

Functions measuring smoothness and the constants in Jackson–Stechkin theorem

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Dedicated to Professor Vitaly Andrienko on the occasion of his 70th birthday

1 Introduction

This paper is devoted to the equivalence of two type direct theorems in Approximation Theory:

- a) for smooth functions (Favard’s estimates).
- b) for arbitrary continuous function (Jackson–Stechkin estimates).

Specifically, we will show that Jackson–Stechkin inequality with optimal respect to the order of smoothness constants follows from Favard’s inequality.

The main tool for this is the function W_{2k} , measuring the smoothness of integrable periodic function. This characteristic is more delicate than standard modulus of continuity of the $2k$ -th order. The function W_{2k} allows us to obtain asymptotically sharp results for approximation by Favard-type operators. For example, we obtain the Jackson–Stechkin inequality for periodic splines with optimal constants.

Two facts play a key role here.

1. Uniform (on k) boundedness of the operators W_{2k} :

$$W_{2k}(f, \delta) \leq 3\|f\|.$$

2. Bernstein–Nikolsky–Stechkin inequality in terms of W_{2k} .

This paper is organized as follows. In the Second Section we consider the smooth characteristic W_{2k} and prove the uniformly boundedness of W_{2k} . Section 3 is devoted to Bernstein–Nikolsky–Stechkin estimate (Theorem 1). Main result of the paper (Theorem 2) claims that the Favard operators gives the Jackson–Stechkin theorem with optimal constants. This result is the consequence of the sharp inequality for the trigonometric approximation (see [1]) and will be present in the Forth Section.

2 Functions measuring smoothness

Let the function f be continued on the one-dimensional torus $T = R/(2\pi Z)$. The standard smooth characteristic of f is the modulus of continuity of r -th order:

$$\omega_r(f, \delta) := \sup_{|h| \leq \delta} \|\Delta_h^r f\| = \sup_{|h| \leq \delta} \sup_x |\Delta_h^r f(x)|,$$

where

$$\Delta_h^r f(x) := \sum_{j=0}^r (-1)^j \binom{r}{j} f(x + jh).$$

We will construct the operators W_{2k} on the base of the even central difference

$$\widehat{\Delta}_h^{2k} f(x) := \sum_{j=-k}^k (-1)^j \binom{2k}{k+j} f(x + jh).$$

Define

$$W_{2k}(f, x, h) := \left(\frac{2k}{k}\right)^{-1} \int_T \widehat{\Delta}_t^{2k} f(x) \phi_h(t) dt, \quad 0 < h < \pi/k,$$

where

$$\phi_h(t) = \begin{cases} \frac{1}{h} \left(1 - \frac{|t|}{h}\right), & t \in [-h, h], \\ 0, & t \notin [-h, h]. \end{cases}$$

Put

$$W_{2k}(f, h) := \|W_{2k}(f, \cdot, h)\|.$$

Introduce two notations. These notations corresponds to sharp maximal function and maximal function of operator W_{2k} .

$$W_{2k}^\sharp(f, x, \delta) := \sup_{0 < h \leq \delta} |W_{2k}(f, x, h)|.$$

$$W_{2k}^*(f, \delta) := \sup_{x \in T} W_{2k}^\sharp(f, x, \delta).$$

One may consider the operators W_{2k} not only for continuous functions. The boundedness at all points and integrability will suffice.

Properties of the operators W_{2k} , W_{2k}^\sharp , W_{2k}^*

2.1. The functions $W_{2k}^\sharp(f, x, \delta) \leq W_{2k}^*(f, \delta)$ are non increasing, as the functions of δ

2.2.

$$W_{2k}(f, \delta) \leq W_{2k}^*(f, \delta) \leq \left(\frac{2k}{k}\right)^{-1} \omega_{2k}(f, \delta).$$

2.3. For the functions that are orthogonal to a space of trigonometric polynomials of degree $\leq n - 1$ (notation $g \in T_{n-1}^\perp$) we have [1]

$$W_{2k}(g, \delta) \approx \|g\|, \quad \delta = \alpha\pi/n, \quad \alpha \in (1, n/k),$$

or more precisely

$$\|g\| \leq c_\alpha W_{2k}(g, \alpha\pi/n) \leq c_\alpha (1 + \pi^2/8) \|g\|, \quad (1)$$

where

$$c_\alpha \leq \sec(\pi/(2\alpha)) \leq \frac{4}{\pi} (1 - \alpha^{-2})^{-1}.$$

In the case $\alpha = 1$ we have

$$W_{2k}(g, \pi/n) \approx \sqrt{k} \|g\|,$$

or, in the explicit form

$$\|g\| \leq c \sqrt{2k} W_{2k}(g, \frac{\pi}{n}) \leq c (6/\pi) \sqrt{2k} \|g\|, \quad (2)$$

and the first inequality is sharp with respect to k .

The estimates (1), (2) are the key estimates of [1], devoted to Jackson–Stechkin inequality with asymptotically sharp constants.

2.4. Lemma 1.

$$W_{2k}^*(f, h) \leq 3\|f\|, \quad h \in (0, \pi/k).$$

Proof. One can rewrite the function $W_{2k}(f, x, h)$ in the following form

$$W_{2k}(f, x, h) = f(x) + (f * \Lambda_{k,h})(x).$$

Here

$$\Lambda_{k,h}(x) = 2 \sum_{j=1}^k (-1)^{j+1} a_j \phi_{jh}(x), \quad \phi_{jh}(x) := \frac{1}{j} \phi_h\left(\frac{x}{j}\right), \quad a_j := \frac{\binom{2k}{k+j}}{\binom{2k}{k}}.$$

For estimate of the convolution it is sufficient to put $h = 1$. In this case Λ_k is the even, piecewise-linear function with the vertexes in the points (i, b_i) , $i = -k, \dots, k$,

$$b_{-i} = b_i, \quad i = 0, \dots, k-1, \quad b_{-k} = b_k = 0,$$

and

$$b_i = 2 \binom{2k}{k}^{-1} \sum_{j=i+1}^k \binom{2k}{k-j} (-1)^{j+1} \frac{1}{j} \left(1 - \frac{i}{j}\right), \quad i = 0, \dots, k-1.$$

The inequalities

$$0 < b_0 < 2 \ln 2, \quad 0 > b_1 > 2 \ln 2 - \pi^2/6, \quad |b_i| < \frac{1}{2i^2}, \quad i = 2, \dots, k-1,$$

imply

$$\int_{\mathbb{R}} |\Lambda_k(t)| dt \leq 2 \left(\int_0^2 + \int_2^\infty \right) \leq 1.5 + 0.5 = 2.$$

□

3 Bernstein–Nikolsky–Stechkin inequality

The Bernstein–Nikolsky–Stechkin inequality (see [3], Theorem 3.1.4 in russian edition) is the generalization of the classical Bernstein's inequality for trigonometric polynomials $\tau \in T_n$:

$$\|D^r \tau\| \leq n^r \|\tau\|,$$

and reads as

$$\|D^r \tau\| \leq n^r (2 \sin(nh/2))^{-r} \|\widehat{\Delta}_h^r \tau\|, \quad h \in (0, 2\pi/n).$$

Note, that quantity $i^r (2 \sin(nh/2))^r$ is the proper value of the operator $\widehat{\Delta}_h^r$ with respect to the eigenfunction $\exp(int) = c_n(t) + is_n(t)$.

Theorem 1. *If $\tau \in T_n$, then*

$$\|D^{2k} \tau\| \leq n^{2k} W_{2k}(c_n, h)^{-1} W_{2k}(\tau, h), \quad h \in (0, 2\pi/n).$$

Proof. Denote by $\chi_h^r(x)$ the convolution power of the normed characteristic function of the interval $[-h/2, h/2]$.

$$\chi_h^r(x) := (\chi_h * \chi_h^{r-1})(x), \quad \int_T \chi_h(t) dt = 1.$$

Note, that in these notations we have $\phi_h(x) = \chi_h^2(x)$. We can use the standard integral representation for the difference

$$\widehat{\Delta}_t^r f(x) = t^r (D^r(f) * \chi_t^r)(x).$$

The Bernstein–Nikolsky–Stechkin inequality for $t \in (0, 2\pi/n)$ is equivalent (see [4]) to the following inequality for $\tau \in T_n$, $\|\tau\| = \tau(x_0) = 1$:

$$(\tau * \chi_t^r)(x_0) \geq (c_n * \chi_t^r)(0).$$

Therefore, after multiplication of last inequality (for $D^{2k} \tau$) by $t^{2k} \chi_h^2(t)$ and integration on t we get

$$\begin{aligned} \int_T \widehat{\Delta}_t^{2k} \tau(x_0) \chi_h^2(t) dt &= \int_T t^{2k} (D^{2k} \tau * \chi_t^{2k})(x_0) \chi_h^2(t) dt \\ &\geq \int_T (c_n * \chi_t^{2k})(0) t^{2k} \chi_h^2(t) dt, \quad \|D^{2k} \tau\| = D^{2k} \tau(x_0) = 1. \end{aligned}$$

□

4 Favard's operators and Jackson–Stechkin theorem

We shall call an operator $A_{n,r}$ Favard's operator, if

$$\|f - A_{n,r}(f)\| \leq F_r n^{-r} \|D^r f\|.$$

Suppose that $\tau_* \in T_{n-1}$ gives the estimate [1, 2] :

$$\|f - \tau_*\| \leq c_\alpha W_{2k}(f, \alpha\pi/n), \quad \alpha \in (1, n/k).$$

Put $C_{k,\alpha} := F_{2k} W_{2k}(c_n, \alpha\pi/n)^{-1}$, $h_\alpha := \alpha\pi/n$.

The Theorem 1 and the Lemma 1 imply

$$\|\tau_* - A_{n,2k}(\tau_*)\| \leq F_{2k} n^{-2k} \|D^{2k} \tau_*\| \leq C_{k,\alpha} W_{2k}(\tau_*, h_\alpha) \leq$$

$$C_{k,\alpha} (W_{2k}(f - \tau_*, h_\alpha) + W_{2k}(f, h_\alpha)) \leq C_{k,\alpha} (3\|f - \tau_*\| + W_{2k}(f, h_\alpha)).$$

Thus

$$\|f - A_{n,2k}(\tau_*)\| \leq \|f - \tau_*\| + \|\tau_* - A_{n,2k}(\tau_*)\| \leq C(\alpha) W_{2k}(f, h_\alpha). \quad (3)$$

The methods of the paper [1] allow us to obtain the following estimates (see [2]):

$$C_{k,\alpha} \leq \mathcal{K}_{2k} c_\alpha, \quad c_\alpha \leq \sec(\pi/(2\alpha)), \quad \mathcal{K}_r = \frac{4}{\pi} \sum_{j=-\infty}^{\infty} (4j+1)^{-r-1}.$$

Therefore, we have

$$C(\alpha) \leq c_\alpha(1 + 3C_{k,\alpha}) + C_{k,\alpha} \leq c(\alpha - 1)^{-2}.$$

In the case $\alpha = 1$ the estimates of the constants are the following (see [2]):

$$C_{k,1} \leq \mathcal{K}_{2k} c_1, \quad c_1 \leq c\sqrt{k},$$

and

$$C(1) = O(k).$$

In the case of approximation of the periodic functions by periodic smooth splines the Favard's type estimates take place [5] with

$$F_r = \mathcal{K}_r.$$

Thus, we have the Jackson–Stechkin inequality for approximation by periodic splines with best (respect to r) constants.

Rewrite the inequality (3) for approximation by Favard's operators in terms of best approximations and standard moduli of smoothness. Let $E_{n-1}^F(f)$ be the best uniform approximation of continuous periodic function by Favard's operators.

Theorem 2. For $r \in N$

$$E_{n-1}^F(f) \leq c \max((\alpha - 1)^{-2}, 1) \sqrt{r} 2^{-r} \omega_r(f, \alpha\pi/n), \quad \alpha \in (1, 2n/r). \quad (4)$$

$$E_{n-1}^F(f) \leq c r^{3/2} 2^{-r} \omega_r(f, \pi/n).$$

The sharpness of (4) with respect to order r gives, for example, the periodic step-functions: $\text{sign}(c_n(t))$.

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