty$, (3.2) is valid with $q = \min(p, 2)$ (see [9]) and use Theorem 3.1. $\square$

REMARK 3.3. As (3.1) with $x = F$ and $y = G$ was shown to be equivalent to (3.2) (see [9]) and (3.3) was shown to be dual to (3.2), the condition

$$\begin{cases} \delta_X(\varepsilon) \geqq K\varepsilon^s \quad \text{where} \\ \delta_X(\varepsilon) \equiv \inf\left(1 - \|\varphi+\psi\|_X/2 : \varphi, \psi \in X,\right. \\ \qquad\qquad \left.\|\varphi\|_X = \|\psi\|_X = 1,\ \|\varphi-\psi\|_X = \varepsilon\right) \end{cases} \tag{3.5}$$

which is dual to (3.1) (see [18, p. 63]) is equivalent to (3.3). Hence we note that the condition (3.3) on (a given norm of) a Banach space X means that X has a modulus of convexity of at least power type s (see [18, p. 63]).

REMARK 3.4. For a space B both the inequalities (1.1) and (1.3) depend on the norm and may not be valid for an equivalent norm. However, the sharp Marchaud inequality or sharp converse inequality is valid if it is valid for an equivalent norm. It will be evident that the validity of the sharp Jackson inequality and of the lower estimate for the modulus of smoothness will, in the situations proved in this paper for one norm of B, imply their validity for any equivalent norm.

REMARK 3.5. On the face of it, it may seem that in Theorem 3.1 we neglected to treat the situation when $q > 2$. However, as (3.2) is equivalent to $\eta_B(\sigma) \leqq k\sigma^q$ (with $\eta_B(\sigma)$ of (3.1)), and as $\eta_B(\sigma)/\sigma^2$ is equivalent to a non-increasing function for any Banach space (see [18, p. 64, Proposition 1.e.5]), a nontrivial Banach space (different from $\mathbb{R}$ or $\{0\}$) for which (3.2) is satisfied with $q > 2$ does not exist.

We outline now the basic notations (and some facts) concerning Orlicz spaces (see [19] and [2, pp. 265–280]) which we will use in this section and later. A Young function Φ is an increasing convex function on $\mathbb{R}_+$ satisfying

$\Phi(0)=0$. For a domain Ω and a (positive) measure $d\mu(x)$ the Orlicz class $\mathcal{M}(\Phi)$ and the Orlicz functional $M_\Phi(f)$ are given by

$$\mathcal{M}(\Phi)\equiv\Big\{f:\ M_\Phi(f)\equiv\int_\Omega\Phi\big(|f(x)|\big)\,d\mu(x)<\infty\Big\}. \tag{3.6}$$

The Luxemburg norm of the Orlicz space is given by

$$\|f\|_{O_L(\Phi)}\equiv\inf\left(a>0:\ M_\Phi\left(\frac{|f|}{a}\right)\leqq 1\right). \tag{3.7}$$

$\Psi(y):\mathbb{R}_+\to\mathbb{R}_+$ is the complementary Young function to $\Phi(x)$ satisfying $\lim\limits_{x\to\infty}\frac{\Phi(x)}{x}=\infty$ if

$$\Psi(y)=\sup\{\,xy-\Phi(x):\ x\geqq 0\},\quad y\geqq 0. \tag{3.8}$$

The Orlicz norm of the Orlicz space is given by

$$\|f\|_{O(\Phi)}\equiv\sup\Big\{\int_\Omega|f(x)g(x)|\,d\mu(x):\ \int_\Omega\Psi\big(|g(x)|\big)\,d\mu(x)\leqq 1\Big\}. \tag{3.9}$$

A Young function Φ satisfies the Δ_2 condition if for some $K>0$

$$\Phi(2x)\leqq K\Phi(x)\quad\text{for}\quad x\geqq x_0\geqq 0\quad\big(x_0=0\quad\text{when}\quad\mu(\Omega)<\infty\big).$$

A Young function Ψ satisfies the ∇_2 condition if for some $a>1$

$$\Psi(x)\leqq\frac{1}{2a}\Psi(ax)\quad\text{for}\quad x\geqq x_0\geqq 0\quad\big(x_0=0\quad\text{when}\quad\mu(\Omega)<\infty\big).$$

It is known that if Φ is a Young function, Ψ given by (3.8) is a Young function as well. Also $\|f\|_{O_L(\Phi)}\leqq\|f\|_{O(\Phi)}\leqq 2\|f\|_{O_L(\Phi)}$ (see [2, Theorem 8.14, p. 272]). Moreover, if Φ satisfies the Δ_2 condition, the complementary Young function Ψ satisfies the ∇_2 condition (see [19, Corollary 4, p. 26]).

LEMMA 3.6. *Suppose Φ is a Young function and that $\Phi(u^{1/s})$ is concave for some $1<s<\infty$. Then $\Psi(t^{1/q})$ is convex where Ψ is the complementary Young function and $\frac{1}{s}+\frac{1}{q}=1$.*

PROOF. Let $g(u)\equiv\Phi(u^{1/s})$, $u\geqq 0$. Then

$$g(u)=\inf_{z\geqq 0}\big(g'_+(z)(u-z)+g(z)\big)$$

and as $\lim_{u\to\infty} g(u) = +\infty$, for every $z \geqq 0$ we have $g'_+(z) > 0$. In other words, $g(u) = \inf_{(a,b)\in L}(au + b)$, where L is a subset of $(0, +\infty) \times \mathbb{R}$. Therefore, $\Phi(x) = \inf_{(a,b)\in L}(ax^s + b)$. By the definition of Ψ,

$$\Psi(y) = \sup_{x\geqq 0}\big(xy - \Phi(x)\big) = \sup_{x\geqq 0}\ \sup_{(a,b)\in L}\ (xy - ax^s - b)$$

$$= \sup_{(a,b)\in L}\ \sup_{x\geqq 0}(xy - ax^s - b).$$

As $s > 1$, the second supremum is achieved at $x = \left(\frac{y}{as}\right)^{\frac{1}{s-1}}$. Hence,

$$\Psi(y) = \sup_{(a,b)\in L}\left(\left(\frac{1}{(as)^{\frac{1}{s-1}}} - a(as)^{-q}\right)y^q - b\right),$$

which means that $\Psi(t^{1/q})$ is a supremum of a family of functions linear in t, and therefore $\Psi(t^{1/q})$ is convex. □

LEMMA 3.7. *Suppose $\Phi(u^{1/s})$ is concave for some s, $2 \leqq s < \infty$, where Φ is a Young function satisfying the ∇_2 condition. Then there exist constants $A, m > 0$ and a Young function $\widetilde{\Phi}(u)$, such that $A^{-1}\Phi(u) \leqq \widetilde{\Phi}(u) \leqq A\Phi(u)$, satisfying*

$$\max\left\{\|f+g\|_{O(\widetilde{\Phi})}, \|f-g\|_{O(\widetilde{\Phi})}\right\} \geqq \left(\|f\|^s_{O(\widetilde{\Phi})} + m\|g\|^s_{O(\widetilde{\Phi})}\right)^{1/s}, \tag{3.10}$$

for all $f, g \in \mathcal{M}(\Phi)$.

PROOF. The complementary Young function Ψ satisfies the Δ_2 condition, and $\Psi(t^{1/q})$ is convex for $\frac{1}{q} + \frac{1}{s} = 1$ by the previous lemma. Thus, we can apply Lemma 2.2 of [16] for $B = O_L(\Psi)$ and $M = \Psi$, to find a Young function $N = \widetilde{\Psi}$, equivalent to Ψ such that

$$\frac{\|f+g\|_{O_L(\widetilde{\Psi})} + \|f-g\|_{O_L(\widetilde{\Psi})}}{2} \leqq \left(\|f\|^q_{O_L(\widetilde{\Psi})} + L\|g\|^q_{O_L(\widetilde{\Psi})}\right)^{1/q}, \tag{3.11}$$

for all $f, g \in \mathcal{M}(\Psi)$ with $L > 0$. Let $\widetilde{\Phi}$ be the complementary Young function of $\widetilde{\Psi}$. The Young function $\widetilde{\Phi}$ is equivalent to Φ ([19, Proposition 2, p. 15]) and the dual of $O_L(\widetilde{\Psi})$ is isometric to $O(\widetilde{\Phi})$ ([19, Corollary 9, p. 111]). Hence, using Theorem 3.1, (3.11) implies (3.10). □

Now we will show examples of Young functions Φ for which there exists an equivalent Young function $\widetilde{\Phi}$ such that $\widetilde{\Phi}(u^{1/s})$ is concave for some s, $2 \leqq s < \infty$, and which satisfies the ∇_2 condition (consequently, the corresponding Orlicz spaces will satisfy (1.2)).

We intend to consider $\Phi(u)=u^r\left(1+|\ln u|\right)$ and $\Phi(u)=\max\{u^\alpha,u^\beta\}$ for appropriate values of r, α, β. Note that these functions themselves (being convex) cannot satisfy the condition that $g(u)\equiv\Phi(u^{1/s})$ is concave for the following reason: $\Phi'_+(1)>\Phi'_-(1)$, and hence $g'_+(1)>g'_-(1)$. However, with proper s, g can be concave near 0 and near ∞. Our task is to "patch" these pieces together to construct an equivalent function $\widetilde{\Phi}$ satisfying the necessary conditions.

Lemma 3.8. *Let Φ be a Young function such that*

$$\Phi\left(u^{1/s}\right) \text{ is concave on } [0,a] \text{ and on } [b,\infty), \tag{3.12}$$

where $0<a<b$, $s\geqq 2$. Then there is a Young function $\widetilde{\Phi}$ satisfying

$$\widetilde{\Phi}(u)=c_1\Phi(u),\quad u\in[0,a], \tag{3.13}$$

and

$$\widetilde{\Phi}(u)=c_2+\Phi(u),\quad u\in[b,\infty), \tag{3.14}$$

with some constants $c_1>0$ and c_2, which is equivalent to $\Phi(u)$ and also $\widetilde{\Phi}(u^{1/s})$ is concave on $[0,\infty)$.

Proof. As Φ is convex, it is absolutely continuous and Φ' exists almost everywhere and is non-decreasing. We choose c_1 to satisfy

$$c_1\Phi'(a-)a^{\frac{1}{s}-1}=\Phi'(b+)b^{\frac{1}{s}-1}.$$

Now define

$$\phi(u):=\begin{cases} c_1\Phi'(u), & u\in[0,a),\\ u^{1-\frac{1}{s}}\Phi'(b+)b^{\frac{1}{s}-1}, & u\in[a,b],\\ \Phi'(u), & u\in(b,\infty),\end{cases}$$

and $\widetilde{\Phi}(x):=\int_0^x\phi(u)\,du$. Clearly, (3.13) and (3.14) are satisfied. Also, as $\phi(a)=c_1\Phi'(a-)$, $\phi(b)=\Phi'(b+)$ and ϕ is increasing on $[a,b]$, $\widetilde{\Phi}$ is a Young function. For $u\in[a,b]$ we obtain

$$\left(\widetilde{\Phi}(u^{1/s})\right)'=\frac{1}{s}\phi(u)u^{\frac{1}{s}-1}=\frac{1}{s}\Phi'(b+)b^{\frac{1}{s}-1}=\text{const}$$

$$=\left(\widetilde{\Phi}(u^{1/s})\right)'\Big|_{u=a-}=\left(\widetilde{\Phi}(u^{1/s})\right)'\Big|_{u=b+}.$$

Hence, $\left(\widetilde{\Phi}(u^{1/s})\right)'$ is non-increasing on $[0,\infty)$.

We observe that the resulting Young function $\widetilde{\Phi}$ is equivalent to Φ. $\square$

Example 3.9. Let $\Phi(u) = \max\{u^\alpha, u^\beta\}$, where $1 < \alpha < \beta$. Then $\Phi(u^{1/s})$ satisfies (3.12) for any $s \geqq \max\{2, \beta\}$.

Proof. We have

$$\Phi(u^{1/s}) = \begin{cases} u^{\alpha/s}, & u \leqq 1, \\ u^{\beta/s}, & u > 1, \end{cases}$$

so both α/s and β/s must not exceed 1. □

Example 3.10. Let $\Phi(u) = u^r(1 + |\ln u|)$, $r \geqq (3 + \sqrt{5})/2$ (which guarantees that Φ is a Young function). Then $\Phi(u^{1/s})$ satisfies (3.12) for any $s > r$ and does not satisfy (3.12) with $s = r$.

Proof. We have

$$\Phi'(u) = \begin{cases} u^{r-1}(r + 1 + r \ln u), & u > 1, \\ u^{r-1}(r - 1 - r \ln u), & u < 1, \end{cases}$$

and

$$\Phi''(u) = \begin{cases} u^{r-2}(r^2 + r - 1 + r(r-1)\ln u), & u > 1, \\ u^{r-2}(r^2 - 3r + 1 - r(r-1)\ln u), & u < 1. \end{cases}$$

Hence, $r \geqq (3 + \sqrt{5})/2$ implies convexity of Φ. We further compute

$$\left(\Phi(u^{\frac{1}{s}})\right)' = \begin{cases} \frac{1}{s} u^{\frac{r}{s}-1}\left(r + 1 + \frac{r}{s}\ln u\right), & u > 1, \\ \frac{1}{s} u^{\frac{r}{s}-1}\left(r - 1 - \frac{r}{s}\ln u\right), & u < 1, \end{cases}$$

and

$$\left(\Phi(u^{\frac{1}{s}})\right)'' = \begin{cases} \frac{1}{s} u^{\frac{r}{s}-2}\left(\frac{r}{s}(r+2) - r - 1 + \frac{r}{s}\left(\frac{r}{s} - 1\right)\ln u\right), & u > 1, \\ \frac{1}{s} u^{\frac{r}{s}-2}\left(\frac{r}{s}(r-2) - r + 1 - \frac{r}{s}\left(\frac{r}{s} - 1\right)\ln u\right), & u < 1. \end{cases}$$

Under the condition $s > r$, the function $\left(\Phi(u^{\frac{1}{s}})\right)''$ is clearly non-positive for $u < 1$ and also non-positive for $u > u_0$, where u_0 is such that $\frac{r}{s}\left(\frac{r}{s} - 1\right)\ln u_0 = -1$. If $r = s$, then $\left(\Phi(u^{\frac{1}{s}})\right)'' = 1$ for $u > 1$. □

Example 3.11. The Zygmund spaces $L_p(\operatorname{Log} L)^\alpha$. Let $\Phi(u) = u^p(\ln(2+u))^{\alpha p}$, $\alpha p \geqq 1$, $p \geqq 1$ (see [2, Definition 6.11, p. 252]). Then $\Phi(u^{1/s})$ satisfies (3.12) for any $s > p$ and does not satisfy (3.12) with $s = p$.

PROOF. We find

$$\Phi'(u) = pu^{p-1} \ln^{\alpha p-1}(2+u) \left(\ln(2+u) + \frac{\alpha u}{2+u} \right),$$

and hence,

$$\left(\Phi\left(u^{\frac{1}{s}}\right)\right)' = \frac{p}{s} u^{\frac{p}{s}-1} \ln^{\alpha p-1}(2+u^{\frac{1}{s}}) \left(\ln\left(2+u^{\frac{1}{s}}\right) + \frac{\alpha u^{\frac{1}{s}}}{2+u^{\frac{1}{s}}} \right).$$

Differentiating once more, we obtain

$$(3.15) \qquad \left(\Phi\left(u^{\frac{1}{s}}\right)\right)''$$

$$= \frac{p}{s} u^{\frac{p}{s}-2} \ln^{\alpha p-1}(2+u^{\frac{1}{s}}) \left(\ln\left(2+u^{\frac{1}{s}}\right) + \frac{\alpha u^{\frac{1}{s}}}{2+u^{\frac{1}{s}}} \right) \left(\frac{p}{s} - 1 + D(u) \right),$$

where

$$D(u) = \frac{u^{\frac{1}{s}}}{s(2+u^{\frac{1}{s}})} \left(\frac{\alpha p - 1}{\ln\left(2+u^{\frac{1}{s}}\right)} + \frac{1 + \frac{2\alpha}{2+u^{\frac{1}{s}}}}{\ln\left(2+u^{\frac{1}{s}}\right) + \frac{\alpha u^{\frac{1}{s}}}{2+u^{\frac{1}{s}}}} \right).$$

The sign of $\left(\Phi\left(u^{\frac{1}{s}}\right)\right)''$ for $u > 0$ is determined by the sign of the last factor of the right hand side of (3.15). As $D(u)$ is positive for $u > 0$, if $s = p$, then $\left(\Phi\left(u^{\frac{1}{s}}\right)\right)'' > 0$. If $s > p$, then $\frac{p}{s} - 1 < 0$ and we can choose a, b to satisfy (3.12) since

$$\lim_{u \to 0+} D(u) = 0 \quad \text{and} \quad \lim_{u \to \infty} D(u) = 0. \qquad \square$$

Note that in all of the above examples it is easy to verify that Φ satisfies the Δ_2 and the ∇_2 conditions.

4. Applications using holomorphic semigroups

The operators $T(t)$, $T(t) : B \to B$ for $t \in [0, \infty) = \mathbb{R}_+$ form a C_0 semigroup if $T(t+s)f = T(t)T(s)f$ and $\lim_{t \to 0+} \|T(t)f - f\|_B = 0$. $\{T(t)\}_{t \geqq 0}$ is a semigroup of contractions if $\|T(t)f\|_B \leqq \|f\|_B$ for all $t \in \mathbb{R}_+$ and $f \in B$. The infinitesimal generator $\mathcal{A}$ related to the semigroup $\{T(t)\}_{t \geqq 0}$ is given by

$$(4.1) \qquad \mathcal{A}f \equiv \text{s-}\lim_{t \to 0+} \frac{T(t)f - f}{t},$$

(where $\operatorname{s-lim}_{t\to 0+} g_t = \varphi$ if $\|g_t - \varphi\|_B \to 0$ as $t \to 0+$) and the domain $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ consists of all f such that the limit in (4.1) exists. A holomorphic semigroup is a semigroup satisfying

$$(4.2)\qquad T(t)f \in \mathcal{D}(\mathcal{A}) \quad \text{for} \quad t > 0 \quad \text{and} \quad t\big\|\mathcal{A}T(t)f\big\|_B \leqq N\|f\|_B$$

with N independent of t and f. (Note that (4.2) is essentially a Bernstein-type inequality.)

It was proved (see [15, Theorem 5.1, p. 74]) that for a holomorphic C_0 semigroup of contractions we have

$$(4.3)\quad \big\|\big(T(t)-I\big)^r f\big\|_B \approx \inf_{g\in\mathcal{D}(\mathcal{A}^r)} \big(\|f-g\|_B + t^r\|\mathcal{A}^r g\|_B\big) \equiv K_{\mathcal{A}^r}(f,t^r)_B,$$

which is a strong converse inequality of type A in the terminology of [15]. We recall that by $A(t) \approx E(t)$ one means $C^{-1}A(t) \leqq E(t) \leqq CA(t)$. Using (4.3) and general properties of K-functionals, we have for holomorphic semigroups

$$(4.4)\qquad \big\|\big(T(t)-I\big)^r f\big\|_B \approx \sup_{0<u\leqq t} \big\|\big(T(u)-I\big)^r f\big\|_B.$$

As a corollary of Theorem 2.1 and (4.3) we obtain the following result.

THEOREM 4.1. *Suppose that* $\{T(t)\}_{t\geqq 0}$ *is a holomorphic* C_0 *semigroup of contractions on a Banach space* B *and that* B *satisfies the condition* (1.2) *for some* $2 \leqq s < \infty$ *and* $m > 0$. *Then for any integer* r

$$(4.5)\qquad K_{\mathcal{A}^r}(f,t^r)_B \geqq C\bigg\{\sum_{j=1}^{\infty} 2^{-jrs} K_{\mathcal{A}^{r+1}}\big(f, 2^{j(r+1)}t^{r+1}\big)_B^s\bigg\}^{1/s}.$$

PROOF. We use (2.1) with $T = T(t)$ and $T^{2^\ell} = T\big(t2^\ell\big)$, to which we apply (4.3) (for both r and $r+1$), which yields

$$\big\|\big(T(t)-I\big)^r f\big\|_B \leqq C_1 K_{\mathcal{A}^r}(f,t^r)_B$$

and $\big\|\big(T(t2^j)-I\big)^{r+1} f\big\|_B \geqq C_2 K_{\mathcal{A}^r}\big(f, 2^{j(r+1)}t^{r+1}\big)_B$ to complete the proof of (4.5). $\square$

The usefulness of Theorem 4.1 is clearly demonstrated by applying it to the Gauss–Weierstrass semigroup of operators (see for instance [3, p. 261]) given by

$$(4.6)\qquad W(t)f(x) \equiv \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \exp\left(\frac{-|x-u|^2}{4t}\right) f(u)\,du.$$

THEOREM 4.2. *Suppose that B, a Banach space of functions on $\mathbb{R}^d$, satisfies (1.2) and (1.3) and that $B \subset \mathcal{S}'$ which means that B is continuously imbedded in the Schwartz space of tempered distribution. Then*

$$\big\|(W(t)-I)^r f\big\|_B \approx \sup_{u \leqq t} \big\|(W(u)-I)^r f\big\|_B \tag{4.7}$$

$$\approx \inf_{g\in\mathcal{D}(\Delta^r)} \left(\|f-g\|_B + t^r\|\Delta^r g\|_B\right) \equiv K_{\Delta^r}(f,t^r)_B,$$

$$K_{\Delta^r}(f,t^r)_B \geqq C\Big\{\sum_{j=1}^{\infty} 2^{-jrs} K_{\Delta^{r+1}}\big(f,2^{j(r+1)}t^{r+1}\big)_B^s\Big\}^{1/s}, \tag{4.8}$$

$$K_{\Delta^r}(f,t^r)_B \geqq C\Big\{\sum_{j=1}^{\infty} 2^{-jrs} E_{2^{j/2}t^{1/2}}(f)_B^s\Big\}^{1/s} \tag{4.9}$$

where Δ is the Laplacian, and

$$E_\lambda(f)_B = \inf\big\{\|f-\varphi_\sigma\|_B : \operatorname{supp}\widehat{\varphi}_\sigma(y) \subset (y : |y| \leqq \lambda)\big\}, \tag{4.10}$$

with $\widehat{\varphi}_\sigma$ as the Fourier transform of φ_σ.

PROOF. For $f \in B$ satisfying (1.3) we may use the Riemann vector valued integration in (4.6) to obtain for all $f \in B$

$$\begin{cases} \|W(t)f\|_B \leqq \|f\|_B, & \lim_{t\to 0+} \|W(t)f - f\|_B = 0, \\ \Delta^r W(t)f \in B \quad \text{and} & \|\Delta^r W(t)f\|_B \leqq \left(\dfrac{d}{t}\right)^r \|f\|_B. \end{cases} \tag{4.11}$$

For $\varphi \in \mathcal{S}$, the Schwartz space of test functions, straightforward computation implies $\frac{W(t)\varphi-\varphi}{t} - \Delta\varphi \to 0$ in $\mathcal{S}$ and hence in B^*, the dual to B. Therefore, whenever $f \in \mathcal{D}(\Delta)$, that is when Δf exists in the $\mathcal{S}'$ sense and $\Delta f \in B$, we have

$$\left\langle \frac{W(t)f-f}{t} - \Delta f, \varphi \right\rangle = \left\langle f, \quad \frac{W(t)\varphi-\varphi}{t} - \Delta\varphi \right\rangle \to 0$$

for all $\varphi \in \mathcal{S}$. For $f \in \mathcal{D}(\mathcal{A})$ with $\mathcal{A}$ the infinitesimal generator of $W(t)$,

$$\left\|\frac{W(t)f-f}{t} - \mathcal{A}f\right\|_B \to 0 \quad \text{and} \quad \left\|\frac{W(t+s)f-W(s)f}{t} - \mathcal{A}W(s)f\right\|_B \to 0.$$

As $W(s)f \in \mathcal{D}(\Delta)$ and $W(s)f \in \mathcal{D}(\mathcal{A})$, we have

$$\lim_{t\to 0+} \left\langle \frac{W(t+s)f-W(s)f}{t} - \Delta W(s)f, \varphi \right\rangle = 0$$

and

$$\lim_{t\to 0+}\left\langle \frac{W(t+s)f-W(s)}{t}-\mathcal{A}W(s)f,\varphi\right\rangle=0$$

for all $\varphi\in\mathcal{S}\cap B^*$, and hence $\Delta W(s)f=\mathcal{A}W(s)f$ for all $s>0$.

For $f\in\mathcal{D}(\Delta)$ we can now write

$$\|W(t+s)f-W(s)f\|_B\leqq t\|\mathcal{A}W(s)f\|_B=t\|\Delta W(s)f\|_B\leqq t\|\Delta f\|_B$$

for all $s>0$, and using (4.11), $\|W(t)f-f\|_B\leqq t\|\Delta f\|_B$.

Similarly, for $g\in\mathcal{D}(\Delta^r)$

$$\big\|\big(W(t)-I\big)^r g\big\|_B\leqq t^r\|\Delta^r g\|_B.$$

The above directly implies the inequality $\big\|\big(W(t)-I\big)^r f\big\|_B\leqq CK_{\Delta^r}(f,t^r)_B$. The proof of the inequality $K_{\Delta^r}(f,t^r)_B\leqq C\big\|\big(W(t)-I\big)^r f\big\|_B$ follows exactly that of Theorem 5.1 in [15], replacing $\mathcal{A}$ by Δ, which as it operates on $g=-\sum_{k=1}^{r}(-1)^k\binom{r}{k}T(kmt)f$, is the same.

For $B=L_p(\mathbb{R}^d)$, $1\leqq p<\infty$, $\Delta=\mathcal{A}$ (see the proof in [3, Theorem 4.3.11, p. 261]).

The inequality (4.8) now follows from Theorem 4.1 as (4.7) implies $K_{\mathcal{A}^r}(f,t^r)_B\approx K_{\Delta^r}(f,t^r)_B$ for all f, r and t. (4.9) follows from (4.8) and the inequality

$$E_\lambda(f)_B\leqq CK_{\Delta^r}(f,\lambda^{-2r})_B. \tag{4.12}$$

The latter was proved in [4, (2.9), p. 271] for $B=L_p(\mathbb{R}^d)$, $1\leqq p\leqq\infty$. In fact, (4.12) follows for any B satisfying (1.3), as all we need in the proof of [4, pp. 271–272] is that the linear convolution operators $R_{\lambda,\ell,b}f$ there satisfy

$$\|R_{\lambda,\ell,b}f\|_B\leqq C_1\|f\|_B\quad\text{and}\quad\|R_{\lambda,\ell,b}g-g\|_B\leqq C_2\frac{1}{\lambda^{2\ell}}\|\Delta^\ell g\|_B. \tag{4.13}$$

We define $F=f*\varphi$ with $\varphi\in B^*$ and $\|\varphi\|_{B^*}=1$ such that φ satisfies

$$\|R_{\lambda,\ell,b}f\|_B-\varepsilon\leqq\big|R_{\lambda,\ell,b}F(0)\big|\leqq\sup_x\big|R_{\lambda,\ell,b}F(x)\big|$$

$$=\|R_{\lambda,\ell,b}F\|_\infty\leqq C_1\|F\|_\infty\leqq C_1\|f\|_B.$$

Similarly, we obtain the second inequality of (4.13) using $G=g*\varphi$. □

REMARK 4.3. For $L_p(\mathbb{R}^d)$ a somewhat more general result than in Theorem 4.2 was proved in [7, Theorem 7.1] using a completely different method.

Here the proof is much simpler and applies to a wide class of Orlicz spaces (see Section 3), and perhaps to other spaces that satisfy (1.2) with some norm of B that satisfies (1.3) at the same time. Orlicz spaces described in Section 3 satisfy (1.2) with the same norm for which (1.3) is valid.

REMARK 4.4. Using the monotonicity of $K_{\Delta^{r+1}}(f,u)_B$ and of $E_u(f)_B$, one can obtain the following equivalent form of (4.8) and (4.9), which may appear more traditional:

$$(4.8)' \qquad K_{\Delta^r}(f,t^r)_B \geqq Ct^r \left\{ \int_t^\infty u^{-rs} K_{\Delta^{r+1}}\big(f,u^{r+1}\big)_B^s \frac{du}{u} \right\}^{1/s}$$

and

$$(4.9)' \qquad K_{\Delta^r}(f,t^r)_B \geqq Ct^r \left\{ \int_{t^{1/2}}^\infty u^{-2rs} E_{u^{-1}}(f)_B^s \frac{du}{u} \right\}^{1/s}.$$

For $B = L_p(\mathbb{R}^d)$, $1 < p < \infty$, $\omega^r(f,t)_p$ and $K_{\Delta^r}(f,t^r)_p$ given by (1.5) and (4.7) respectively satisfy $\omega^{2r}(f,t)_p \approx K_{\Delta^r}\big(f,t^{2r}\big)_p$ and hence (4.8) can take the form

$$(4.8)'' \quad \omega^{2r}(f,t)_p \geqq Ct^{2r} \left\{ \int_t^\infty u^{-2rs} \omega^{2r+2}(f,u)_p^s \frac{du}{u} \right\}^{1/s}, \quad s = \max(p,2).$$

In fact, the result of Theorem 4.2 is given as an example of use of Theorem 4.1, and the same method can be used for many semigroups that are given by positive convolution operators on $\mathbb{R}^d$ or $\mathbb{T}^d$, $d = 1, 2, \dots$.

In the next section we will give applications relating to holomorphic semigroups generated by multipliers.

5. Cesàro summability and holomorphic semigroups

For the purpose of this section, H_k are eigenspaces of a self-adjoint operator $P(D)$, and λ_k the eigenvalues of $P(D)$, satisfy $0 \leqq \lambda_k$, $\lambda_k < \lambda_{k+1}$. Furthermore, for our space B we assume that $H_k \subset B$, $H_k \subset B^*$ and that $\operatorname{span}(\cup H_k)$ is dense in B. The expansion of f is given by

$$(5.1) \qquad f \sim \sum_{k=0}^\infty P_k f$$

where $P_k f$ is the projection of f on H_k in the L_2 sense (see [12, (2.2)]). It was shown in [5] that if the Cesàro summability of some order ℓ is a contraction in B, that is

$$(5.2) \qquad \|C_n^\ell f\|_B \leqq \|f\|_B$$

for

$$C_n^\ell f \equiv \frac{1}{A_n^\ell} \sum_{k=0}^{\infty} A_{n-k}^\ell P_k f \quad \text{where} \quad A_m^\ell \equiv \frac{(m+\ell)!}{\ell! m!},$$

then $T(t)f$ given by

$$T(t)f = \sum_{k=0}^{\infty} e^{-kt} P_k f \quad \text{for} \quad t > 0 \quad \text{and} \quad T(0)f = f \tag{5.3}$$

is a holomorphic C_0 semigroup of contractions with its infinitesimal generator given by

$$\mathcal{A}f \sim \sum_{k=1}^{\infty} -kP_k f, \tag{5.4}$$

$$\mathcal{D}(\mathcal{A}) = \{f \in B : \exists\, g \in B \text{ such that } P_k g = -kP_k f \text{ for all } k\}.$$

The following theorem will establish among other facts that the positivity of $C_n^\ell f$ implies that it is a contraction in Orlicz spaces with the Luxemburg norm as well as with the Orlicz norm. We remind the reader that if an operator is a contraction on a space with respect to a given norm, it does not imply that it is a contraction with an equivalent norm.

THEOREM 5.1. *Suppose $Of(x)$ is given by*

$$Of(x) = \int_\Omega f(y) G(x,y) w(y)\, dy \tag{5.5}$$

where $G(x,y) = G(y,x) \geqq 0$, $w(y) \geqq 0$ and $\int_\Omega G(x,y)w(y)\,dy = 1$. Then Of is a contraction with respect to the Luxemburg norm given by

$$\|f\|_{O_L(\Phi)} = \inf\left\{a \in \mathbb{R}_+ : \int_\Omega \Phi\left(\frac{|f(x)|}{a}\right) w(x)\,dx \leqq 1\right\} \tag{5.6}$$

and with respect to the Orlicz norm given by

$$\|f\|_{O(\Phi)} = \sup\left\{\int_\Omega |f(x)g(x)|\,w(x)\,dx : \int_\Omega \Psi\big(|g(x)|\big) w(x)\,dx \leqq 1\right\} \tag{5.7}$$

where Φ and Ψ are associate Young functions.

PROOF. For $a \in \mathbb{R}_+$, which is close to the infimum in (5.6), we write

$$\int_\Omega \Phi\left(\left|\frac{1}{a} Of(x)\right|\right) w(x)\,dx \leqq \int_\Omega \Phi\left(\frac{1}{a}\int_\Omega |f(y)|\,G(x,y)w(y)\,dy\right) w(x)\,dx \equiv I.$$

Using Jensen's inequality, the convexity of Φ and $\int G(x,y)w(y)\,dy = 1$, we have

$$I \leqq \int_\Omega \int_\Omega \Phi\left(\frac{|f(y)|}{a}\right) G(x,y)w(y)\,dy\,w(x)\,dx$$

$$= \int_\Omega \Phi\left(\frac{|f(y)|}{a}\right) w(y)\,dy \int_\Omega G(x,y)w(x)\,dx = \int_\Omega \Phi\left(\frac{|f(y)|}{a}\right) w(y)\,dy,$$

which completes the proof for the Luxemburg norm of the Orlicz space. We now write

$$\int_\Omega |Of(x)|\,|g(x)|\,w(x)\,dx \leqq \int_\Omega |g(x)| \int_\Omega G(x,y)|f(y)|\,w(y)\,dy\,w(x)\,dx$$

$$= \int_\Omega |f(y)|\,w(y) \int_\Omega |g(x)|\,G(x,y)w(x)\,dx\,dy.$$

As Ψ is also a Young function and is convex, we have

$$\int_\Omega \Psi\left\{\int_\Omega |g(x)|\,G(x,y)w(x)\,dx\right\} w(y)\,dy \leqq \int_\Omega \Psi\big(|g(y)|\big)w(y)\,dy \leqq 1,$$

and hence our result follows. □

For $L_p(\Omega)$ with weight $w(x) \geqq 0$ the proof is easier as it follows directly from Hölder's inequality, but the result for L_p is included in the more intricate proof of Theorem 5.1.

Clearly, the positivity of the Cesàro summability in the above context implies that

$$C_n^\ell f(x) = \int_\Omega f(y)G_{n,\ell}(x,y)w(y)\,dy$$

where $G_{n,\ell}(x,y) = G_{n,\ell}(y,x)$, $G_{n,\ell}(x,y) \geqq 0$, $w(y) \geqq 0$, and when $1 \in H_0$, also $\int G_{n,\ell}(x,y)w(y)\,dy = 1$.

THEOREM 5.2. *Suppose H_k, λ_k and $P_k f$ are as described at the beginning of this section, B is an Orlicz space which satisfies* (1.2) (*for some s, $2 \leqq s < \infty$*) *with a Luxemburg norm or Orlicz norm, C_n^ℓ is positive for some ℓ, $1 \in H_0$ and λ_k is a polynomial in k of degree b. Then*

$$K_{\mathcal{A}^r}(f,t^r)_B \approx \inf\left\{\|f-g\|_B + t^r \left\|P(D)^{r/b}g\right\|_B : g \in \mathcal{D}\big(P(D)^{r/b}\big)\right\}, \tag{5.8}$$

$$K_{\mathcal{A}^r}(f,t^r)_B \geqq C\left\{\sum_{j=1}^\infty 2^{-jrs}K_{\mathcal{A}^{r+1}}\big(f,t^{r+1}2^{j(r+1)}\big)_B^s\right\}^{1/s} \tag{5.9}$$

and

$$K_{\mathcal{A}^r}\big(f, 2^{-nr}\big)_B \geqq C\Big\{\sum_{j=1}^{n} 2^{-jrs} E_{2^{n-j}}(f)_B^s\Big\}^{1/s} \tag{5.10}$$

where

$$E_n(f) = \inf\Big\{\|f-\varphi\|_B : \varphi \in \operatorname{span}\bigcup_{k=0}^{n} H_k\Big\}.$$

PROOF. The proof of (5.8) follows the proof in [6, Theorem 4.3, p. 83] where the result is proved for L_p spaces. In fact, the same proof works for Banach spaces B for which some order of the Cesàro summability is bounded, which implies the realization result (see [12, Theorem 6.2 and Theorem 7.1]), and that result is the key ingredient for the proof in [6, Theorem 4.3]. We now show

(5.11)

$$E_n(f)_B \leqq C \inf\Big(\|f-g\|_B + n^{-r-1}\big\|P(D)^{(r+1)/b} g\big\|_B\Big) \approx K_{\mathcal{A}^{r+1}}\big(f, n^{-r-1}\big)_B.$$

The first inequality of (5.11) follows from [12, Theorem 4.1] when we recall that $\lambda_k \geqq 0$, and $\lambda_k \approx k^b$ (essentially $P(D)$, $(r+1)/b$ and n^b are $-P(D)$, αm and λ respectively in [12]). The second equivalence is treated in detail in [6, Section 4]. Using (5.11), we may deduce (5.10) from (5.9), which in turn is a direct application of (4.5). $\square$

REMARK 5.3. Similar to what we stated in Remark 4.4, one can give a different form of (5.9) and (5.10). For example, we have

$$K_{\mathcal{A}^r}(f, t^r)_B \geqq C t^r\Big\{\sum_{n=1}^{[1/t]} n^{-rs-1} E_n(f)_B^s\Big\}^{1/s}. \tag{5.10$'$}$$

6. Sharp Jackson theorem for polynomials on a simplex

For the simplex $S \in \mathbb{R}^d$

$$S = \Big\{\mathbf{x} = (x_1, \dots, x_d) : x_i \geqq 0 \quad x_0 = 1 - \sum_{i=1}^{d} x_i \geqq 0\Big\}, \tag{6.1}$$

the Jacobi weight is given by

$$W_{\boldsymbol{\alpha}}(\mathbf{x}) = x_0^{\alpha_0} \dots x_d^{\alpha_d}, \quad \boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_d), \quad \alpha_i > -\frac{1}{2}. \tag{6.2}$$

The self-adjoint differential operator (see [11, p. 226]) on S with weight $w_{\boldsymbol{\alpha}}(\mathbf{x})$ is given by

$$(6.3)\qquad P_{\boldsymbol{\alpha}}(D)=-\sum_{\boldsymbol{\xi}\in E_S} w_{\boldsymbol{\alpha}}(\mathbf{x})^{-1}\frac{\partial}{\partial\boldsymbol{\xi}}\widetilde{d}(\boldsymbol{\xi},\mathbf{x})w_{\boldsymbol{\alpha}}(\mathbf{x})\frac{\partial}{\partial\boldsymbol{\xi}}\equiv\sum_{\boldsymbol{\xi}\in E_S}P_{\boldsymbol{\alpha},\boldsymbol{\xi}}(D)$$

where E_S is the set of directions parallel to the edges of S, and $\widetilde{d}(\boldsymbol{\xi},\mathbf{x})$ is given by

$$(6.4)\qquad \widetilde{d}(\boldsymbol{\xi},\mathbf{x})=\sup_{\substack{\lambda\geqq 0\\ \mathbf{x}+\lambda\boldsymbol{\xi}\in S}} d(\mathbf{x},\mathbf{x}+\lambda\boldsymbol{\xi})\sup_{\substack{\lambda\geqq 0\\ \mathbf{x}-\lambda\boldsymbol{\xi}\in S}} d(\mathbf{x},\mathbf{x}-\lambda\boldsymbol{\xi})$$

using the Euclidean distance $d(\mathbf{x},\mathbf{y})$.

For Π_k the polynomials of total degree $\leqq k$ we have $\Pi_k=H_0\oplus\ldots\oplus H_k$ where

$$(6.5)\qquad P_{\boldsymbol{\alpha}}(D)\varphi=\ell\Big(\ell+d+\sum_{i=0}^{d}\alpha_i\Big)\varphi\equiv\lambda_\ell\varphi,\quad \varphi\in H_\ell.$$

Defining the K-functional on S by

$$(6.6)\qquad K_r\big(f,P_{\boldsymbol{\alpha}}(D)^{r/2},t^r\big)_B$$
$$=\inf\big(\|f-g\|_B+t^r\big\|P_{\boldsymbol{\alpha}}(D)^{r/2}g\big\|_B:\ g\in\mathcal{D}\big(P_{\boldsymbol{\alpha}}(D)^{r/2}\big)\big)$$

where for $\beta\in[0,\infty)$ $P_{\boldsymbol{\alpha}}(D)^\beta$ is given for $\beta>0$ by

(6.7)
$$P_{\boldsymbol{\alpha}}(D)^\beta f=\sum_{\ell=1}^{\infty}\lambda_\ell^\beta P_\ell f,\quad f\in\mathcal{D}\big(P_{\boldsymbol{\alpha}}(D)^\beta\big)\quad\text{if}\quad\exists\,\psi\in B\quad P_\ell\psi=\lambda_\ell^\beta P_\ell f$$

with $P_\ell\varphi$ the L_2 projection of φ onto H_ℓ. We can now deduce the sharp Jackson inequality for polynomials and lower estimate for K-functionals on the simplex.

THEOREM 6.1. *Suppose B is a weighted L_p or an Orlicz space on the simplex S satisfying* (1.2) *for some $2\leqq s<\infty$. Then*

$$(6.8)\qquad K_r\big(f,P_{\boldsymbol{\alpha}}(D)^{r/2},t^r\big)_B$$
$$\geqq C\Big\{\sum_{j=1}^{\infty}2^{-jrs}K_{r+1}\big(f,P_{\boldsymbol{\alpha}}(D)^{(r+1)/2},t^{r+1}2^{j(r+1)}\big)_B^s\Big\}^{1/s}$$

and

$$K_r\big(f, P_{\boldsymbol{\alpha}}(D)^{r/2}, 2^{-nr}\big)_B \geqq C_1 2^{-nr}\Big\{\sum_{j=0}^{n} 2^{jrs} E_{2^j}(f)_B^s\Big\}^{1/s} \tag{6.9}$$

where S, $P(D)$, $K_r\big(f, P_{\boldsymbol{\alpha}}(D)^{r/2}, t\big)_B$ *and* $P_{\boldsymbol{\alpha}}(D)^{r/2}$ *are given by* (6.1), (6.3), (6.6) *and* (6.7) *respectively and* $E_n(f)_B$ *is given by*

$$E_n(f)_B = \inf\big(\|f - P\|_B : P \in \Pi_n\big). \tag{6.10}$$

Proof. We follow [17, Corollary 7.4.2, p. 273], which implies the positivity of the Cesàro summability C_n^δ, provided that δ is large enough. The use of Theorem 5.2 will complete the proof of (6.8), when we recall that $\lambda_\ell = \ell\big(\ell + d + \sum_{i=0}^{d} \alpha_i\big)$ is a polynomial of degree $b = 2$ in ℓ. The proof of (6.9) follows from the boundedness of the Cesàro summability which implies (see [12, Theorem 6.1])

$$E_n(f)_B \leqq C K_{r+1}\big(f, P_{\boldsymbol{\alpha}}(D)^{(r+1)/2}, n^{-r-1}\big)_B \tag{6.11}$$

and hence (6.9) can be deduced from (6.8). $\square$

For $d = 1$ and $B = L_p$ with Jacobi weights, Theorem 6.1 was proved in [7, Theorem 6.1].

7. Sharp Jackson inequality on the sphere

The result of this section was proved for $L_p(S^{d-1})$, $1 < p < \infty$ in [7, Theorem 8.1]. Here we will give an alternative proof which yields an extension to a class of Banach spaces that include many Orlicz spaces.

The Laplace–Beltrami operator $\widetilde{\Delta}$ on the unit sphere $S^{d-1} = \big\{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}|^2 \equiv x_1^2 + \dots + x_d^2 = 1\big\}$ is given by

$$\widetilde{\Delta} f(\mathbf{x}) = \Delta F(\mathbf{x}), \quad \mathbf{x} \in S^{d-1} \quad \text{where} \tag{7.1}$$

$$F(\mathbf{x}) = f\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \quad \text{and} \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2}.$$

The eigenspace H_k of spherical harmonic polynomials of degree k on S^{d-1} is given by

$$H_k = \big\{\varphi : \widetilde{\Delta}\varphi = -k(k + d - 2)\varphi\big\}, \quad \lambda_k \equiv k(k + d - 2). \tag{7.2}$$

For a Banach space of functions on S^{d-1} the K-functional is given by

(7.3)
$$K_r\big(f,(-\widetilde{\Delta})^{r/2},t^r\big)_B=\inf\Big\{\|f-g\|_B+t^r\big\|(-\widetilde{\Delta})^{r/2}g\big\|_B:\,g\in\mathcal{D}\big((-\widetilde{\Delta})^{r/2}\big)\Big\}$$

where

$$(-\widetilde{\Delta})^{r/2}f\sim\sum_{\ell=1}^{\infty}\big(\ell(\ell+d-2)\big)^{r/2}P_\ell f, \tag{7.4}$$

$f\in\mathcal{D}\big((-\widetilde{\Delta})^{r/2}\big)$ if $\exists\,\psi\in B$ satisfying $P_\ell\psi=\lambda_\ell^{r/2}P_\ell f$ and $P_\ell\varphi$ is the L_2 projection of f on H_ℓ.

We can now state and prove the result of this section.

THEOREM 7.1. *Suppose that B is an Orlicz space of functions on S^{d-1} satisfying* (1.2) *for some $2\leqq s<\infty$, and for $\rho\in SO(d)$*

$$\big\|f(\rho\,\cdot)\big\|_B=\big\|f(\,\cdot\,)\big\|_B,\quad \big\|f(\rho\,\cdot)-f(\,\cdot\,)\big\|_B\to 0\quad \textit{as}\quad |\rho-I|\to 0, \tag{7.5}$$

where $|\rho-I|=\max\big\{|\rho\mathbf{x}-\mathbf{x}|:\,\mathbf{x}\in S^{d-1}\big\}$. Then for $r=1,2,\dots$

(7.6)
$$K_r\big(f,(-\widetilde{\Delta})^{r/2},t^r\big)_B\geqq C\Big\{\sum_{j=1}^{\infty}2^{-jrs}K_{r+1}\big(f,(-\widetilde{\Delta})^{(r+1)/2},t^{r+1}2^{j(r+1)}\big)_B^s\Big\}^{1/s}$$

and

$$K_r\big(f,(-\widetilde{\Delta})^{r/2},2^{-nr}\big)_B\geqq C_1\,2^{-nr}\Big\{\sum_{j=0}^{n}2^{jrs}E_{2^j}(f)_B^s\Big\}^{1/s} \tag{7.7}$$

where $\widetilde{\Delta}$, $K_r\big(f,(-\widetilde{\Delta})^{r/2},t^r\big)_B$ and $(-\widetilde{\Delta})^{r/2}$ are given by (7.1), (7.3) *and* (7.4) *respectively, and $E_n(f)_B$ is given by*

$$E_n(f)_B=\inf\Big(\|f-P\|_B:\,P\in\operatorname{span}\bigcup_{k=0}^{n-1}H_k\Big) \tag{7.8}$$

with H_k of (7.2).

We remind the reader that $SO(d)$ is the collection of $d\times d$ orthogonal matrices whose determinant equals 1.

PROOF. We first recall that the Cesàro summability of order $\ell>d-1$ is a positive operator (see for instance [17, Corollary 7.2.5, p. 266]). This already implies that C_n^ℓ is a contraction operator on $L_p(S^{d-1})$. Furthermore, the above and [14, Theorem 2.1] imply that C_n^ℓ is a contraction on many other Banach spaces of functions on S^{d-1}, including all Orlicz spaces. We now use the semigroup given in (5.3) and Theorem 5.2 to obtain (7.6) when we observe that, using the technique of [6, Section 4], $K_{\mathcal{A}^r}(f,t^r)_B \approx K_r\big(f,(-\widetilde{\Delta})^{r/2},t^r\big)_B$ for that semigroup for any Banach space B for which the Cesàro summability is bounded. The inequality (7.7) follows using [12, Theorem 6.1], which is applicable here as the Cesàro summability is bounded and implies

$$E_{2^k}(f)_B \leqq CK_{r+1}\big(f,(-\widetilde{\Delta})^{(r+1)/2},2^{-k(r+1)}\big)_B. \tag{7.9}$$ □

8. Non-holomorphic semigroups and averaged moduli of smoothness

For a semigroup $\{T(u)\}_{u\geqq 0}$ on a Banach space B the averaged moduli of smoothness are given by

$$\mathrm{w}_T^r(f,t)_B \equiv \frac{1}{t}\int_0^t \big\|\big(T(u)-I\big)^r f\big\|_B\,du. \tag{8.1}$$

We recall that the moduli $\omega_T^r(f,t)_B$ are given by

$$\omega_T^r(f,t)_B \equiv \sup_{0\leqq u\leqq t}\big\|\big(T(u)-I\big)^r f\big\|_B, \tag{8.2}$$

and we have the following equivalence.

THEOREM 8.1. *Suppose $\{T(u)\}_{u\geqq 0}$ is a C_0 semigroup of contractions on a Banach space B. Then*

$$\mathrm{w}_T^r(f,t)_B \leqq \omega_T^r(f,t)_B \leqq C(r)\mathrm{w}_T^r(f,t)_B. \tag{8.3}$$

PROOF. We now follow verbatim the proof in [8, pp. 184–185]. There the result refers only to L_p and translations, but the proof is the same and the identity (5.3) in [8, p. 184] is replaced by the identity

(8.4)

$$\big(T(h)-I\big)^r = \sum_{k=1}^r (-1)^k\binom{r}{k}\big\{T(kh)\big(T(ks)-I\big)^r - \big(T(h+ks)-I\big)^r\big\},$$

the proof of which is the same. □

As a corollary, we obtain the following result.

THEOREM 8.2. *Suppose* $\{T(u)\}_{u\geqq 0}$ *is a* C_0 *semigroup of contractions on a Banach space* B *which satisfies* (1.2) *for some* $2\leqq s<\infty$. *Then*

$$\omega_T^r(f,t)_B \geqq C\Big\{\sum_{j=1}^{\infty} 2^{-jrs}\omega_T^{r+1}(f,2^j t)_B^s\Big\}^{1/s}. \tag{8.5}$$

PROOF. Using the definition of $\omega_T^r(f,t)_B$, we have

$$\omega_T^r(f,t)_B^s \geqq \frac{1}{t}\int_0^t \big\|\big(T(u)-I\big)^r f\big\|_B^s\,du.$$

Theorem 2.1 now implies (by setting $v=2^j u$)

$$\omega_T^r(f,t)_B^s \geqq \frac{m_1}{t}\int_0^t \sum_{j=0}^{\ell} 2^{-rsj}\big\|\big(T(2^j u)-I\big)^{r+1} f\big\|_B^s\,du$$

$$= m_1\sum_{j=0}^{\ell} 2^{-rsj}\frac{1}{2^j t}\int_0^{2^j t} \big\|\big(T(v)-I\big)^{r+1} f\big\|_B^s\,dv$$

$$\geqq m_1\sum_{j=0}^{\ell} 2^{-rsj}\left(\frac{1}{2^j t}\int_0^{2^j t} \big\|\big(T(v)-I\big)^{r+1} f\big\|_B\,dv\right)^s$$

$$= m_1\sum_{j=0}^{\ell} 2^{-rsj}\mathrm{w}_T^{r+1}(f,2^j t)_B^s \geqq \frac{m_1}{\big(C(r+1)\big)^s}\sum_{j=0}^{\ell} 2^{-rsj}\omega_T^{r+1}(f,2^j t)_B^s. \qquad \square$$

As an immediate application, we obtain the following result.

THEOREM 8.3. *Suppose* B *is a Banach space of functions on* $\mathbb{R}_+$, $\mathbb{R}$ *or* $\mathbb{T}$ *satisfying* (1.2) *with some* $2\leqq s<\infty$ *and* $\|f(\cdot+\xi)\|_B \leqq \|f(\cdot)\|_B$ *for* $\xi\geqq 0$. *Then*

$$\omega^r(f,t)_B \geqq C\Big\{\sum_{j=1}^{\infty} 2^{-jrs}\omega^{r+1}(f,2^j t)_B^s\Big\}^{1/s} \tag{8.6}$$

where $\omega^k(f,t)_B$ *is* $\omega_T^k(f,t)_B$ *with* $T(u)f(x)=f(x+u)$.

We remark that for $\mathbb{R}_+$, Theorem 8.3 was not deduced in [7] even for $L_p(\mathbb{R}_+)$, with $1<p<\infty$. Of course (8.6) is valid for other spaces, not just L_p.

9. Results for spaces of functions on $\mathbb{R}^d$ or $\mathbb{T}^d$, $d > 1$

For $d > 1$ we use a result on averaged moduli that stems from the work [4] which is different from the averaged moduli in Section 8.

We define

$$V_t f(x) = \frac{1}{m_t} \int_{|x-y|=t} f(y)\,dy, \quad V_t 1 = 1 \tag{9.1}$$

where $|x-y|$ is the Euclidean distance between x and y for which we have the following result.

THEOREM 9.1. *Suppose B is a Banach space of functions on $\mathbb{R}^d$ or $\mathbb{T}^d$ with $d > 1$ which satisfies* (1.3). *Then*

$$\|V_{\ell,t} f - f\|_B \approx \inf_g \left(\|f-g\|_B + t^{2\ell} \|\Delta^\ell g\|_B \right) \equiv K_{\Delta^\ell}\left(f, t^{2\ell} \right)_B \tag{9.2}$$

where

$$V_{\ell,t} f \equiv \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} V_{jt} f \tag{9.3}$$

and $\Delta f \equiv \frac{\partial^2 f}{\partial x_1^2} + \dots + \frac{\partial^2 f}{\partial x_d^2}$ *is the Laplacian.*

PROOF. For $L_p(\mathbb{R}^d)$ $1 \leqq p \leqq \infty$ and $d > 1$, Theorem 9.1 was proved in [4, Theorem 3.1, pp. 273–276], and in fact all we do here is show how to deduce our theorem from [4, Theorem 3.1]. We note that (9.2) for $L_\infty(\mathbb{R}^d)$ implies the validity of (9.2) for $C(\mathbb{R}^d)$. (Perhaps the only interesting situation of (9.2) in case $B = L_\infty(\mathbb{R}^d)$ is when $B = C(\mathbb{R}^d)$, because only when $f \in C(\mathbb{R}^d)$ do both sides of (9.2) tend to zero as $t \to 0$.)

Using [1, Theorem 6.2, p. 97] with $m_1 > \frac{2(d+2)}{d-1}\ell$, we have

$$\|\Delta^\ell V_{kt}^{m_1} F\|_{C(\mathbb{R}^d)} \leqq \frac{A_1(m_1, \ell, k)}{t^{2\ell}} \|F\|_{C(\mathbb{R}^d)},$$

and hence for m large enough, $m > \frac{2(d+2)}{d-1}\ell^2$ for example, we have

$$\|\Delta^\ell V_{\ell,t}^m F\|_{C(\mathbb{R}^d)} \leqq \frac{A_2(m,\ell)}{t^{2\ell}} \|F\|_{C(\mathbb{R}^d)}. \tag{9.4}$$

We now show that for $F \in C(\mathbb{R}^d)$

$$A^{-1} \|F - V_{\ell,t} F\|_{C(\mathbb{R}^d)} \leqq \|F - V_{\ell,t}^m F\|_{C(\mathbb{R}^d)} + t^{2\ell} \|\Delta^\ell V_{\ell,t}^m F\|_{C(\mathbb{R}^d)} \tag{9.5}$$
$$\leqq A \|F - V_{\ell,t} F\|_{C(\mathbb{R}^d)}.$$

The left hand inequality of (9.5) is clear using (9.2) for $C(\mathbb{R}^d)$ (already proved in [4, Theorem 3.1]), and recalling the definition of $K_{\Delta^\ell}(f,t^{2\ell})_{C(\mathbb{R}^d)}$. Using $\|V_{\ell,t}F\|_{C(\mathbb{R}^d)} \leqq A_1\|F\|_{C(\mathbb{R}^d)}$, we have

$$\|F - V_{\ell,t}^m F\|_{C(\mathbb{R}^d)} \leqq A_2\|F - V_{\ell,t}F\|_{C(\mathbb{R}^d)}.$$

To conclude the proof of (9.5) we have to estimate $t^{2\ell}\|\Delta^\ell V_{\ell,t}^m F\|_{C(\mathbb{R}^d)}$. We choose G_1 such that

$$\|F - G_1\|_{C(\mathbb{R}^d)} + t^{2\ell}\|\Delta^\ell G_1\|_{C(\mathbb{R}^d)} \leqq 2K_{\Delta^\ell}(F,t^{2\ell})_{C(\mathbb{R}^d)}$$

and write

$$\begin{aligned} t^{2\ell}\|\Delta^\ell V_{\ell,t}^m F\|_{C(\mathbb{R}^d)} &\leqq t^{2\ell}\|\Delta^\ell V_{\ell,t}^m (F-G_1)\|_{C(\mathbb{R}^d)} + t^{2\ell}\|\Delta^\ell V_{\ell,t}^m G_1\|_{C(\mathbb{R}^d)} \\ &\leqq A_2(m,\ell)\|F-G_1\|_{C(\mathbb{R}^d)} + t^{2\ell}\|V_{\ell,t}^m\Delta^\ell G_1\|_{C(\mathbb{R}^d)} \\ &\leqq A_2(m,\ell)\|F-G_1\|_{C(\mathbb{R}^d)} + t^{2\ell}A_3^m\|\Delta^\ell G_1\|_{C(\mathbb{R}^d)} \\ &\leqq A_4K_{\Delta^\ell}(F,t^{2\ell})_{C(\mathbb{R}^d)} \leqq A_5\|F - V_{\ell,t}F\|_{C(\mathbb{R}^d)}, \end{aligned}$$

which concludes the proof of (9.5). To prove (9.2) for a Banach space on $\mathbb{R}^d$ or $\mathbb{T}^d$, we proceed first by showing

$$A^{-1}\|f - V_{\ell,t}f\|_B \leqq \|f - V_{\ell,t}^m f\|_B + t^{2\ell}\|\Delta^\ell V_{\ell,t}^m f\|_B \leqq 2A\|f - V_{\ell,t}f\|_B. \tag{9.6}$$

We first attend to Banach spaces B of functions on $\mathbb{R}^d$. To prove the left hand inequality of (9.6), choose $g \in B^*$ satisfying $\|g\|_{B^*} = 1$ and define $F(x) = f * g(x) = \langle f(x - \cdot), g(\cdot)\rangle$. Using (1.3) we have $F \in C(\mathbb{R}^d)$ and recalling (9.5), we have

$$A^{-1}\|F - V_{\ell,t}F\|_{C(\mathbb{R}^d)} \leqq \|F - V_{\ell,t}^m F\|_{C(\mathbb{R}^d)} + t^{2\ell}\|\Delta^\ell V_{\ell,t}F\|_{C(\mathbb{R}^d)}$$

(so using $\|g\|_{B^*} = 1$ and the convolution structure of $V_{\ell,t}$ will imply)

$$\leqq \|f - V_{\ell,t}^m f\|_B + t^{2\ell}\|\Delta^\ell V_{\ell,t}f\|_B.$$

For appropriate g_ε and $F = F_\varepsilon = f * g_\varepsilon$

$$\|F - V_{\ell,t}F\|_{C(\mathbb{R}^d)} \geqq |F(0) - V_{\ell,t}F(0)| \geqq \|f - V_{\ell,t}f\|_B - \varepsilon,$$

and as $\varepsilon > 0$ is arbitrary, the left inequality of (9.6) is proved.

We now follow the same technique to deduce from

$$\|F - V_{\ell,t}^m F\|_{C(\mathbb{R}^d)} \leqq A\|F - V_{\ell,t}F\|_{C(\mathbb{R}^d)}$$

and

$$t^{2\ell}\|\Delta^\ell V_{\ell,t}^m F\|_{C(\mathbb{R}^d)} \leqq A\|F - V_{\ell,t}F\|_{C(\mathbb{R}^d)}$$

the inequalities

$$\|f - V_{\ell,t}^m f\|_B \leqq A\|f - V_{\ell,t}f\|_B \quad \text{and} \quad t^{2\ell}\|\Delta^\ell V_{\ell,t}^m f\|_B \leqq A\|f - V_{\ell,t}f\|_B,$$

which together with the above, imply (9.6) and hence (9.2) for a Banach space of functions on $\mathbb{R}^d$ satisfying (1.3).

To prove the result for a Banach space of functions satisfying (1.3) on $\mathbb{T}^d$, we observe that $C(\mathbb{T}^d) \subset C(\mathbb{R}^d)$ and that for $F \in C(\mathbb{T}^d)$, $V_{\ell,t}^k F \in C(\mathbb{T}^d)$ for all k, ℓ and t.

Moreover, (9.5) is satisfied with the norm $C(\mathbb{T}^d)$ replacing $C(\mathbb{R}^d)$, since if $G \in C(\mathbb{T}^d)$, $\|G\|_{C(\mathbb{T}^d)} = \|G\|_{C(\mathbb{R}^d)}$. We now use the same technique to deduce (9.6) for Banach spaces of functions on $\mathbb{T}^d$ from (9.5) with $C(\mathbb{T}^d)$ instead of $C(\mathbb{R}^d)$.

To show that the inequality (9.6) implies (9.2), we observe that the right hand inequality implies $\|f - V_{\ell,t}f\|_B \geqq \frac{1}{2A}K_{\Delta^\ell}\big(f, t^{2\ell}\big)_B$. Choosing g such that $\Delta^\ell g \in B$ and $\|f-g\|_B + t^{2\ell}\|\Delta^\ell g\|_B \leqq 2K_{\Delta^\ell}\big(f, t^{2\ell}\big)_B$, and using the left inequality of (9.6), we write

$$A^{-1}\|f - V_{\ell,t}f\|_B \leqq \|f-g\|_B + \big\|V_{\ell,t}^m(f-g)\big\|_B$$

$$+ t^{2\ell}\big\|\Delta^\ell V_{\ell,t}^m(f-g)\big\|_B + t^{2\ell}\|\Delta^\ell V_{\ell,t}^m g\|_B.$$

We now follow the method used earlier to deduce

$$\|V_{\ell,t}^m f\|_B \leqq A_6\|f\|_B \quad \text{and} \quad \big\|\Delta^\ell V_{\ell,t}^m f\big\| \leqq \frac{A_2(m,\ell)}{t^{2\ell}}\|f\|_B \quad \text{for all} \quad f \in B,$$

from the corresponding inequalities for $B = C(\mathbb{R}^d)$ or $B = C(\mathbb{T}^d)$. We also need to recall that $\big\|\Delta^\ell V_{\ell,t}^m g\big\|_B = \big\|V_{\ell,t}^m \Delta^\ell g\big\|_B$ whenever $\Delta^\ell g \in B$ to complete the proof. □

THEOREM 9.2. *Suppose B is a Banach space of functions on $\mathbb{R}^d$ or $\mathbb{T}^d$ and its norm satisfies (1.2) for some s, $2 \leqq s < \infty$, and (1.3). Then for any ℓ such that $2\ell > r$*

$$\omega^r(f,t)_B \geqq C\bigg\{\sum_{j=1}^{\infty} 2^{-jrs}K_{\Delta^\ell}\big(f, (2^j t)^{2\ell}\big)_B^s\bigg\}^{1/s}. \tag{9.7}$$

PROOF. We write

$$\omega^r(f,t)_B^s = \sup_{|u|\leqq t} \|\Delta_u^r f\|_B^s \geqq \sup_{|u|=t} \|\Delta_u^r f\|_B^s \geqq \frac{1}{m_t}\int_{|u|=t} \|\Delta_u^r f\|_B^s\,du$$

with m_t of (9.1) i.e. $\int_{|u|=t} du = m_t$. We now use Theorem 2.1 with $T = T(u)$ and $T(u)f(x) = f(x+u)$ to obtain

$$\omega^r(f,t)_B^s \geqq \frac{C}{m_t}\int_{|u|=t}\sum_{j=1}^{L} 2^{-jrs}\|\,\Delta_{2^j u}^{r+1} f\,\|_B^s\,du$$

$$\geqq \frac{C_1}{m_t}\int_{|u|=t}\sum_{j=1}^{L} 2^{-jrs}\|\,\Delta_{2^j u}^{2\ell} f\,\|_B^s\,du$$

$$= C_1\sum_{j=1}^{L} 2^{-jrs}\frac{1}{m_t 2^{j(d-1)}}\int_{|v|=2^j t}\|\,\Delta_v^{2\ell} f\,\|_B^s\,dv.$$

As translations are isometries (see (1.3)), we have

$$\|\,\Delta_v^{2\ell} f(\,\cdot\,)\|_B = \left\|\sum_{k=-\ell}^{\ell}(-1)^k\binom{2\ell}{\ell-k} f(\,\cdot\,+kv)\right\|_B.$$

Therefore, using the Hölder and the triangle inequality we have

$$\omega^r(f,t)_B^s \geqq C_1\sum_{j=1}^{L} 2^{-jrs}\left\|\frac{1}{m_t 2^{j(d-1)}}\int_{|v|=2^j t}\sum_{k=-\ell}^{\ell}(-1)^k\binom{2\ell}{\ell-k} f(\,\cdot+kv)\,dv\right\|_B^s.$$

Since $\int_{|v|=2^j t} dv = m_t 2^{j(d-1)}$, we now have (using Theorem 9.1)

$$\omega^r(f,t)_B^s \geqq C_1\sum_{j=1}^{L} 2^{-jrs}\binom{2\ell}{\ell}\|V_{\ell,2^j t} f - f\|_B^s$$

$$\geqq C_2\sum_{j=1}^{L} 2^{-jrs} K_{\Delta^\ell}\big(f,(2^j t)^{2\ell}\big)_B^s.$$

□

The sharp-Jackson result can now be deduced from Theorem 9.2.

THEOREM 9.3. *Suppose B is a Banach space of functions on $\mathbb{R}^d$ or $\mathbb{T}^d$ satisfying* (1.2) *and* (1.3). *Then*

$$\omega^r(f,t)_B \geqq C\Big\{\sum_{j=1}^{\infty} 2^{-jrs} E_{1/(t2^j)}(f)_B^s\Big\}^{1/s} \tag{9.8}$$

where $E_\lambda(f)_B$ is given in (4.10) when B is a space of functions on $\mathbb{R}^d$ and by

$$E_\lambda(f)_B = \inf\Big\{\|f-\varphi\|_B : \varphi(\mathbf{x}) = \sum_{|\boldsymbol{n}|<\lambda} a_{\boldsymbol{n}} e^{i\boldsymbol{n}\mathbf{x}}\Big\} \tag{9.9}$$

when B is a space of functions on $\mathbb{T}^d$.

Proof. When $E_\lambda(f)_B$ is given by (4.10), we use (4.12) to deduce (9.4) from (9.3), writing

$$f = f - \varphi_{1/t} + (\varphi_{1/2t} - \varphi_{1/t}) + \cdots + (\varphi_{1/2^i t} - \varphi_{1/2^{i-1}t}) + \varphi_{1/2^i t}$$

where φ_λ is a near best approximant i.e. $\|f-\varphi_\lambda\|_B \leqq aE_\lambda(f)_B$. When E_λ is given by (9.9), we use the analogue of (4.12) and the same expansion to obtain (9.8). □

The lower estimate of $\omega^r(f,t)_B$ is given in the following theorem.

Theorem 9.4. *Under the conditions of Theorem 9.3, we have*

$$\omega^r(f,t)_B^s \geqq C_1 \sum_{j=1}^{L} 2^{-jrs}\omega^{r+1}(f,t2^j)_B^s \tag{9.10}$$

where $L = \min(\ell : 2^{-\ell} \leqq t)$ and B is a space of functions on $\mathbb{T}^d$ or $\mathbb{R}^d$.

Proof. Since when $2^{-\ell} \leqq t < 2^{-\ell+1}$,

$$\omega^k(f,2^{-\ell}) \leqq \omega^k(f,t)_B \leqq \omega^k(f,2^{-\ell+1})_B \leqq 2^k\omega^k(f,2^{-\ell})_B,$$

it is sufficient to prove (9.10) for $t = 2^{-n}$ and $L = n$. For a Banach space of functions on $\mathbb{R}^d$ or $\mathbb{T}^d$ satisfying (1.3), the weak converse inequality yields

(9.11)

$$\omega^{r+1}(f,2^{-n+j})_B \leqq C_2\Big\{\sum_{k=0}^{n-j} 2^{-k(r+1)}E_{2^{n-j-k}}(f)_B + \frac{1}{2^{(n-j)(r+1)}}\|f\|_B\Big\}.$$

Therefore, recalling $2 \leqq s < \infty$, we have

$$\sum_{j=1}^{n} 2^{-jrs}\omega^{r+1}(f,2^{-n+j})_B^s$$

$$\leqq C_2^s\Big\{\sum_{j=1}^{n} 2^{-jrs}\Big(\sum_{k=0}^{n-j} 2^{-k(r+1)}E_{2^{n-j-k}}(f)_B + \frac{1}{2^{(n-j)(r+1)}}\|f\|_B\Big)^s\Big\}$$

$$\leqq C_3\left\{\sum_{j=1}^{n} 2^{-jrs}\left(\sum_{k=0}^{n-j} 2^{-ks(r+1)} E_{2^{n-j-k}}(f)_B^s\right) + \sum_{j=1}^{n} 2^{-jrs}2^{-(n-j)(r+1)s}\|f\|_B^s\right\}$$

$$\leqq C_3\Big[\sum_{j=1}^{n} 2^{-jrs}\sum_{m=0}^{n-j} 2^{-(n-j-m)s(r+1)} E_{2^m}(f)_B^s\Big] + C_3 2^{-nrs}\|f\|_B^s$$

$$= C_3\Big[\sum_{m=0}^{n} E_{2^m}(f)_B^s 2^{-(n-m)s(r+1)}\sum_{j=1}^{n-m} 2^{js}\Big] + C_3 2^{-nrs}\|f\|_B^s$$

$$\leqq C_4\sum_{m=1}^{n} E_{2^m}(f)_B^s 2^{-(n-m)sr} + C_3 2^{-nrs}\|f\|_B^s.$$

In view of (9.8) (for $t = 2^{-n}$), we have

$$\omega^r(f,t)_B^s + t^{rs}\|f\|_B^s \geqq C_5\sum_{j=1}^{L} 2^{-jrs}\omega^{r+1}(f,t2^j)_B^s. \tag{9.12}$$

We choose g so that $\|f-g\|_B = E_1(f)_B$ where $E_\lambda(f)_B$ is given in (4.10) and (9.11) for function spaces on $\mathbb{R}^d$ or $\mathbb{T}^d$ respectively. Using (9.12), we now write

$$\sum_{j=1}^{L} 2^{-jrs}\omega^{r+1}(f-g,t2^j)_B^s < C_5^{-1}\{\omega^r(f-g,t)_B^s + t^{rs}\|f-g\|_B^s\}$$

$$\leqq C_6\omega^r(f,t)_B$$

since $\|f-g\|_B + C\omega^r(f,1)_B \leqq C_1 t^{-r}\omega^r(f,t)_B$ (see [13, Theorem 2.1]).

For $g\in C^r$ one has

$$\omega^r(g,\tau)_B \leqq \tau^r \max_{|\xi|=1}\left\|\left(\frac{\partial}{\partial\xi}\right)^r g\right\|_B. \tag{9.13}$$

This follows from $\|\Delta_h^r g\|_\infty \leqq |h|^r\Big\|\big(\frac{\partial}{\partial\xi}\big)^r g\Big\|_\infty$ for h in the ξ direction and hence following the arguments used in Theorem 9.1 (and elsewhere), $\|\Delta_h^r g\|_B \leqq |h|^r\Big\|\big(\frac{\partial}{\partial\xi}\big)^r g\Big\|_B$ (with h still in the ξ direction). Using (1.5) we now have (9.13).

Therefore, as $g\in C^\infty$ and $2^{-L}\approx t$, we have

$$\sum_{j=1}^{L} 2^{-jrs}\omega^{r+1}(g,t2^j)_B^s \leqq \sum_{j=1}^{L} 2^{-jrs}(t2^j)^{(r+1)s}\max_{|\xi|=1}\left\|\left(\frac{\partial}{\partial\xi}\right)^{r+1} g\right\|_B^s$$

$$\leqq C_7 t^{rs} \max_{|\xi|=1} \left\| \left(\frac{\partial}{\partial\xi}\right)^{r+1} g \right\|_B^s .$$

For function spaces on $\mathbb{T}^d$, $\left\|\left(\frac{\partial}{\partial\xi}\right)^{r+1} g\right\|_B^s = 0$. For function spaces on $\mathbb{R}^d$ we note that $\operatorname{supp}\hat{g}(y) \subset \{y : |y| \leqq 1\}$ implies $\operatorname{supp}\widehat{\left(\frac{\partial}{\partial\xi}\right)^r g}(y) \subset \{y : |y| \leqq 1\}$ and using [4, Theorem 2.1] with $R = 1$ and $\ell = 1$, we have

$$\left\|\Delta \left(\frac{\partial}{\partial\xi}\right)^r g\right\|_\infty \leqq C \left\|\left(\frac{\partial}{\partial\xi}\right)^r g\right\|_\infty$$

and hence

$$\left\|\Delta \left(\frac{\partial}{\partial\xi}\right)^r g\right\|_B \leqq C \left\|\left(\frac{\partial}{\partial\xi}\right)^r g\right\|_B .$$

We now use [10, Theorem 6.2] to obtain

$$\left\|\left(\frac{\partial}{\partial\xi}\right)^{r+1} g\right\|_B \leqq C_8 \left\|\Delta \left(\frac{\partial}{\partial\xi}\right)^r g\right\|_B^{1/2} \left\|\left(\frac{\partial}{\partial\xi}\right)^r g\right\|_B^{1/2} \leqq C_9 \left\|\left(\frac{\partial}{\partial\xi}\right)^r g\right\|_B .$$

Therefore,

$$t^{rs} \max_{|\xi|=1} \left\|\left(\frac{\partial}{\partial\xi}\right)^{r+1} g\right\|_B^s \leqq t^{rs} \max_{|\xi|=1} \left\|\left(\frac{\partial}{\partial\xi}\right)^r g\right\|_B^s$$

$$\leqq C_{10} t^{rs} \omega^r(f,1)_B^s \leqq C_{11} \omega^r(f,t)_B^s . \qquad \square$$

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