L-approximation of B-splines by trigonometric polynomials

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Dedicated to Professor Eleonora Storozenko on the occasion of her eighteenth birthday

arXiv:0811.0686v1 [math.CA] 5 Nov 2008

Denote by T_{2n-1} a space of real trigonometric polynomials

$$\tau(x) = \sum_{j=-n+1}^{n-1} \alpha_j \exp(2\pi i\, jx), \quad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, \quad \alpha_{-j} = \overline{\alpha}_j,$$

and denote by $T_{2n-1}^{\perp}$ a subspace of real functions $f \in L_\infty = L_\infty(\mathbb{T})$ that are orthogonal to T_{2n-1} with respect to the "scalar product "

$$(f,g) = \int_{-1/2}^{1/2} f(u)g(u)\, du = \int_{\mathbb{T}} f(u)g(u)\, du.$$

It is known that the best approximation in $L = L(\mathbb{T})$ by trigonometric polynomials from T_{2n-1} may be calculate as follows (see [7])

$$E_n(f)_1 := \inf_{\tau \in T_{2n-1}} \|f - \tau\|_1 = \sup_{g \in T_{2n-1}^{\perp},\ \|g\|=1} \int_{\mathbb{T}} g(u) f(u)\, du.$$

Here $\|\cdot\|_1$ and $\|\cdot\|$ denotes norms in L and L_∞ respectively.

This note is a continuation of our papers [1, 2], devoted to L-approximation of L-normed characteristic function

$$\chi_h(x) := \begin{cases} h^{-1}, & x \in (-h/2, h/2), \\ 0, & x \notin (-h/2, h/2), \end{cases}$$

of the interval $(-h/2, h/2)$ by trigonometric polynomials. In the paper [1] the sharp values of the best approximation for the special values of h were found. In [2] we gave the complete solution of the problem for arbitrary values of $h \in (0,1]$. In general case [2] the situation is more deep and results are not so simple as in [1]. For applications to the problem of optimal constants in the Jackson-type inequalities we need, however, results on L-approximation of B-splines and linear combinations of B-splines (see [6, 3]). Here we present some simple results about L-approximation of B-splines as well as give the the proof of its sharpness for the special values of h. In some sense we give the appendix to the paper [1].

The B-splines are the convolutions of function χ_h with itself:

$$\chi_h^1(x) := \chi_h(x),$$

$$\chi_h^k(x) := \int_{\mathbb{R}} \chi_h(u) \chi_h^{k-1}(x-u)\, du = (\chi_h * \chi_h^{k-1})(x).$$

The B-splines are the functions with the $k-1$ order smoothness and the supports $\operatorname{supp}\,\chi_h^k = (-kh/2, kh/2)$, $\quad |\operatorname{supp}\,\chi_h^k| = kh$.
Particularly

$$\chi_h^2(x) = \begin{cases} h^{-1}(1-|x|h^{-1}), & x \in (-h,h), \\ 0, & x \notin (-h,h). \end{cases}$$

It is easy to check that the operator of k-th order differentiation transforms k-th B-splines to k-th central differences:

$$D^k(f * \chi_h^k)(x) = h^{-k}\Delta_h^k f(x),$$

$$\Delta_h^k f(x) := \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x + kh/2 - jh).$$

Since

$$\sup_x |\Delta_h^k f(x)| = \|\Delta_h^k f\| \le \sum_{j=0}^{k} \binom{k}{j} \|f\| \le 2^k \|f\|,$$

then

$$\|D^k(f * \chi_h^k)\| \le (h/2)^{-k}\|f\|. \tag{1}$$

One of the main tools in approximation theory is the classical Favard's [4] inequality:

$$\|g\| \le \mathcal{K}_k (2\pi n)^{-k} \|D^k g\|, \quad g \in T_{2n-1}^{\perp}, \quad \mathcal{K}_k := \frac{4}{\pi} \sum_{j=-\infty}^{\infty} \frac{1}{(4j+1)^{k+1}}. \tag{2}$$

The Favard's constants $\mathcal{K}_k$ have the following properties

$$1 = \mathcal{K}_0 < \mathcal{K}_2 = \pi^2/8 < \cdots < 4/\pi < \cdots < \mathcal{K}_3 = \pi^3/24 < \mathcal{K}_1 = \pi/2.$$

Direct consequence of (2) and (1) is

$$\|g * \chi_h^k\| \le \mathcal{K}_k (2\pi n)^{-k} \|D^k(g * \chi_h^k)\| \le \mathcal{K}_k (\pi n h)^{-k} \|g\|, \quad g \in T_{2n-1}^{\perp}.$$

Therefore, we have

$$\begin{aligned} E_n(\chi_h^k)_1 &= \sup_{g \in T_{2n-1}^{\perp},\ \|g\|=1} \int_T g(u)\chi_h^k(-u)\,du \\ &\le \sup_{g \in T_{2n-1}^{\perp},\ \|g\|=1} |(\chi_h^k * g)(0)| \le \mathcal{K}_k (\pi n h)^{-k}. \end{aligned} \tag{3}$$

Theorem. *Let* $k, n \in \mathbb{N}$, $\quad k \le n$, $\quad h(\alpha) = \alpha/(2n)$, $\quad 0 < \alpha \le 2n/k$. *Then*

$$E_n(\chi_{h(\alpha)}^k)_1 \le F_k \alpha^{-k}, \quad \textit{where} \quad F_k := (2/\pi)^k \mathcal{K}_k. \tag{4}$$

For example for $k = 1, 2, 3$ *we have*

$$E_n(\chi_{h(\alpha)})_1 \le \frac{1}{\alpha},$$

$$E_n(\chi^2_{h(\alpha)})_1 \le \frac{1}{2\alpha^2},$$

$$E_n(\chi^3_{h(\alpha)})_1 \le \frac{1}{3\alpha^3}.$$

The inequalities (4) *become equalities if* $\alpha = 2j+1,\ j \in \mathbb{Z}_+,\ j \le \dfrac{2n-k}{2k}$.

The question about the value of the best L-approximation of B-spline for *arbitrary* $0 < \alpha \le 2n/k$ is not so simple (see [2] for the case $k=1$) .

Proof. The estimate (4) follows from the inequality (3). We need to prove equalities for $\alpha = 2j+1$ only. At first consider the case $k=1$. We will use notation

$$c_y(x) := \cos(2\pi xy), \quad y \in \mathbb{R}.$$

The function $\pm\,\mathrm{sign}\,(c_n)$, $n \in \mathbb{N}$ gives equality in (4). In other words, for $k=1$ and $h_j = (2j+1)/(2n),\ j = 0, \ldots, n-1$ we have

$$E_n(\chi_{h_j})_1 \ge \int_{\mathbb{R}} \chi_{h_j}(u)(-1)^j \mathrm{sign}\, c_n(u)\, du = 1/(2j+1). \tag{5}$$

One can rewrite the equality in (5) as

$$\int_{\mathbb{R}} \chi_{2j+1}(u)(-1)^j \mathrm{sign}\, c_{1/2}(u)\, du = 1/(2j+1). \tag{6}$$

Note, that $\mathrm{sign}\,(c_{1/2}(x)) \equiv \mathcal{E}_0(x)$, where $\mathcal{E}_0(x)$ is the first Euler's spline (see [5], pp. 148–151). The Euler splines $\mathcal{E}_k(x)$ are defined as follows:

$$\mathcal{E}_{j+1}(t) = \gamma_j \int_{\mathbb{T}} \mathcal{E}_j(x+u)\, du, \quad \gamma_j^{-1} = \int_{-1/2}^{1/2} \mathcal{E}_j(u)\, du,$$

and have the following properties:

$$\mathcal{E}_j(x+2) = \mathcal{E}_j(x), \quad \mathcal{E}_j(x+1) = -\mathcal{E}_j(x),$$

$$\int_{-1}^{1} \mathcal{E}_j(u+x)\, du = 0,$$

$$\mathcal{E}_j(-x) = \mathcal{E}_j(x), \quad \mathcal{E}_j(-x-1/2) = \mathcal{E}_j(x+1/2),$$

$$\|\mathcal{E}_j\| = 1, \quad \mathcal{E}_j(\nu) = (-1)^\nu, \quad \nu \in \mathbb{N},$$

$$D\mathcal{E}_j(x) = \pi \mathcal{K}_{j-1}\mathcal{K}_j^{-1}\mathcal{E}_{j-1}(x+1/2). \tag{7}$$

Come back to (6). Integrating by parts (7) we get

$$(-1)^j \int_{\mathbb{R}} \chi_{2j+1}(u)\mathcal{E}_0(u)\,du =$$

$$\frac{(-1)^j}{2}\int_{\mathbb{R}} \chi_{2j+1}(u)\ D\mathcal{E}_1(u-1/2)\,du = \frac{(-1)^{j+1}}{2}\int_{\mathbb{R}} D\chi_{2j+1}(u)\ \mathcal{E}_1(u-1/2)\,du =$$

$$\frac{(-1)^{j+1}}{2}(2j+1)^{-1}\int_{\mathbb{R}} \Delta^1_{2j+1}\delta(u)\ \mathcal{E}_1(u-1/2)\,du = \frac{(-1)^{j+1}}{2}(2j+1)^{-1}(-\Delta^1_{2j+1}\ \mathcal{E}_1(-1/2)) =$$

$$\frac{(-1)^{j+1}}{2}(2j+1)^{-1}(\mathcal{E}_1(-j-1)-\mathcal{E}_1(j)) =$$

$$\frac{(-1)^{j+1}}{2}(2j+1)^{-1}((-1)^{j+1}-(-1)^j) = (2j+1)^{-1}.$$

One can rewrite the proof for odd k without essential modifications:

$$\int_{\mathbb{R}} \chi^k_{h_j}(u)(-1)^j \operatorname{sign} c_n(u)\,du = \int_{\mathbb{R}} \chi^k_{2j+1}(u)(-1)^j \operatorname{sign} c_{1/2}(u)\,du =$$

$$\frac{(-1)^j\mathcal{K}_k}{\pi^k}\int_{\mathbb{R}} \chi^k_{2j+1}(u)\ D^k\mathcal{E}_k(u-k/2)\,du = \frac{(-1)^{j+1}\mathcal{K}_k}{\pi^k}\int_{\mathbb{R}} D^k\chi^k_{2j+1}(u)\ \mathcal{E}_k(u-k/2)\,du =$$

$$\frac{(-1)^{j+1}\mathcal{K}_k}{\pi^k}(2j+1)^{-k}\int_{\mathbb{R}} \Delta^k_{2j+1}\delta(u)\ \mathcal{E}_k(u-k/2)\,du =$$

$$\frac{(-1)^{j+1}\mathcal{K}_k}{\pi^k}(2j+1)^{-k}(-\widehat{\Delta}^k_{2j+1}\mathcal{E}_k(-k/2)) = \frac{(-1)^{j+1}\mathcal{K}_k}{\pi^k}(2j+1)^{-k}2^k(-1)^{j+1} = \frac{F_k}{(2j+1)^k}.$$

Consider the case of even k. We will use the equality

$$D^k\mathcal{E}_k(x) = \frac{\pi^k}{\mathcal{K}_k}(-1)^{k/2}\mathcal{E}_0(x),$$

which implies

$$\int_{\mathbb{R}} \chi^k_{2j+1}(u)\ \operatorname{sign}(c_{1/2}(u))\,dt = (-1)^{k/2}\frac{\mathcal{K}_k}{\pi^k}\int_{\mathbb{R}} \chi^k_{2j+1}(u)\ D^k\mathcal{E}_k(u)\,du.$$

The integration by parts gives

$$\int_{\mathbb{R}} \chi^k_{2j+1}(u)\ D^k\mathcal{E}_k(u)\,du = \int_{\mathbb{R}} D^k\chi^k_{2j+1}(u)\ \mathcal{E}_k(u)\,du.$$

Since

$$D^k\chi^k_{2j+1}(u) = (2j+1)^{-k}\Delta^k_{2j+1}\delta(u),$$

then

$$(2j+1)^k\int_{\mathbb{R}} D^k\chi^k_{2j+1}(u)\mathcal{E}_k(u)\,du = (\Delta^k_{2j+1}\delta * \mathcal{E}_k)(0) = \Delta^k_{2j+1}\mathcal{E}_k(0) = 2^k(-1)^{k/2}. \quad \square$$

Remark 1. The restriction $\alpha k \le 2n$ in the Theorem means that we work with usual B-splines with support in $[-1/2, 1/2]$. The inequality (4) is true without this restriction, but we do not consider the sharpness of (4) for other values of αk.

Remark 2. The approximations of characteristic functions ($k = 1$) were considered with details in [1, 2]. Note, that one can obtain the nontrivial approximation of step–function iff $\alpha > 1$. For small α the polynomial of the best approximation of the step–function is equal to 0, and the best approximation is equal to 1. As in the case of step–function we can indicate the value of parameter α for the nontrivial estimates of the best approximation. But we do not know what is the critical value of α_0 for nontrivial approximation if $k > 1$. For $\alpha > k^{-1/k}$, $k = 2, 3$, we have $E_n(\chi^k_{h(\alpha)})_1 < 1$ but we do not know the best approximation in the case $\alpha = k^{-1/k}$. The $x_{minimum}$ of the function $x^{-1/x}$ lies in $[2, 3]$. Probably, there are some links between this fact and the optimal smoothness of the averaging operators. The averaging of the second order (convolution with the hat function χ^2_h) often gives the most useful and sharp results.

Remark 3. This remark is close to Remark 2. We gave the Theorem in simple form. We can present here the more precise version of (4):

$$E_n(\chi^k_{h(\alpha)})_1 \leq \min\left(1, \frac{F_k}{\alpha^k}\right). \tag{4'}$$

Note, that for $\alpha \leq F_k^{1/k}$ the inequality (4′) gives trivial estimate. For $\alpha > F_k^{1/k}$ (in other words, if the support of the k-th B-spline χ^k_h is greater then $kF_k^{1/k}/(2n)$) the best approximation of χ^k_h is less then 1.

References

[1] Babenko A.G., Kryakin Yu.V. *On approximation of step functions by trigonometric polynomials in the integral metric* // Izvestia of the Tula State University. Ser. Mathematics. Mechanics. Informatics. Tula: TSU, 2006. V. 12, N 1. P. 27–56.

[2] Babenko A.G., Kryakin Yu.V. *Integral approximation of characteristic function of interval by trigonometric polynomials*// Trudy Instituta Matematiki i Mekhaniki, 2008. Vol. 14, N 3. P. 19–37.

[3] Babenko A.G., Kryakin Yu.V. *On $T^{\perp}_{2n-1}$ spaces.* //ArXiv:math CA/xxxxxxxx, (2008).

[4] Favard J. *Application de la formule summatoire d'Euler à la démostration de quelques propriétés extrémales des intégrales des fonctions périodiques et presque–périodiques* // Matematisk Tidskrift København B. H. 4, 1936, 81–94.

[5] DeVore R.A., Lorentz G. G. *Constructive approximation.* Fundamental Principles of Mathematical Sciences, 303. Springer-Verlag, Berlin, 1993. 449 pp.

[6] Foucart S., Kryakin Yu., Shadrin A. *On the exact constant in Jackson-Stechkin inequality for the uniform metric* // ArXiv:math CA/0612283, (2006), 1–20 (will appear in Constructive Approximation).

[7] Nikolsky S.M. *Approximation of functions in the mean by trigonometrical polynomials (Russian. English summary)* // Bull. Acad. Sci. URSS. Ser. Math. [Izvestia Akad. Nauk SSSR] 10, (1946). 207–256.

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