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On a Result by Geronimus

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Abstract—In 1935, Ya.L. Geronimus found the best integral approximation on the period $[-\pi, \pi)$ of the function $\sin(n+1)t - 2q\sin nt$, $q \in \mathbb{R}$, by the subspace of trigonometric polynomials of degree at most $n-1$. This result is an integral analog of the known theorem by E.I. Zolotarev (1868). At present, there are several methods of proving this fact. We propose one more variant of the proof. In the case $|q| \geq 1$, we apply the $(2\pi/n)$-periodization and the fact that the function $|\sin nt|$ is orthogonal to the harmonic $\cos t$ on the period. In the case $|q| < 1$, we use the duality relations for Chebyshev's theorem (1859) on a rational function least deviating from zero on a closed interval with respect to the uniform metric.
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1. NOTATION

Let $\mathcal{P}_n$ be a subspace of algebraic polynomials $P(x) = d_0 + d_1x + d_2x^2 + \ldots + d_nx^n$ of degree at most n with real coefficients, and let $\mathcal{T}_n$ be a subspace of real-valued trigonometric polynomials of degree at most n:

$$\tau(t) = a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu t + b_\nu \sin \nu t) = \sum_{\nu=-n}^{n} c_\nu e^{i\nu t} \tag{1.1}$$

$$(a_0 = c_0 \in \mathbb{R}, \quad c_\nu = \overline{c}_{-\nu} = (a_\nu - ib_\nu)/2, \quad a_\nu, b_\nu \in \mathbb{R}, \quad \nu = 1, 2, \ldots, n).$$

In the case when one of the leading coefficients a_n or b_n of polynomial (1.1) is different from zero, the degree $\deg \tau$ of the polynomial τ is exactly n. We denote by $\mathbb{T}$ the period of length 2π (i.e., the half-open interval $[\alpha, \alpha + 2\pi)$ with identified end points, where α is an arbitrary real number); we denote by $\chi_{(a,b)}$ the 2π-periodic extension to $\mathbb{R}$ of the characteristic function of an open interval (a, b) of length $b - a \leq 2\pi$. Below, we use the following definition (see [21, Definition 2; 16, Ch. 3, Sect. 10, (10.7); 2, Definitions 1, 2]).

Definition. *The sign function corresponding to a set* of points $t_1 < t_2 < \ldots < t_{2r}$ from $[t_1, t_1 + 2\pi)$ is defined to be the function $\sigma(t) = \varepsilon \sum_{k=1}^{2r} (-1)^k \chi_{(t_k, t_{k+1})}(t)$, $t \in \mathbb{R}$, $\varepsilon = \pm 1$; here,

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$t_{2r+1} := t_1 + 2\pi$. The set $\{t_k\}_{k=1}^{2r}$ is called *canonical* for $\mathcal{T}_{n-1}$ if the sign function σ corresponding to this set is orthogonal to $\mathcal{T}_{n-1}$, i.e., if $\int_{\mathbb{T}} \sigma(t)\,\tau(t)\,dt = 0$ for all $\tau \in \mathcal{T}_{n-1}$.

Remark 1. Necessary and sufficient conditions for a set of points $t_1 < t_2 < \ldots < t_{2r}$ from $[t_1, t_1 + 2\pi)$ to be canonical for $\mathcal{T}_{n-1}$ are that

$$\sum_{j=1}^{r} t_{2j} - \sum_{j=1}^{r} t_{2j-1} = \pi, \qquad \sum_{j=1}^{r} e^{i\nu t_{2j}} - \sum_{j=1}^{r} e^{i\nu t_{2j-1}} = 0 \quad \text{for} \quad \nu = 1, 2, \ldots, n-1. \tag{1.2}$$

Moreover, the first condition in (1.2) is equivalent to the orthogonality of the corresponding sign function σ to constants and the other conditions are equivalent to the orthogonality of σ to all polynomials $\tau \in \mathcal{T}_{n-1}$ with zero mean value on the period. Indeed, let the sign function σ corresponding to the set of points $t_1 < t_2 < \ldots < t_{2r}$ from $[t_1, 2\pi + t_1)$ be orthogonal to $\mathcal{T}_{n-1}$. This is equivalent to each of the relations

$$\sum_{k=1}^{2r} (-1)^k \int_{t_k}^{t_{k+1}} e^{i\nu t}\,dt = 0 \quad \text{for} \quad \nu = 0, 1, 2, \ldots, n-1 \quad (t_{2r+1} := t_1 + 2\pi),$$

$$\sum_{k=1}^{2r} (-1)^k \,(t_{k+1} - t_k) = 0, \qquad \sum_{k=1}^{2r} (-1)^k \left(e^{i\nu t_{k+1}} - e^{i\nu t_k}\right) = 0 \quad \text{for} \quad \nu = 1, 2, \ldots, n-1,$$

where the last relation is equivalent to (1.2).

We will use the following notation: $C(\mathbb{T})$ is the space of continuous 2π-periodic functions $f : \mathbb{T} \to \mathbb{R}$ with uniform norm $\|f\|_{C(\mathbb{T})} = \max\{|f(t)| \colon t \in \mathbb{T}\}$ and $L = L(\mathbb{T})$ is the space of Lebesgue integrable functions $f \colon \mathbb{T} \mapsto \mathbb{R}$ with integral norm $\|f\|_L = \int_{\mathbb{T}} |f(t)|\,dt$.

The number $E_{n-1}(f)_L := \min\{\|f - \tau\|_L : \tau \in \mathcal{T}_{n-1}\}$ is the value of the best integral approximation of a function $f \in L$ by the subspace $\mathcal{T}_{n-1}$, and an element $\tau^* \in \mathcal{T}_{n-1}$ providing the minimum on the right-hand side of this equality is called a polynomial of the *best integral approximation* for f.

2. CHEBYSHEV'S RESULT

Let us present one of Chebyshev's results, which will be used in the present paper together with Bernstein's remark.

In 1859, for every $n \in \mathbb{Z}_+ = \{0, 1, 2 \ldots\}$, Chebyshev found [15, Sects. 9–11] a fraction

$$\bigl(x^{n+1} - P^*(x)\bigr)/Q(x) \tag{2.1}$$

least deviating from zero with respect to the metric of the space $C[-1, 1]$ among all fractions of the form

$$\bigl(x^{n+1} - P(x)\bigr)/Q(x), \quad P \in \mathcal{P}_n, \tag{2.2}$$

where Q is a fixed polynomial of degree $\ell \le n + 1$ with real coefficients without zeros on $[-1, 1]$.

In the special case when the denominator Q has the form

$$Q(x) = x - a \quad (a > 1),$$

the numerator of extremal fraction (2.1) satisfies the equality [15, Sect. 11, Subsect. 38]

$$2^n \left(a + \sqrt{a^2-1}\right) \left(x^{n+1} - P^*(x)\right)$$
$$= \left(ax - 1 + S(a)S(x)\right)\left(x + S(x)\right)^n + \left(ax - 1 - S(a)S(x)\right)\left(x - S(x)\right)^n, \quad S(x) := \sqrt{x^2-1}.$$

Bernstein (1912) noted [4, Paper 7] that Chebyshev's result mentioned above contains a solution of the problem on the polynomial $P_a \in \mathcal{P}_n$ of the best uniform approximation on $[-1,1]$ for the simplest fraction $F_a(x) := 1/(x-a)$, i.e., on the polynomial providing the minimum

$$\min_{P\in\mathcal{P}_n} \|F_a - P\|_{C[-1,1]} = \|F_a - P_a\|_{C[-1,1]} = \frac{1}{(a^2-1)(a+\sqrt{a^2-1})^n}. \tag{2.3}$$

Indeed, let us take the coefficient γ such that the following equality holds:

$$\gamma \frac{x^{n+1} - P^*(x)}{x-a} = \frac{1}{x-a} - R_a(x), \quad R_a \in \mathcal{P}_n. \tag{2.4}$$

By (2.1) and (2.2), the left-hand side of (2.4) has an $(n+2)$-point alternance on $[-1,1]$. Hence, the polynomial R_a coincides with P_a; i.e., it is the polynomial of the best uniform approximation on $[-1,1]$ for the fraction $1/(x-a)$. Multiplying both sides of equality (2.4) by $x-a$ and setting $x = a$, we arrive at the relations $\gamma\left(a^{n+1} - P^*(a)\right) = 1$, $\gamma = \dfrac{2^{n-1}}{(a^2-1)(a+\sqrt{a^2-1})^{n-1}}$.

3. AUXILIARY STATEMENTS

Using the change $x = \cos t$ in (2.1) and (2.2), it is easy to see that, for every $n \in \mathbb{Z}_+$, Chebyshev found the fraction

$$\left(\cos{(n+1)t} - \tau^*(t)\right)/\vartheta(t) \tag{3.1}$$

least deviating from zero with respect to the uniform metric on the period $\mathbb{T} = [-\pi, \pi)$ among all fractions of the form[4]

$$\left(\cos{(n+1)t} - \tau(t)\right)/\vartheta(t), \quad \tau \in \mathcal{T}_n, \tag{3.2}$$

where ϑ is a fixed cosine polynomial of degree $\ell \le n+1$ without real zeros.

Remark 2. Extremal fraction (3.1) has a $2(n+1)$-point alternance on the period $\mathbb{T} = [-\pi, \pi)$. Therefore, the numerator of this fraction has $2(n+1)$ simple zeros on $\mathbb{T}$. The denominator ϑ has no zeros on $\mathbb{T}$. Consequently, dividing the numerator $\cos{(n+1)t} - \tau^*(t)$ by $\vartheta(t)$, we obtain the remainder $f^*(t)$ different from identical zero if $\deg\vartheta \ge 1$; i.e.,

$$(\cos{(n+1)t} - \tau^*(t))/\vartheta(t) = f^*(t) - g^*(t),$$

where

$$f^*(t) = h^*(t)/\vartheta(t) \not\equiv 0, \quad h^* \in \mathcal{T}_{\ell-1}, \quad \ell = \deg\vartheta \ge 1, \quad g^* \in \mathcal{T}_{n+1-\ell} \subseteq \mathcal{T}_n;$$

[4] Statement (3.1), (3.2) is equivalent to Chebyshev's result (2.1), (2.2) under the condition that polynomials $\tau \in \mathcal{T}_n$ are even. However, we can neglect this condition, since extremal fraction (3.1) satisfies it automatically by the known reasoning (see [6, Ch. 1, Sect. 4, the lemma]) based on extracting the even part.

moreover, $g^* \in \mathcal{T}_n$ is the unique polynomial of the best uniform approximation on $\mathbb{T}$ for f^*. If the denominator $\vartheta \equiv \text{const}$ is nonzero, then, as Chebyshev proved[5] [14] (1854), we have the polynomial $\tau^* \equiv 0$ and the remainder $f^* \equiv 0$.

Let us consider result (3.1), (3.2) for the denominator

$$\vartheta(t) = 1 + q^2 - 2q\cos t, \quad q \in (-1, 1), \tag{3.3}$$

in detail. In this case, by Remark 2, the remainder $f^* = f_q^*$ takes the form

$$f_q^*(t) = \frac{c(q,n)}{1 + q^2 - 2q\cos t}, \quad t \in \mathbb{R},$$

where the value $c(q,n)$ depends only on q and n; moreover,

$$c(q,n) = 0 \quad \text{for} \quad q = 0, \qquad c(q,n) \neq 0 \quad \text{for} \quad 0 < |q| < 1.$$

Using the reasoning mentioned above (see footnotes 4 and 5), it is easy to see that assertion (2.3) implies a solution of the problem on the value

$$E_n(f_q) := \min_{\tau \in \mathcal{T}_n} \|f_q - \tau\|_{C(\mathbb{T})} = \|f_q - \tau_q\|_{C(\mathbb{T})} = \frac{4|q|^{n+2}}{(1-q^2)^2}$$

of the best uniform approximation on $[-\pi, \pi)$ of the fraction

$$f_q(t) = \frac{1}{\beta - \cos t}, \quad \beta = \frac{1}{2}\Big(q + \frac{1}{q}\Big),$$

by the subspace $\mathcal{T}_n$. In [4, Papers 7–9], there is a representation for the difference

$$f_q(t) - \tau_q(t) = s_q\, B(t),$$

where the value[6] $s_q \geq 0$ is independent of t and

$$B(t) = \frac{\cos(n+1)t - 2q\cos nt + q^2\cos(n-1)t}{1 + q^2 - 2q\cos t} = -\cos\left[nt - \arccos\frac{2q - (1+q^2)\cos t}{1 + q^2 - 2q\cos t}\right]. \tag{3.4}$$

The latter equality in (3.4) is defined for $t \in [0, \pi]$. Let us give a representation of the fraction B defined on $[-\pi, \pi)$:

$$B(t) = \cos\psi(t) \quad \text{for} \quad t \in [-\pi, \pi), \quad \text{where} \quad \psi(t) = nt + 2\arctan\Big(\frac{1+q}{1-q}\tan\frac{t}{2}\Big); \tag{3.5}$$

the value of the function $\arctan\Big(\frac{1+q}{1-q}\tan\frac{t}{2}\Big)$ at the point $t = -\pi$ is considered to be equal to $-\pi/2$, i.e., to its right-hand one-sided limit. The function ψ is increasing on $[-\pi, \pi)$, since

$$\psi'(t) = n + \frac{1-q^2}{1 + q^2 - 2q\cos t} > 0, \quad t \in [-\pi, \pi). \tag{3.6}$$

[5] This fact was obtained by Chebyshev in terms of algebraic polynomials. Here, this fact is formulated in trigonometric form in view of the reasoning mentioned above (see footnote 4).

[6] It will be seen from the reasoning below that $s_q = E_n(f_q)$.

Hence, it is seen that the value of the function $\psi(t)$ is strictly increasing and runs over the half-open interval $[-(n+1)\pi, (n+1)\pi)$ as t runs over the half-open interval $[-\pi, \pi)$. Consequently, B has an alternance A consisting of $2(n+1)$ points on the period $\mathbb{T} = [-\pi, \pi)$:

$$A = \big\{ -\pi = t_{-(n+1)} < t_{-n} < \ldots < t_{-2} < t_{-1} < 0 = t_0 < t_1 < t_2 < \ldots < t_n \big\}. \qquad (3.7)$$

Therefore, by Chebyshev's theorem (see [6, Ch. 1, Sects. 2, 5]), the fraction B least deviates from zero on $\mathbb{T}$ with respect to the uniform metric among all fractions of form (3.2) with the denominator $\vartheta(t)$ defined by formula (3.3).

We denote by Φ_n the subspace

$$\Phi_n = \operatorname{Lin}\left\{ \frac{1}{\vartheta(t)}, \quad \frac{\cos t}{\vartheta(t)}, \quad \frac{\sin t}{\vartheta(t)}, \quad \frac{\cos 2t}{\vartheta(t)}, \quad \frac{\sin 2t}{\vartheta(t)}, \quad \ldots, \quad \frac{\cos nt}{\vartheta(t)}, \quad \frac{\sin nt}{\vartheta(t)} \right\}; \qquad (3.8)$$

it is a Chebyshev subspace on $\mathbb{T}$, since $\vartheta(t) > 0$ for all $t \in \mathbb{R}$. Thus, we have

$$\min_{Y \in \Phi_n} \left\| \frac{\cos(n+1)t}{\vartheta(t)} - Y(t) \right\|_{C(\mathbb{T})} = \|B\|_{C(\mathbb{T})} = 1, \quad n \in \mathbb{N}. \qquad (3.9)$$

Let us describe the properties of alternance (3.7) that will be used in Sections 5 and 6. Since

$$B'(t) = -\psi'(t) \sin\psi(t) = -\left(n + \frac{1-q^2}{1+q^2-2q\cos t} \right) \sin\psi(t), \quad t \in [-\pi, \pi),$$

the alternance A coincides with the set of zeros of the function

$$\sin\psi(t) = \frac{\sin(n+1)t - 2q\sin nt + q^2 \sin(n-1)t}{\vartheta(t)} \qquad (3.10)$$

on $[-\pi, \pi)$, i.e., with the set of zeros of the sine polynomial

$$\widetilde{\tau}(t) := \sin(n+1)t - 2q\sin nt + q^2\sin(n-1)t \qquad (3.11)$$

on $[-\pi, \pi)$. Consequently, set (3.7) has the property $t_{-k} = -t_k$ for $k = 1, 2, \ldots, n$. In other words, the alternance A has the following structure[7]:

$$A = \big\{ -\pi, \ -t_n, \ \ldots, \ -t_2, \ -t_1, \ 0, \ t_1, \ t_2, \ \ldots, \ t_n \big\}. \qquad (3.12)$$

In [17, Theorem 2], Geronimus gave the formula

$$\|\widetilde{\tau}\|_L = 4(1+q^2), \quad q \in (-1, 1). \qquad (3.13)$$

This formula is a special case of statement[8] (3.14) from Lemma 1, which, in its turn, is contained in a more general statement proved in [9, Theorem 1].

[7]Note that the set $e^{iA} := \{z = e^{it} : t \in A\}$ located on the unit circle of the complex plane is symmetric with respect to the real axis; i.e., it is self-conjugate.

[8]Setting $p(z) := z - q$, let us rewrite expressions (3.3) and (3.11): $\vartheta(t) = 1 + q^2 - 2q\cos t = |p(z)|^2$, $\widetilde{\tau}(t) = \operatorname{Re}\big\{e^{-i\pi/2} z^{n-1} p^2(z)\big\}$, where $z = e^{it}$, $t \in \mathbb{R}$. Applying Lemma 1, we arrive at equality (3.13).

Lemma 1 (Geronimus). *Suppose that* $\xi \in \mathbb{R}$, $r, n \in \mathbb{N}$, $n \le r < 3n$, $z = e^{it}$, $t \in \mathbb{R}$, *and a trigonometric polynomial* $\tau(t) := \mathrm{Re}\{e^{i\xi} z^{2n-r} p^2(z)\}$ *is given, where*[9] $p(z) = \prod_{j=1}^{r-n}(z - z_j)$, $z_j \in \mathbb{C}$, *and all* $|z_j| < 1$. *Then,*

$$\|\tau\|_L = \frac{2}{\pi}\int_0^{2\pi} |p(e^{it})|^2\,dt. \tag{3.14}$$

Below, we will need Lemma 2; we use in its statement points from the set A (see (3.10)–(3.12)); notation (3.3), (3.8); and the fraction

$$\mathcal{R}(t) = \frac{\vartheta(t)}{\vartheta_a(t)}, \quad \text{where} \quad \vartheta_a(t) = 1 - a\cos t, \quad a = \frac{2qn}{1 - q^2 + n(1+q^2)}.$$

It is easy to verify that, for all $q \in (-1,1)$, $n \in \mathbb{N}$, and $t \in \mathbb{R}$, the following inequalities are valid:

$$|a| < 1, \qquad 0 < \vartheta(t), \qquad 0 < \vartheta_a(t), \qquad 0 < \mathcal{R}(t).$$

Lemma 2. *Let* $q \in (-1,1)$ *and* $n \in \mathbb{N}$. *Then, the following formula holds for all* $Y \in \Phi_n$:

$$(-1)^{n+1}\mathcal{R}(-\pi)\,Y(-\pi) + \mathcal{R}(0)\,Y(0) + \sum_{k=1}^{n}(-1)^k\,\mathcal{R}(t_k)\big\{Y(-t_k) + Y(t_k)\big\} = 0. \tag{3.15}$$

Lemma 2 is proved in Section 6, where we also present (for completeness) a proof of Lemma 1 different from the original proof.

4. A THEOREM BY GERONIMUS

Geronimus [17, Theorem 2; 18; 9] proved the following statement.

Theorem (Geronimus). *Let* $f_{n,q}(t) := \sin(n+1)t - 2q\sin nt$, $n \in \mathbb{N}$, *and* $q \in \mathbb{R}$. *Then,*

$$E_{n-1}(f_{n,q})_L = \|f_{n,q}(t) + \sin(n-1)t)\|_L = 8|q| \quad \text{for} \quad |q| \ge 1, \tag{4.1}$$

$$E_{n-1}(f_{n,q})_L = \|f_{n,q}(t) + q^2\sin(n-1)t)\|_L = 4(1+q^2) \quad \text{for} \quad |q| < 1. \tag{4.2}$$

We recall that, by Jackson's theorem [19] (see [1, Ch. 2, Sect. 49, the theorem; 16, Ch. 3, Theorem 10.9]), the polynomial of the best integral approximation for $f_{n,q}$ is unique.

At present, in addition to the original proof [9, 17, 18] of this theorem, there are several distinct methods of justifying assertions (4.1) and (4.2) (see [7; 20; 8; 13, Ch. 2, Sect. 2.2, Subsect. 2.2.4; 2, Theorem 4]).

Below, we present one more variant of the proof of the Geronimus theorem. To prove assertion (4.1), we apply the fact that the function $|\sin nt|$ is orthogonal to the harmonic $\cos t$ on the period; we also apply the operator

$$\Lambda F(t) := \frac{1}{2n}\sum_{k=0}^{2n-1}(-1)^k F\Big(t + \frac{k\pi}{n}\Big), \tag{4.3}$$

[9] Here and subsequently, if the lower index in a product is greater than the upper one, then the product is considered to be equal to 1.

which preserves harmonics of degree $(2m-1)n$, where m is an arbitrary positive integer, and eliminates the other harmonics. This operator was used implicitly in [5, Paper 51, Ch. 1, Sect. 3] and explicitly in [10]. The proof of assertion (4.2) is based on representation (3.5) of the polynomial $\cos(n+1)t - \tau^*(t)$, $\tau^* \in \mathcal{T}_n$, least deviating from zero on $\mathbb{T}$ with respect to the uniform metric with weight $w(t) = 1/(1+q^2-2q\cos t)$, $q \in (-1,1)$; on the duality relations [13, Ch. 2, Sect. 2.7, Subsect. 2.7.4]; and on the classical techniques of interpolation theory. In other words, (3.5) contains a *nonlinear* component of the solution of the problem, i.e., the information about an *alternance, which is a hypothetical arrangement of points of the canonical set.* To prove the assumption, it remains to solve a *linear* problem on the coefficients of a linear combination of delta functions with supports at the points of the alternance.

5. THE PROOF OF THE GERONIMUS THEOREM

Let us first prove assertion (4.1). For $|q| \ge 1$, we consider the polynomial

$$F_q(t) = \sin(n+1)t + 2q\sin nt + \tau(t), \quad \tau \in \mathcal{T}_{n-1}.$$

Applying the operator Λ to F_q (see (4.3)), we obtain

$$\Lambda F_q(t) = 2q\sin nt, \quad \|2q\sin nt\|_L = \|\Lambda F_q(t)\|_L = \left\| \frac{1}{2n}\sum_{k=0}^{2n-1}(-1)^k F_q\Big(t+\frac{k\pi}{n}\Big)\right\|_L \le \|F_q\|_L. \tag{5.1}$$

The function $|\sin nt|$ is orthogonal to the harmonic $\cos t$ on $\mathbb{T} = [-\pi,\pi)$. Since $|q| \ge 1$, the function $|q| \pm \cos t$ is nonnegative and the function $q + \cos t$ has a constant sign; consequently,

$$\begin{aligned} 8|q| = \|2q\sin nt\|_L &= \int_{\mathbb{T}} |2\sin nt|\,|q|\,dt = \int_{\mathbb{T}} |2\sin nt| \big\{(\operatorname{sgn} q)\,q + (\operatorname{sgn} q)\cos t\big\}\,dt \\ &= \int_{\mathbb{T}} |2(q+\cos t)\sin nt|\,dt = \int_{\mathbb{T}} |\sin(n+1)t + 2q\sin nt + \sin(n-1)t|\,dt. \end{aligned}$$

From this and (5.1), we obtain the relations

$$8|q| = \|\sin(n+1)t + 2q\sin nt + \sin(n-1)t\|_L \le \|F_q\|_L, \tag{5.2}$$

which remain valid if we replace q by $-q$. In addition, taking into account the arbitrariness of the choice of a polynomial $\tau \in \mathcal{T}_{n-1}$ in the definition of F_q, we conclude that (5.2) implies (4.1).

Now, let us prove assertion (4.2). Applying formula (3.15) from Lemma 2 to the function

$$Y(t) = \vartheta_a(t)\tau(t)/\vartheta(t) \in \Phi_n, \quad \tau \in \mathcal{T}_{n-1},$$

we obtain

$$(-1)^{n+1}\tau(-\pi) + \tau(0) + \sum_{k=1}^{n}(-1)^k\big\{\tau(-t_k) + \tau(t_k)\big\} = 0, \quad \tau \in \mathcal{T}_{n-1},$$

where the nodes $-\pi,\ 0,\ \pm t_1,\ \pm t_2,\ \ldots,\ \pm t_n$ are zeros of the polynomial $\widetilde{\tau}$ defined by formula (3.11) on $[-\pi,\pi)$. Hence (see the second group of conditions in (1.2)), the function $\operatorname{sgn}\widetilde{\tau}(t)$ is orthogonal to all trigonometric polynomials from $\mathcal{T}_{n-1}$ with zero mean value on the period $\mathbb{T}$. The function

$\operatorname{sgn}\widetilde{\tau}(t)$ is odd; hence, it is also orthogonal to constants. Thus, the function $\operatorname{sgn}\widetilde{\tau}(t)$ is orthogonal to the whole subspace $\mathcal{T}_{n-1}$.

Consequently, the set of zeros of the polynomial $\widetilde{\tau}(t) = \sin(n+1)t - 2q\sin nt + q^2\sin(n-1)t$ on $\mathbb{T}$ is canonical for $\mathcal{T}_{n-1}$. Hence, using Markov's theorem (see, for example, [2, Theorem 1]) and equality (3.13), we arrive at assertion (4.2). The theorem is proved. □

6. PROOF OF AUXILIARY STATEMENTS

Proof of Lemma 2. The duality relations [13, Ch. 2, Sect. 2.7, Subsect. 2.7.4] for (3.9) imply the existence of $2(n+1)$ real numbers

$$\mu(-\pi),\ \mu(-t_n),\ \ldots,\ \mu(-t_1),\ \mu(0),\ \mu(t_1),\ \ldots,\ \mu(t_n)$$

such that

$$\mu(-\pi)Y(-\pi) + \mu(0)Y(0) + \sum_{k=1}^{n}\big(\mu(-t_k)Y(-t_k) + \mu(t_k)Y(t_k)\big) = 0, \quad Y \in \Phi_n. \tag{6.1}$$

Assertion (6.1) allows us to prove the coincidence of the functions $\mathcal{R}$ and μ (up to a constant nonzero factor) on the alternance (see (3.12))

$$A = \big\{-\pi,\ -t_n,\ \ldots, -t_1,\ 0,\ t_1,\ \ldots,\ t_n\big\}. \tag{6.2}$$

Let us construct several trigonometric fractions from Φ_n that vanish at almost all points of the alternance. Such fractions are similar to fundamental interpolation polynomials.

Consider the following even function from Φ_n:

$$Y^*(t) := \frac{\sin\psi(t)}{\sin t} = \frac{\sin(n+1)t - 2q\sin nt + q^2\sin(n-1)t}{(1-2q\cos t + q^2)\sin t}. \tag{6.3}$$

This function vanishes at all points of the alternance A except for the points 0 and $-\pi$. We find the values of the function Y^* at these two points, using the l'Hospital rule:

$$Y^*(0) = \lim_{t\to 0}\frac{\sin\psi(t)}{\sin t} = \lim_{t\to 0}\frac{\psi'(t)\cos\psi(t)}{\cos t} = n + d, \quad \text{where} \quad d := \frac{1+q}{1-q}; \tag{6.4}$$

$$Y^*(-\pi) = (-1)^n(n + 1/d). \tag{6.5}$$

Applying (6.1) to Y^*, we obtain $\mu(-\pi)Y^*(-\pi) + \mu(0)Y^*(0) = 0$, which, together with (6.4) and (6.5), implies the equality

$$(-1)^n(n+1/d)\mu(-\pi) + (n+d)\mu(0) = 0. \tag{6.6}$$

For a given $k \in \{1, 2, \ldots, n\}$, we consider the function

$$Y_k(t) := \frac{\sin\psi(t)}{\cos t - \cos t_k} = \frac{\tau_k(t)}{\vartheta(t)}, \quad \text{where} \quad \tau_k(t) = \frac{\sin(n+1)t - 2q\sin nt + q^2\sin(n-1)t}{\cos t - \cos t_k}. \tag{6.7}$$

Since τ_k is a sine polynomial of degree n, Y_k belongs to Φ_n and vanishes at all points from alternance (6.2) except for the points $\pm t_k$. The function Y_k is odd. Substituting this function into (6.1), we obtain $\mu(-t_k)Y_k(t_k) = \mu(t_k)Y_k(t_k)$, which, together with the inequality $Y_k(t_k) \neq 0$, implies the equality $\mu(-t_k) = \mu(t_k)$. This equality and (6.6) allow us to transform (6.1):

$$\left((-1)^{n+1}\,\frac{n+d}{n+1/d}\,Y(-\pi) + Y(0)\right)\mu(0) + \sum_{k=1}^{n}\big(Y(-t_k) + Y(t_k)\big)\mu(t_k) = 0, \quad Y \in \Phi_n. \tag{6.8}$$

For even functions $Y \in \Phi_n$, formula (6.8) can be simplified:

$$\left((-1)^{n+1}\,\frac{n+d}{n+1/d}\,Y(-\pi) + Y(0)\right)\mu(0) + 2\sum_{k=1}^{n} Y(t_k)\mu(t_k) = 0. \tag{6.9}$$

Let us consider the even function $Y_k^*(t) := \dfrac{Y_k(t)}{\sin t} = \dfrac{Y^*(t)}{\cos t - \cos t_k} \in \Phi_n$, $k = 1, 2, \ldots, n$, where Y^* and Y_k are defined above (see (6.3) and (6.7)). The function Y_k^* vanishes at all points of alternance (6.2) except for the points $-\pi$, $-t_k$, 0, and t_k. With the help of (6.4) and (6.5), we arrive at the equalities

$$Y_k^*(-\pi) = \frac{(-1)^{n+1}(n+1/d)}{1+\cos t_k}, \qquad Y_k^*(0) = \frac{n+d}{1-\cos t_k}.$$

Using the l'Hospital rule, in view of (3.6), we find

$$Y_k^*(-t_k) = Y_k^*(t_k) = \frac{(-1)^{k+1}\psi'(t_k)}{\sin^2 t_k} = \frac{(-1)^{k+1}}{\sin^2 t_k}\left(n + \frac{1-q^2}{1+q^2-2q\cos t_k}\right).$$

Applying (6.9) to Y_k^*, we obtain

$$(n+d)\left(\frac{1}{1+\cos t_k} + \frac{1}{1-\cos t_k}\right)\mu(0) = 2\frac{(-1)^k}{\sin^2 t_k}\left(n + \frac{1-q^2}{1+q^2-2q\cos t_k}\right)\mu(t_k).$$

This, for $k = 1, 2, \ldots, n$, implies the equalities

$$(n+d)\,\mu(0) = (-1)^k\left(n + \frac{1-q^2}{\vartheta(t_k)}\right)\mu(t_k), \qquad \mu(t_k) = (-1)^k\,\frac{\gamma\,\vartheta(t_k)}{\vartheta_a(t_k)}\,\mu(0),$$

where $\vartheta_a(t) = 1 - a\cos t$, $a = \dfrac{2qn}{1-q^2+n(1+q^2)}$, and $\gamma = \dfrac{n+d}{1-q^2+n(1+q^2)}$.

Substituting the expression for $\mu(t_k)$ into (6.8), after elementary transformations, we obtain a formula equivalent to (3.15). Lemma 2 is proved. □

In the proofs of Lemmas 3 and 4 below, we use reasoning similar to that applied in [12, Ch. 2, Subsect. 2.6, Theorem 2.6] (see also [3, Ch. 2, Sect. 9, Theorem 9.1]) and in [1, Appendices and Problems, Problem 15; 21, Theorem 1].

Lemma 3. *Suppose that $\xi \in \mathbb{R}$; $\ell, m \in \mathbb{N}$, $\ell \geq m$; and a polynomial $G(z) = e^{i\xi} z^{2m-\ell} h^2(z)$ is given, where $h(z) = \prod_{j=1}^{\ell-m}(z - z_j)$, $z, z_j \in \mathbb{C}$, and all $|z_j| < 1$. Then, the fraction $\dfrac{G(z)}{|G(z)|}$ satisfies the following orthogonality relations on the unit circle $z = e^{it}$, $t \in [0, 2\pi)$:*

$$\int_0^{2\pi} \frac{G(z)}{|G(z)|}\,z^k\,dt = e^{i\xi}\int_0^{2\pi} \frac{z^{2m-\ell}h^2(z)}{|h(z)|^2}\,z^k\,dt = 0 \quad \textit{for all integer} \quad k \geq 1-(m+s), \tag{6.10}$$

where s is the number of z_j equal to zero.

Proof. It is clear that it is sufficient to consider only the case when all z_j are different from zero (i.e., $s = 0$). We set $\gamma = e^{i\xi}$. For $z = e^{it}$, where $t \in \mathbb{R}$, we have

$$\frac{G(z)}{|G(z)|} = \frac{\gamma z^{2m-\ell}h^2(z)}{|\gamma z^{2m-\ell}h^2(z)|} = \frac{\gamma z^{2m-\ell}h^2(z)}{|h(z)|^2} = \frac{\gamma z^{2m-\ell}h^2(z)}{h(z)\overline{h(z)}} = \frac{\gamma z^{2m-\ell}h(z)}{\overline{h(z)}} = \frac{\gamma z^{2m-\ell}h(z)}{\prod\limits_{j=1}^{\ell-m}(\overline{z} - \overline{z}_j)}$$

$$= \frac{\gamma z^{2m-\ell}h(z)}{\prod\limits_{j=1}^{\ell-m}\left(\dfrac{1}{z} - \overline{z}_j\right)} = \frac{\gamma z^m h(z)}{\prod\limits_{j=1}^{\ell-m}\left(1 - z\overline{z}_j\right)} = \frac{\gamma z^m}{\prod\limits_{j=1}^{\ell-m}\overline{z}_j}\,\frac{h(z)}{\prod\limits_{j=1}^{\ell-m}\left(\dfrac{1}{\overline{z}_j} - z\right)} = \frac{\gamma z^m R(z)}{\prod\limits_{j=1}^{\ell-m}\overline{z}_j}, \quad R(z) = \frac{h(z)}{\prod\limits_{j=1}^{\ell-m}\left(\dfrac{1}{\overline{z}_j} - z\right)}.$$

Hence, in the case under consideration, to prove assertion (6.10), it is sufficient to show that

$$\frac{1}{2\pi}\int_0^{2\pi} z^{k+m}R(z)\,dt = 0, \quad k \in \mathbb{Z}, \quad k \geq 1 - m, \quad z = e^{it}. \tag{6.11}$$

For integer $k \geq -m$, the function $z^{k+m}R(z)$ is analytic in the unit disk $|z| \leq 1$ (including its boundary); therefore (see [11, Part 3, Ch. 3, Sect. 2, Problem 118]), its mean value on the unit circle coincides with its value at the point $z = 0$. Consequently, assertion (6.11) is valid. Lemma 3 is proved. □

Lemma 4. *Suppose that $\xi \in \mathbb{R}$; $r, n \in \mathbb{N}$, $r \geq n$; $z = e^{it}$, $t \in \mathbb{R}$; and the fraction*

$$M(t) := \frac{\mathrm{Re}\{e^{i\xi}z^{2n-r}P(z)\}}{|P(z)|}, \quad \textit{where} \quad P(z) = \prod_{j=1}^{2(r-n)}(z - z_j), \quad z_j \in \mathbb{C}, \quad \textit{and all} \quad |z_j| < 1,$$

is given. Then, the function $F(t) := |M(t)| - 2/\pi$ is orthogonal to the subspace $\mathcal{T}_{2n-1}$.

Proof. Let us transform the function $M(t)$:

$$M(t) = \mathrm{Re}\left\{\frac{Q(z)}{|P(z)|}\right\} = \mathrm{Re}\left\{e^{i\xi}z^{2n-r}\frac{P(z)}{|P(z)|}\right\} = \mathrm{Re}\{e^{i\varphi(t)}\} = \cos\varphi(t); \tag{6.12}$$

here,

$$Q(z) := e^{i\xi}z^{2n-r}P(z), \quad \varphi(t) = \arg\{Q(z)\}, \quad z = e^{it}, \quad t \in \mathbb{R}.$$

Further reasoning is based on the representation of the function $|\cos\varphi(t)|$ as the series

$$|\cos\varphi(t)| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{4\nu^2 - 1}\cos 2\nu\varphi(t). \tag{6.13}$$

Fix an arbitrary positive integer ν and consider the function

$$\cos 2\nu\varphi(t) = \mathrm{Re}\{e^{i2\nu\varphi(t)}\} = \mathrm{Re}\left\{\left(\frac{e^{i\xi}z^{2n-r}P(z)}{|P(z)|}\right)^{2\nu}\right\}. \tag{6.14}$$

Introduce the notation

$$m = 2\nu n, \quad \ell = 2\nu r, \quad h(z) = P^\nu(z) = \prod_{j=1}^{2(r-n)} (z - z_j)^\nu. \tag{6.15}$$

Let us represent the polynomial h in the form

$$h(z) = \prod_{\mu=1}^{\ell-m} (z - \zeta_\mu) \qquad \big(\zeta_\mu \in \mathbb{C}, \quad \text{all} \quad |\zeta_\mu| < 1\big). \tag{6.16}$$

Rewrite formula (6.14) in terms of (6.15)–(6.16):

$$\cos 2\nu\varphi(t) = \operatorname{Re}\left\{\frac{e^{i2\nu\xi} z^{2m-\ell} h^2(z)}{|h(z)|^2}\right\}. \tag{6.17}$$

By Lemma 3, the following orthogonality relations hold on the unit circle $z = e^{it}$, $t \in [0, 2\pi)$:

$$\int_0^{2\pi} \frac{e^{i2\nu\xi} z^{2m-\ell} h^2(z)}{|h(z)|^2}\, z^k\, dt = 0 \quad \text{for all integer} \quad k \geq 1 - m = 1 - 2\nu n \geq 1 - 2n.$$

Therefore (see (6.17)), the function $\cos 2\nu\varphi(t)$ is orthogonal to the subspace $\mathcal{T}_{2n-1}$ for any $\nu \in \mathbb{N}$. Hence, using (6.12) and (6.13), we obtain the statement of the lemma. □

Proof of Lemma 1. The statement of the lemma follows from Lemma 4 and from the fact that the function $f(t) := |p(e^{it})|^2$ is a trigonometric polynomial of degree $\deg f = r - n \leq 2n - 1$.

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