On $T_{2n-1}^{\perp}$ spaces

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Dedicated to Victor Kolyada on the occasion of his 61^{th} *birthday*

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1 Problem

This paper is devoted to the inequalities for mean values of functions from $T_{2n-1}^{\perp}$. By T_{2n-1} we denote a space of 1–periodic real trigonometric polynomials of degree $\leq n-1$ and by $T_{2n-1}^{\perp}$ we denote its orthogonal factor in decomposition of space $L_\infty(T) = T_{2n-1} \bigoplus T_{2n-1}^{\perp}$, $T := R/Z$. The typical result is:

$$g \in T_{2n-1}^{\perp} \implies |(g * \chi_h)(x)| \leq c(h,n) \operatorname*{ess\,sup}_{x\in T} |g(x)|, \quad n \in N, \quad h > 0, \tag{1}$$

where χ_h is the characteristic function of the interval $(-h/2, h/2)$ normed by condition

$$\int_{-h/2}^{h/2} \chi_h(t)\, dt = 1.$$

We describe the procedure that gives the sharp values of constants

$$c(h,n) := \sup_{g \in T_{2n-1}^{\perp}} \frac{\|g * \chi_h\|}{\|g\|}$$

for all values n and h. We present the results of calculations for $c(h,n)$ in the nontrivial principal case $h = 1/n$, $n \geq 2$.

The simple proof of the classical Jackson inequality in the case of the second modulus of continuity may be considered as the consequence of the estimates $c(h,n) < 1$. The problems of the sharp constants in classical Stechkin's inequality are also discussed.

2 Notation and equivalent form of problem

Let C_n (S_{n-1}) be a space of real even (odd) trigonometric polynomials

$$\sum_{j=0}^{n-1} a_j \cos(2\pi j x) \quad \left(\sum_{j=1}^{n-1} b_j \sin(2\pi j x)\right).$$

Denote by $T_{2n-1} := C_n \bigoplus S_{n-1}$ a space of all real trigonometric polynomials.

Let $C_n^{\perp}$ be an orthogonal complement of trigonometric space C_n. In other words, a space of even 1–periodic, functions from $L_\infty(T)$ which are orthogonal to C_n with respect to the scalar product

$$(f,g) = \int_{-1/2}^{1/2} f(t)g(t)\, dt.$$

One can define spaces $S_n^\perp$, $T_{2n-1}^\perp$ in the same manner. Central Steklov's means are the convolutions of integrable functions with $\chi_h(t)$:

$$S_h(f,x) := \frac{1}{h}\int_{x-h/2}^{x+h/2} f(t)\,dt = \frac{1}{h}\int_{-h/2}^{h/2} f(x+t)\,dt =$$

$$\frac{1}{h}\int_{-h/2}^{h/2} f(x-t)\,dt = \int_R f(t)\chi_h(x-t)\,dt =: (f*\chi_h)(x).$$

If

$$f(x+1) = f(x), \quad \widetilde{\chi}_h(x) := \sum_{j=-\infty}^{\infty} \chi_h(x+j),$$

then

$$(f*\chi_h)(x) = \int_T f(t)\widetilde{\chi}_h(x-t)\,dt =: f \odot \widetilde{\chi}_h(x).$$

For $g \in T_{2n-1}^\perp, \quad \tau \in T_{2n-1}$,

$$|S_h(g,x)| = |(g*\chi_h)(x)| = |g \odot (\widetilde{\chi}_h - \tau)(x)| \le \inf_\tau \|\widetilde{\chi}_h - \tau\|_1 \|g\|_\infty = E_n(\widetilde{\chi}_h)_1 \|g\|.$$

If τ_h is the polynomial of the best L–approximation of the characteristic function $\widetilde{\chi}_h$, then

$$g_h := \text{sign } (\widetilde{\chi}_h - \tau_h) \in T_{2n-1}^\perp \quad \text{(A.A. Markov (1898) [5])},$$

and

$$|(g_h*\chi_h)(0)| = |(g_h \odot \widetilde{\chi}_h)(0)| = \int_T |\widetilde{\chi}_h(t) - \tau_h(t)|\,dt = E_n(\widetilde{\chi}_h)_1 \|g_h\|.$$

So, the best constant in the inequality (1) is

$$c(h,n) = E_n(\widetilde{\chi}_h)_1. \tag{1'}$$

3 About the value $E_n(\widetilde{\chi}_h)_1$.

Firstly note that for $h > 1$

$$E_n(\widetilde{\chi}_h)_1 = \frac{\{h\}}{h} E_n(\widetilde{\chi}_{\{h\}})_1, \tag{1''}$$

where $\{h\}$ is the fractional part of h.

It is not difficult to find $E_n(\widetilde{\chi}_h)_1$ for the special values of h. Classical signum-function

$$\text{sign } (\cos 2\pi n t) \in C_n^\perp$$

allows us (see [1, Theorem 1.3.1], [2, (5.1), (5.2)]) to find the value $E_n(\widetilde{\chi}_h)_1$ for

$$h \in M_n = \left(0, \tfrac{1}{2n}\right] \bigcup \left(1 - \tfrac{1}{2n}, 1\right] \bigcup_{j=2}^{n} \left\{\tfrac{2j-1}{2n}\right\} :$$

$$E_n(\widetilde{\chi}_h)_1 = \begin{cases} 1, & h \in \left(0, \frac{1}{2n}\right], \\ \frac{1}{2nh}, & h = \frac{2j-1}{2n}, \quad j = 2, \ldots, n, \\ \frac{1-h}{h}, & h \in \left(1 - \frac{1}{2n}, 1\right], \end{cases}$$

and to prove that

$$E_n(\widetilde{\chi}_h)_1 \le \frac{1}{2nh} < 1, \quad h > \frac{1}{2n}. \tag{2}$$

For $h \in (0,1] \setminus M_n$ we used the precise description of the signum–functions from $C_n^{\perp}$. Denote by G_n a class of functions $g(t)$ with the following properties:

- $|g| = 1$,
- $g \in C_n^{\perp}$,
- function g has $n+1$ breakpoints on $(0, 1/2)$: $t_{0,n} < t_{1,n} < \ldots < t_{n,n}$.

The following Lemma has direct links to results of P.Thchebyshev (1859), A.Markov (1906), S.Bernstein(1912), Y.Geronimus (1935), G.Szego (1964), F.Peherstorfer (1979) (see [2] for some details).

Lemma A. *The set of the zeros of the equation*

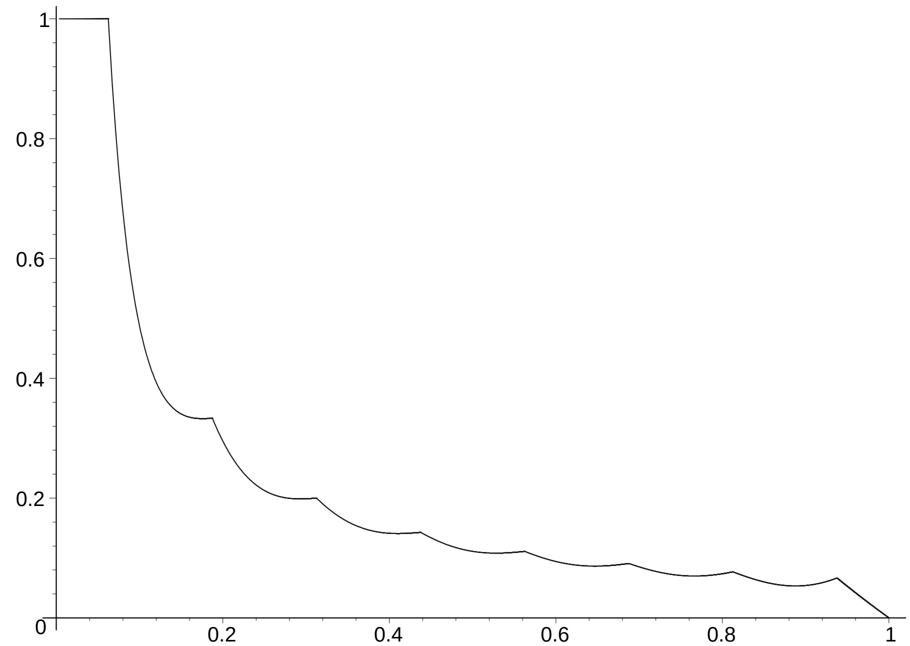
$$\cos 2\pi(n+1)t - 2q\cos 2\pi nt + q^2 \cos 2\pi(n-1)t = 0, \quad q \in (-1,1), \tag{3}$$

on $(0, 1/2)$ *is equal to the set of the breakpoints of some function from* G_n*. In the converse direction: for any function* g_0 *from* G_n *there is* $q_0 \in (-1,1)$ *such that the set of zeros of* (3) *on* $(0,1/2)$ *is equal to the set of breakpoints of* g_0*.*

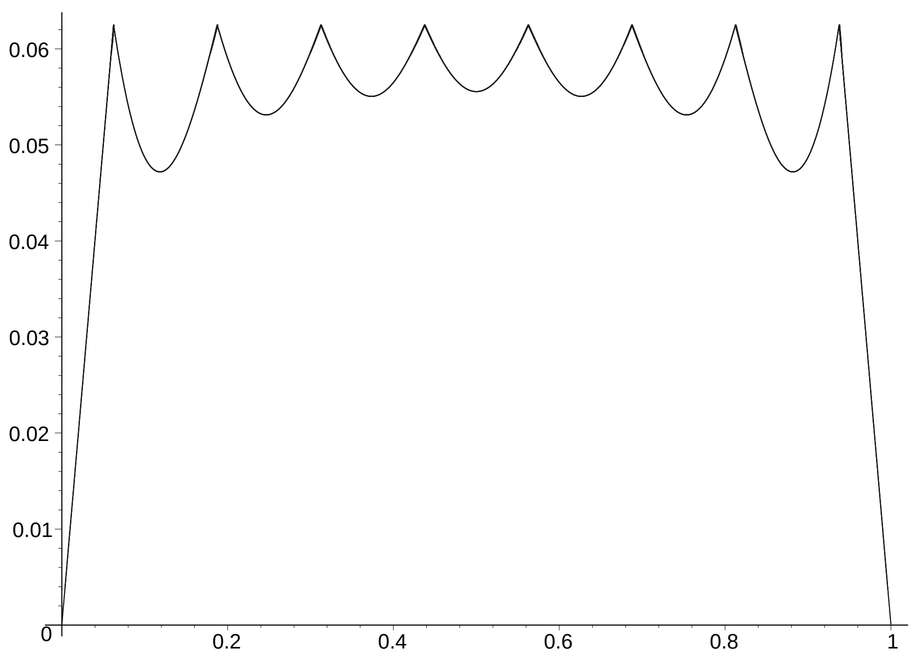
Lemma A and $(1'')$ give formula for the best approximations of characteristic function for arbitrary $h > 0$ ([2], Theorem 5). In particular, the following statement (see [2, section 5, p. 30]) is true.

Theorem B. For $n \ge 2$, $h = t_{1,n} \in (1/(2n), 3/(2n))$

$$E_n(\widetilde{\chi}_h)_1 = 1 - 2t_{0,n}/t_{1,n}.$$



Plot of $\Psi_8(h) := E_8(\widetilde{\chi}_h)_1$ for $h \in [0,1]$.



Plot of $\Phi_8(h) := h{\cdot}E_8(\widetilde{\chi}_h)_1$ for $h \in [0,1]$.

4 Evaluation of $E_n(\widetilde{\chi}_h)_1$ for $h = 1/n$

The case $h = 1/n$ is important for two reasons.

1. This case is the start point of our investigations on approximation of concrete functions in L–metrics. The answer to this question is the principal particular case of Theorem B.

2. We have nice formula in this case (see [2, section 5, p. 30]).

Theorem C. *For $n \geq 2,\ h(n) = 1/n$ we have*

$$c(h(n), n) = E_n(\widetilde{\chi}_{h(n)})_1 < E_{n+1}(\widetilde{\chi}_{h(n+1)})_1 < \cdots < \lim_{k\to\infty} E_k(\widetilde{\chi}_{h(k)})_1 = 1 - 2v_0,$$

where v_0 is the first positive zero of equation

$$\sec \pi v - \tan \pi v = v.$$

Note that (see (2))

$$1 - 2v_0 = 0.3817350529 \cdots < 1/2.$$

5 Favard and Jackson type theorems

Put

$$W_2(f, h, x) := f(x) - (f * \chi_h)(x) = -\frac{1}{h}\int_0^{h/2} (f(x-t) - 2f(x) + f(x+t))\, dt, \quad (4)$$

$$W_2(f, h) := \sup_x |f(x) - (f * \chi_h)(x)|. \quad (5)$$

It is clear that

$$2W_2(f, h) \leq \omega_2(f, h/2) := \sup_{x,\, 0<t<h} |f(x - t/2) - 2f(x) + f(x + t/2)|. \quad (6)$$

Theorem 1 (Favard type). *Let $g \in T_{2n-1}^{\perp}$. Then*

$$\|g\| \leq (1 - c(h, n))^{-1} W_2(g, h), \quad h > \frac{1}{2n}. \quad (7)$$

Proof. It is a direct consequence of $(4), (5), (1), (1'), (2)$. The identity

$$g(x) = (g * \chi_h)(x) + W_2(g, h, x)$$

yields

$$\|g\| \leq \|g * \chi_h\| + W_2(g, h) \leq c(n, h)\|g\| + W_2(g, h).$$

□

Denote a space of continuous functions on T by $C(T)$.

Theorem 2 (Jackson type). *Let $f \in C(T)$. Then for $h > \frac{1}{2n}$*

$$E_n(f) := \inf_{\tau \in T_{2n-1}} \|f - \tau\| \leq (1 - c(h, n))^{-1} W_2(f, h). \quad (8)$$

Proof. The inequality (8) is a modification of (7). If τ_h is the best L–approximation of $\widetilde{\chi}_h$, then for suitable choose of $\tau_f \in T_{2n-1}$ we have

$$(f - \tau_f)(x) = (f - \tau_f) \odot (\widetilde{\chi}_h - \tau_h)(x) + W_2(f, h, x),$$

and we can repeat the proof of Theorem 1. □

6 On exact constants in Favard and Jackson theorems

The constants in Theorems 1 and 2 are not sharp.

Consider the principal case $h = (2n)^{-1}$. Theorems 1 and 2 take place in this case too, but for the proof we need more complex ideas. (see chapter 8).

Conjecture. *The following inequalities are true*

$$\|g\| \leq 3\, W_2(g, 1/(2n)), \quad g \in T^{\perp}_{2n-1}, \tag{9}$$
$$E_n(f) \leq 3\, W_2(f, 1/(2n)), \quad f \in C(T). \tag{10}$$

We can not replace the constants 3 in the inequalities (9), (10) with smaller constants. The inequalities (9), (10) imply the sharp Favard's inequalities as follows.

$$\|g\| \leq 3W_2(g, (2n)^{-1}) \leq 3 \cdot 2n\, \|D^2 g\| \int_0^{1/(4n)} t^2\, dt = 2^{-5} n^{-2} \|D^2 g\|, \quad g \in T^{\perp}_{2n-1},$$

$$E_n(f) \leq 3W_2(f, (2n)^{-1}) \leq 3 \cdot 2n\, \|D^2 f\| \int_0^{1/(4n)} t^2\, dt = 2^{-5} n^{-2} \|D^2 f\|, \quad f \in C^2(T).$$

7 On the classical Favard and Jackson inequalites

The inequalities (6), (8) and

$$W_2(f, h) \leq \frac{\|D^2 f\|}{h} \int_0^{h/2} t^2\, dt = \frac{h^2}{24} \|D^2 f\|,$$

give the classical Favard and Jackson inequalities for the second derivative and the second modulus of smoothness (this means that it is also true for the first modulus of continuity).

8 On the extrapolation of Favard and Jackson inequalities

Despite the fact that approximation of the characteristic function $\widetilde{\chi}_h$ is possible only from some value of support h we can prove the inequality (7) and (8) for small values of h too. Let us show how to do this.

Write the identity

$$\begin{aligned} g &= g - g * \chi_h + \chi_h * (g - g * \chi_h) + \chi_h^2 * g = \\ &\quad g - g * \chi_h + \widetilde{\chi}_h \odot (g - g * \chi_h) + \widetilde{\chi}_h^2 \odot g. \end{aligned} \tag{11}$$

Denote by $E_n(\widetilde{\chi}^2)_1$ the best L–approximation of

$$\widetilde{\chi}_h^2 := \widetilde{\chi}_h \odot \widetilde{\chi}_h$$

by T_{2n-1}.

Then, using slightly modified proof of Theorem 1 we get:

$$\|g\| \le (1+1)W_2(g,h) + E_n(\widetilde{\chi}_h^2)\|g\|, \quad g \in T_{2n-1}^{\perp}$$

and

$$\|g\| \le 2(1 - E_n(\widetilde{\chi}_h^2))^{-1} W_2(g,h).$$

The last inequality is valid for some h such that

$$E_n(\widetilde{\chi}_h^2) < 1.$$

It is known that (see [3])

$$E_n(\widetilde{\chi}_h^2) = 1/2, \quad h = 1/(2n).$$

So, we can take the constant in the inequality (9) equal to 4. We can use the father extrapolation of (11). Identity

$$\begin{aligned} g = g - g * \chi_h + \chi_h * (g - g * \chi_h) + \chi_h^2 * (g - g * \chi_h) + \dots = \\ g - g * \chi_h + \widetilde{\chi}_h \odot (g - g * \chi_h) + \widetilde{\chi}_h^2 \odot (g - g * \chi_h) + \dots \end{aligned} \tag{12}$$

gives

$$\|g\| \le \left(1 + \sum_{j=1}^{\infty} E_n(\widetilde{\chi}_h^j)_1\right) W_2(g,h).$$

In the case $h = 1/(2n)$ (see [3])

$$E_n(\widetilde{\chi}_h^j)_1 = F_j,$$

where

$$F_j = 2\left(\frac{2}{\pi}\right)^{j+1} \sum_{k=-\infty}^{\infty} (4k+1)^{-j-1} = \left(\frac{2}{\pi}\right)^j \mathcal{K}_j.$$

In particular,

$$F_0 = 1, \quad F_1 = 1, \quad F_2 = 1/2, \quad F_3 = 1/3, \quad F_4 = 5/24, \quad F_5 = 2/15.$$

From

$$\sum_{j=0}^{\infty} F_j = \sum_{j=0}^{\infty} \mathcal{K}_j (2/\pi)^j = \sec(1) + \tan(1)$$

we deduce the estimate (9) with the constant $\sec(1) + \tan(1) = 3.408223443\ldots$

The proof of Jackson's inequality with the same constant (Favard's constant) one can obtain in the following way. Let $\tau_h^j \in T_{2n-1}$ be the polynomial of the best L–approximation of $\widetilde{\chi}_h^j$. Put

$$\tau_{f,N} := \sum_{j=0}^{N-1} \tau_h^j \odot (f - f * \chi_h) + \tau_h^N \odot f, \quad \tau_h^0 = 0. \tag{13}$$

Then by subtraction (13) from

$$f = \sum_{j=0}^{N-1} \widetilde{\chi}_h^j \odot (f - f * \chi_h) + \widetilde{\chi}_h^N \odot f, \quad \widetilde{\chi}_h^0 = \widetilde{\delta}, \tag{14}$$

we get Jackson's type theorem:

$$\|f - \tau_{f,N}\| \le \left(\sum_{j=0}^{N-1} E_n(\widetilde{\chi}_h^j)_1 \right) \, W_2(f,h) + E_n(\chi_h^N)_1 \, \|f\|,$$

and

$$E_n(f) \le \left(\sum_{j=0}^{\infty} E_n(\widetilde{\chi}_h^j)_1 \right) \, W_2(f,h). \tag{15}$$

9 On Stechkin's theorem

9.1 Introduction

Stechkin's theorem is the generalization of Jackson's theorem to differences of higher orders:

$$\Delta_t^r f(x) := \sum_{j=0}^{k} (-1)^j \binom{r}{j} f(x + jt).$$

Classical Stechkin's inequality is formulated in notation of r–th modulus of smoothness

$$\omega_r(f,h) := \sup_{0<t\le h} \|\Delta_t^r f\|,$$

and has the following form:

$$E_n(f) \le K_{n,r}(h) \; \omega_r(f,h).$$

The behavior of the sharp constant (as the function of n, r, h)

$$K_{n,r}(h) := \sup_{f \in C} \frac{E_n(f)}{\omega_r(f,h)}$$

is not clear in details. Put

$$\gamma_r^* := \begin{cases} \binom{2k}{k}^{-1}, & r = 2k, \\ \binom{2k-1}{k-1}^{-1}, & r = 2k-1 \end{cases} \qquad \left(\gamma_r^* \asymp \frac{r^{1/2}}{2^r} \right).$$

It was recently proved (see [4]) that

$$K_{n,r}(\alpha/(2n)) \le C_\alpha \gamma_r^*, \quad \alpha > 1, \tag{16}$$

$$K_{n,r}(1/(2n)) \le C\sqrt{r}\ln(r+1)\gamma_r^*. \tag{17}$$

In particular, for $\alpha = 2$

$$K_{n,r}(1/n) \le 5\gamma_r^*.$$

Inequality (16) is not true for $\alpha < 1$ [4]. So, we have the intrinsic open question: is the inequality

$$K_{n,r}(1/(2n)) \le C\gamma_r^*.$$

true?

We will show that the method of chapter 8 allows us to prove that

$$K_{n,r}(1/n) \le \sqrt{2}\gamma_r^*, \quad K_{n,r}(1/(2n)) \le C\sqrt{r}\,\gamma_r^*. \tag{18}$$

It is known ([4]) that for $h \le 1/(2k)$

$$K_{n,r}(h) \ge c'\gamma_r^*,$$

where

$$c' := \begin{cases} \frac{r}{r+1}, & r = 2k-1; \\ 1, & r = 2k. \end{cases}$$

Therefore, in the classical case $\delta = 1/n$ we have the narrow interval for the value $K_{n,r}(\delta)$.

9.2 Smoothness and general results

We assume that the smoothness order is an even number. In other words, we suppose that $r = 2k$. It is convenient to consider the symmetric differences:

$$\widehat{\Delta}_t^{2k} f(x) = \sum_{j=-k}^{k} (-1)^j \binom{2k}{k+j} f(x+jt).$$

Introduce a class of even, integrable functions Φ. This is a class of the convolution kernels. We write $\phi \in \Phi$ if ϕ is integrable on R function with compact support and

$$\phi(x) = \phi(-x), \qquad \int_R \phi(t)\,dt = 1.$$

We will use notation

$$\phi_j(x) := \frac{1}{j}\phi\left(\frac{x}{j}\right), \quad \phi_0(t) := \delta(x).$$

Define the function, measuring the $2k$–th ϕ–th smoothness of f at the point x.

$$W_{2k}(f,\phi,x) := \binom{2k}{k}^{-1} \int_R \widehat{\Delta}_t^{2k} f(x)\phi(t)\,dt.$$

The function $W_{2k}(f,\phi,x)$ can be written as the convolution of f with the function

$$W(x) := W_{2k}(\phi,x) := \binom{2k}{k}^{-1} \sum_{j=-k}^{k} (-1)^j \binom{2k}{k+j} \phi_j(x) =$$

$$\delta(x) - 2\sum_{j=1}^{k} (-1)^{j+1} a_j \phi_j(x), \quad a_j := \frac{\binom{2k}{k+j}}{\binom{2k}{k}}.$$

Put

$$U(x) := U_{2k}(f, \phi, x) := 2 \sum_{j=1}^{k} (-1)^{j+1} a_j \phi_j(x).$$

and denote by U^j the convolution power of U:

$$U^0(x) := \delta(x), \quad U^j(x) := (U * U^{j-1})(x).$$

The identity (see (12))

$$g = \sum_{j=0}^{\infty} U^j * (g - U * g) = \sum_{j=0}^{\infty} \widetilde{U}^j \odot (g - U * g)$$

and equalities

$$W_{2k}(f, \phi, x) = (W * f)(x) = (f - U * f)(x)$$

give the following result

Theorem 3 (Favard type). *Let* $g \in T_{2n-1}^{\perp}$. *Then*

$$\|g\| \leq \left(\sum_{j=0}^{\infty} E_n(\widetilde{U}^j)_1 \right) \|W_{2k}(g, \phi, \cdot)\|.$$

The passage from Theorem 3 to Stechkin's inequality is described in the last lines of chapter 8 (see (13) – (15)).

Theorem 4 (Stechkin type). *Let* $f \in C(T)$. *Then*

$$E_n(f) \leq \left(\sum_{j=0}^{\infty} E_n(\widetilde{U}^j)_1 \right) \|W_{2k}(f, \phi, \cdot)\|.$$

9.3 Concrete results

By choosing

$$\phi(x) = \chi_h^2(x)$$

one can obtain the estimate [4]

$$\sum_{j=0}^{\infty} E_n(\widetilde{U}^j)_1 \leq \left(\cos \frac{\pi}{2} \rho \right)^{-1}, \quad \rho = \frac{\mu_{2k}}{2nh} < 1, \quad \mu_{2k} \approx (1 - (2k)^{-1/2})^{1/2}.$$

Since

$$\|W_{2k}(f, \chi_h^2, \cdot)\| \leq \gamma_{2k}^* \omega_{2k}(f, h),$$

we have

Theorem 5. *Let* $f \in C(T)$, $\alpha > 1$. *Then*

$$E_n(f) \leq (\cos(\pi/(2\alpha))^{-1} \; \gamma_{2k}^* \; \omega_{2k} \left(f, \frac{\alpha}{2n} \right).$$

In particular, for $\alpha = 2$

$$(\cos(\pi/(2\alpha))^{-1} = \sqrt{2}.$$

From

$$\gamma_{2k}^{*}\omega_{2k}(f,\delta) \leq \gamma_{2k-1}^{*}\omega_{2k-1}(f,\delta),$$

the first inequality (18) follows.

Consider the case $\alpha = 1$. Choose $\phi = \chi_h^2$. We can obtain good estimates in this case, however only in terms of the characteristic $W_{2k}(f, \chi_h^2, x)$. Put

$$K_{n,2k}^{*}(\phi) := \sup_{f\in C} \frac{E_n(f)}{\|W_{2k}(f,\phi,\cdot)\|}.$$

We can present now the corollary of Theorem 7.1 from [4].

Theorem 6. For $r = 2k$ we have

$$\frac{\gamma_r^{*}}{1-\mu_r^2} \leq K_{n,r}^{*}(\chi_{1/(2n)}^2) \leq \frac{4}{\pi}\,\frac{\gamma_r^{*}}{1-\mu_r^2},$$

where

$$\frac{\gamma_r^{*}}{1-\mu_r^2} \asymp \sqrt{r}\gamma_r^{*} \asymp \frac{r}{2^r}.$$

So we can omit the factor $\ln(r+1)$ in (17).

10 Comments

1. Equivalence of convolutions

Denote by $\widetilde{g}$ the 1-periodization of $g \in L(R)$, $\quad \operatorname{supp} g < \infty$:

$$\widetilde{g}(t) := \sum_{k\in Z} g(t+k).$$

Let $f \in L(T)$ be an arbitrary 1-periodic function. It is well–known that

$$(f \odot \widetilde{g})(x) := \int_T f(x-t)\widetilde{g}(t)\,dt = \int_{-\infty}^{+\infty} f(x-t)g(t)\,dt =: (f * g)(x).$$

Indeed,

$$I(x) := \int_T f(x-t)\widetilde{g}(t)\,dt = \int_0^1 f(x-t)\sum_{k\in Z} g(t+k)\,dt = \sum_{k\in Z}\int_0^1 f(x-t)g(t+k)\,dt.$$

The change of variable $t+k = u$, $t = u-k$ gives

$$I(x) = \sum_{k\in Z}\int_0^1 f(x-t)g(t+k)\,dt = \sum_{k\in Z}\int_k^{k+1} f(x-u+k)g(u)\,du =$$

$$\sum_{k\in Z}\int_k^{k+1} f(x-u)g(u)\,du = \int_{-\infty}^{+\infty} f(x-t)g(t)\,dt. \quad \square$$

2. How to choose τ_f in Theorem 2

Lemma 1. *Let* $g \in L(T)$ *and*

$$|\widehat{g}(k)| < 1 \quad \textit{for all} \quad k \in Z, \quad \textit{where} \quad \widehat{g}(k) := \int_T g(t) e^{-2\pi i k t}\, dt. \tag{1c}$$

Then for arbitrary $n \in N$, $\varphi \in T_{2n-1}$ *there exist* $\tau \in T_{2n-1}$ *such that*

$$\varphi = \tau - \tau \odot g. \tag{2c}$$

Proof. The equation (2c) is equivalent to

$$\varphi = \tau \odot (D_n - g), \quad \text{where} \quad D_n(x) = \sum_{k=-(n-1)}^{n-1} e^{2\pi i k x},$$

and

$$\widehat{\varphi}(k) = \widehat{\tau}(k)(1 - \widehat{g}(k)), \quad |k| \le n - 1.$$

Thus

$$\tau(x) = \sum_{k=-(n-1)}^{n-1} \frac{\widehat{\varphi}(k)}{1 - \widehat{g}(k)} e^{2\pi i k x}.$$

Remark 1. Condition $\|g\|_L < 1$ imply (1c).

Remark 2. We proved Theorem 2 in the following form:

$$\|f - \tau_f\| \le \frac{W_2(f,h)}{1 - c(h,n)}, \quad f \in C(T), \quad n \in N, \quad \frac{1}{2n} < h \le 1, \tag{10$'$}$$

where

$$\tau_f(x) = \sum_{k=-(n-1)}^{n-1} \frac{\widehat{f}(k)\widehat{\tau}_h(k)}{1 - \widehat{\chi}_h(k) + \widehat{\tau}_h(k)} e^{2\pi i k x}.$$

3. About the equality $E_n(\widetilde{\chi}^j_{1/(2n)})_1 = F_j$.

In [3] we proved this equality with some restrictions on h. Equivalence of convolutions $*$ and $\odot$ for periodic functions gives the proof without restrictions. Namely, we do not need to modify anything in [3].

4. About the inequality $\sum_{j=0}^{\infty} E_n(\widetilde{U}^j)_1 \le \left(\cos \frac{\pi}{2}\rho\right)^{-1}, \quad \rho < 1$.

In [4] inequality had been proved in another form

$$\sum_{j=0}^{\infty} \|U^j\|_{T^{\perp}_{2n-1}} \le \left(\cos \frac{\pi}{2}\rho\right)^{-1}, \quad \rho < 1.$$

The equality

$$\|U^j\|_{T^{\perp}_{2n-1}} = E_n(\widetilde{U}^j)_1$$

allows us to simplify the approach to Stechkin's theorem and give new estimates of constants. Namely, in this paper we proved that Stechkin's constants (in Theorem 4) are equal to Favard's constants (in Theorem 3).

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