

On the Exact Constant in the Jackson–Stechkin Inequality for the Uniform Metric

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Abstract The classical Jackson–Stechkin inequality estimates the value of the best uniform approximation of a 2π -periodic function f by trigonometric polynomials of degree $\leq n - 1$ in terms of its r -th modulus of smoothness $\omega_r(f, \delta)$. It reads

$$E_{n-1}(f) \leq c_r \omega_r\left(f, \frac{2\pi}{n}\right),$$

where c_r is *some* constant that depends only on r . It has been known that c_r admits the estimate $c_r < r^{ar}$ and, basically, nothing else has been proved.

The main result of this paper is in establishing that

$$\left(1 - \frac{1}{r+1}\right) \gamma_r^* \leq c_r < 5\gamma_r^* \quad \gamma_r^* = \frac{1}{\binom{r}{\lfloor \frac{r}{2} \rfloor}} \asymp \frac{r^{1/2}}{2^r},$$

i.e., that the Stechkin constant c_r , far from increasing with r , does in fact decay exponentially fast. We also show that the same upper bound is valid for the constant $c_{r,p}$ in the Stechkin inequality for L_p -metrics with $p \in [1, \infty)$, and for small r we present upper estimates which are sufficiently close to $1 \cdot \gamma_r^*$.

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1 Introduction

The classical Jackson–Stechkin inequality estimates the value of the best uniform approximation of a 2π -periodic function f by trigonometric polynomials of degree $\leq n - 1$ in terms of its r -th modulus of smoothness $\omega_r(f, \delta)$. It reads

$$E_{n-1}(f) \leq c_r \omega_r\left(f, \frac{2\pi}{n}\right), \quad (1.1)$$

where c_r is *some* constant which depends only on r (see [10] or [3, p. 205]).

Besides the case $r = 1$, hardly any attempts have been made to find the best value of this constant c_r , or even to determine its dependence on r . Stechkin's original proof [10] (as well as alternative ones) allows us to obtain the estimate $c_r < r^{ar}$, and, basically, nothing else has been proved.

The main result of this paper is in establishing that

$$\left(1 - \frac{1}{r+1}\right) \gamma_r^* \leq c_r < 5\gamma_r^*, \quad \gamma_r^* = \frac{1}{\binom{r}{\lfloor \frac{r}{2} \rfloor}} \asymp \frac{r^{1/2}}{2^r}, \quad (1.2)$$

i.e., that the Stechkin constant c_r , far from increasing with r , does in fact decay exponentially fast. (Here and elsewhere, the asymptote relation $p(r) \asymp q(r)$ means that, with some absolute constants a_1, a_2 , we have $a_1 p(r) \leq q(r) \leq a_2 p(r)$ for all r .)

We also show that the same upper bound is valid for the constant $c_{r,p}$ in the Stechkin inequality for L_p -metrics with $p \in [1, \infty)$, and for small r we present upper estimates which are sufficiently close to $1 \cdot \gamma_r^*$.

In retrospect, such a result could have been anticipated, since for trigonometric approximation in the L_2 -metric, Chernykh [2] in 1967 already established that

$$E_{n-1}(f)_2 \leq c_{r,2} \omega_r\left(f, \frac{2\pi}{n}\right)_2, \quad c_{r,2} = \frac{1}{\sqrt{\binom{2r}{r}}} \asymp \frac{r^{1/4}}{2^r}, \quad (1.3)$$

proving also that such a $c_{r,2}$ is the best possible (for the argument $\delta = \frac{2\pi}{n}$ in $\omega_r(f, \delta)$). However, this result was based on the Fourier technique for L_2 -approximation, and that does not work in other L_p -metrics.

Our method of proving (1.2) is based on deriving first the intermediate inequality

$$\|f\| \leq c_{n,r}(\delta) \omega_r(f, \delta), \quad f \in T_{n-1}^\perp, \quad (1.4)$$

which is valid for the functions f which are orthogonal to the trigonometric polynomials of degree $\leq n - 1$. This may be viewed as a difference analogue of the classical

Bohr-Favard inequality for differentiable functions

$$\|f\| \leq \frac{F_r}{n^r} \|f^{(r)}\|, \quad f \in T_{n-1}^\perp,$$

and is of independent interest.

We proceed from the Bohr-Favard-type inequality (1.4) to the Stechkin one (1.1) by approximating f with the de la Vallée Poussin sums $v_{m,n}(f)$ and using the fact that

$$f - v_{m,n}(f) \in T_m^\perp, \quad \omega_r(f - v_{m,n}(f), \delta) \leq (1 + \|v_{m,n}\|)\omega_r(f, \delta).$$

With that we arrive at the inequality

$$E_{n-1}(f) \leq \|f - v_{m,n}(f)\| \leq c_{m,n,r}(\delta)\omega_r(f, \delta),$$

where we finally minimize the resulting constant over m , for given r , n and δ .

2 Results

Let $C[-\pi, \pi]$ be the space of continuous 2π -periodic functions. For $f \in C[-\pi, \pi]$, we denote by $E_{n-1}(f)$ the value of best approximation of f by trigonometric polynomials of degree $\leq n - 1$ in the uniform norm,

$$E_{n-1}(f) := \inf_{\tau \in T_{n-1}} \|f - \tau\|,$$

and by $\omega_r(f, \delta)$ its r -th modulus of smoothness with the step δ ,

$$\omega_r(f, \delta) := \sup_{0 < h \leq \delta} \|\Delta_h^r(f, \cdot)\|,$$

where $\Delta_h^r(f, x)$ is the forward difference of order r of f at x with the step h :

$$\Delta_h^r(f, x) = \sum_{i=0}^r (-1)^i \binom{r}{i} f(x + ih).$$

We will study the best constant $K_{n,r}(\delta)$ in the Stechkin inequality

$$E_{n-1}(f) \leq K_{n,r}(\delta)\omega_r(f, \delta),$$

i.e., the quantity

$$K_{n,r}(\delta) := \sup_{f \in C} \frac{E_{n-1}(f)}{\omega_r(f, \delta)},$$

which depends on the given parameters $n, r \in \mathbb{N}$ and $\delta \in [0, 2\pi]$.

In such a setting (which goes back to Korneichuk and Chernykh) we may safely consider $\delta = \frac{\alpha\pi}{n}$ with some α not necessarily 1 or 2. The choice of particular δ 's can be based on two criteria:

- (1) “nice” look and/or tradition: $\delta = \frac{\pi}{n}$, or $\delta = \frac{2\pi}{n}$, or (why not) $\delta = \frac{1}{n}$, and the like;
- (2) “nice” result:

$$\sup_{f \in C} \frac{E_{n-1}(f)}{\omega_r(f, \delta)} \asymp c_{n,r}(\delta).$$

Ideally, both approaches should be combined to provide nice results for nice δ 's, but that doesn't happen very often.

In this paper we obtain the following results:

- (1) First of all, we show that the exact order of the Stechkin constant $K_{n,r}(\delta)$ at $\delta = \frac{2\pi}{n}$ (and in fact at any $\delta \in [\frac{2\pi}{n}, \frac{\pi}{r})$) is $r^{1/2}2^{-r}$, namely

$$K_{n,r}\left(\frac{2\pi}{n}\right) \asymp \gamma_r^* \asymp \frac{r^{1/2}}{2^r},$$

where

$$\gamma_r^* = \frac{1}{\binom{r}{\lfloor \frac{r}{2} \rfloor}} = \begin{cases} \frac{1}{\binom{2k}{k}}, & r = 2k; \\ \frac{1}{\binom{2k-1}{k-1}}, & r = 2k - 1. \end{cases}$$

Moreover, we locate the exact value of this constant within quite a narrow interval.

Theorem 2.1 *We have*

$$c'_r\left(\frac{2\pi}{n}\right)\gamma_r^* \leq \sup_{f \in C} \frac{E_{n-1}(f)}{\omega_r(f, \frac{2\pi}{n})} \leq c_r\left(\frac{2\pi}{n}\right)\gamma_r^*,$$

where

$$c'_r\left(\frac{2\pi}{n}\right) = \begin{cases} 1 - \frac{1}{r+1}, & r = 2k - 1; \\ 1, & r = 2k; \end{cases} \quad n > 2r,$$

and

$$c_r\left(\frac{2\pi}{n}\right) = 5, \quad n \geq 1.$$

It is surprising that, in this theorem, the upper estimate is provided by one and the same linear method of approximation that works for all r simultaneously. Namely, for any r , the de la Vallée Poussin operator $v_{m,n}$ with $m = \lfloor \frac{8}{9}n \rfloor$ provides

$$\|f - v_{m,n}(f)\| \leq 5\gamma_r^* \omega_r\left(f, \frac{2\pi}{n}\right), \quad \forall r \in \mathbb{N}.$$

Perhaps it makes sense to try to derive such an estimate from the properties of $v_{m,n}$ directly.

- (2) Next, we show that the value of the constant $c_{n,r}(\delta)$ remains bounded uniformly in r and n also for $\frac{\pi}{n} < \delta < \frac{2\pi}{n}$ (but it grows to infinity as δ approaches $\frac{\pi}{n}$).

Theorem 2.2 *For any $\alpha > 1$, there exists a constant c_α which depends only on α such that*

$$E_{n-1}(f) \leq c_\alpha \gamma_r^* \omega_r \left(f, \frac{\alpha\pi}{n} \right), \quad n \geq 1.$$

(The constant $c'_\alpha = c'_r(\frac{\alpha\pi}{n})$ in a similar lower bound for $E_{n-1}(f)$ admits, of course, the same lower estimate $c'_\alpha \geq 1 - \frac{1}{r+1}$ as in Theorem 2.1.)

(3) Thirdly, although we did not succeed in reaching the argument $\delta = \frac{\pi}{n}$ with an absolute constant in front of $\gamma_r^* \omega_r(f, \delta)$, we prove that this constant grows like $\mathcal{O}(\sqrt{r} \ln r)$ at most.

Theorem 2.3 *For $\delta = \frac{\pi}{n}$, we have the estimate*

$$E_{n-1}(f) \leq c_r \left(\frac{\pi}{n} \right) \gamma_r^* \omega_r \left(f, \frac{\pi}{n} \right), \quad c_r \left(\frac{\pi}{n} \right) = \mathcal{O}(\sqrt{r} \ln r), \quad n \geq 1.$$

(4) Fourthly, for small r , the general upper bound $c_r(\frac{2\pi}{n}) = 5$ can be decreased to the values that are quite close to the lower bound $c'_r \approx 1$, thus giving support to the (upcoming) conjecture that $K_{n,r}(\delta) = 1 \cdot \gamma_r^*$ for $\delta \geq \frac{\pi}{n}$.

Theorem 2.4 *For $\delta = \frac{\pi}{n}$ and $\delta = \frac{2\pi}{n}$, we have*

$$E_{n-1}(f) \leq c_r(\delta) \gamma_r^* \omega_r(f, \delta),$$

where $c_{2k-1}(\delta) = c_{2k}(\delta)$, and the values of $c_{2k}(\delta)$ are given below:

$$\frac{c_2(\frac{\pi}{n})}{1\frac{1}{4}} \Big| \frac{c_4(\frac{\pi}{n})}{2\frac{7}{10}}, \quad \frac{c_2(\frac{2\pi}{n})}{1\frac{1}{16}} \Big| \frac{c_4(\frac{2\pi}{n})}{1\frac{1}{9}} \Big| \frac{c_6(\frac{2\pi}{n})}{1\frac{1}{2}}.$$

(5) Finally, all upper estimates in Theorems 2.1–2.4 remain valid for any $p \in [1, \infty]$. (There is no need to give a separate proof of this statement, since all the inequalities we used in the text still hold for the L_p -metrics, $1 \leq p < \infty$, in particular the Bohr-Favard inequality (4.1) and the inequalities of Sect. 5 involving the norms of the de la Vallée Poussin operator.)

Theorem 2.5 *For any $p \in [1, \infty]$, we have*

$$E_{n-1}(f)_p \leq c_r(\delta) \gamma_r^* \omega_r(f, \delta)_p,$$

with the same constants $c_r(\delta)$ and the same δ 's as in Theorems 2.1–2.4. In particular,

$$E_{n-1}(f)_p \leq 5 \gamma_r^* \omega_r \left(f, \frac{2\pi}{n} \right)_p.$$

The latter L_p -estimate is hardly of the right order for $1 < p < \infty$ because $\gamma_r^* \asymp r^{1/2} 2^{-r}$, while, for $p = 2$, Chernykh's result (1.3) says that $K_{n,r}(\frac{2\pi}{n})_2 \asymp r^{1/4} 2^{-r}$, so

one may guess that

$$K_{n,r} \left(\frac{2\pi}{n} \right)_p \asymp r^{\max(1/2p, 1/2p')} 2^{-r}.$$

This guess is partially based on the results of Ivanov [7] who obtained such an upper bound for the values $K_{n,r}(\delta)_p$ with relatively large $\delta = \frac{\pi r^{1/3}}{n}$, and proved that, for $p \in [2, \infty]$, the order of the lower bounds is the same.

(6) The value $\delta = \frac{\pi}{n}$ is critical in the sense that the Stechkin constant $K_{n,r}(\delta)$ and the constant γ_r^* are no longer of the same (exponential) order for $\delta = \frac{\alpha\pi}{n}$ with $\alpha < 1$. Indeed, in this case, with $f_0(x) := \cos nx$, we have $\omega_r(f_0, \delta) = 2^r \sin^r \frac{\alpha\pi}{2}$ (see (7.2)) and $E_{n-1}(f_0) = 1$, so that, for $\alpha < 1$, we have

$$\frac{K_{n,r}(\frac{\alpha\pi}{n})}{\gamma_r^*} > \frac{c}{r^{1/2} \sin^r \frac{\alpha\pi}{2}} > c_\alpha \lambda_\alpha^r, \quad \lambda_\alpha > 1.$$

This being said, a natural question arises from the two estimates

$$K_{n,r} \left(\frac{2\pi}{n} \right) \asymp \gamma_r^*, \quad K_{n,r} \left(\frac{\pi}{n} \right) \leq c\sqrt{r} \ln r \cdot \gamma_r^*$$

whether an extra factor at $\delta = \frac{\pi}{n}$ is essential. We believe it is not, and we are making the following conjecture.

Conjecture 2.6 *For all $r \in \mathbb{N}$, we have*

$$\sup_{n \in \mathbb{N}} K_{n,r} \left(\frac{\pi}{n} \right) := \sup_{n \in \mathbb{N}} \sup_{f \in C} \frac{E_{n-1}(f)}{\omega_r(f, \frac{\pi}{n})} = 1 \cdot \gamma_r^*, \quad \gamma_r^* = \frac{1}{\lfloor \frac{r}{2} \rfloor}.$$

(Our point is mainly about the upper bound, namely that $K_{n,r}(\delta) \leq 1 \cdot \gamma_r^*$, for any $\delta \geq \frac{\pi}{n}$. The lower bound for even $r = 2k$ is established in this paper, while for odd r we guess that $K_{n,r}(\delta)$ tends to γ_r^* at $\delta = \frac{\pi}{n}$ for large n , but for $\delta > \frac{\pi}{n}$ it takes smaller values.)

This conjecture is true for $r = 1$, for in this case we have Korneichuk’s result [9]:

$$1 - \frac{1}{2n} \leq K_{n,1} \left(\frac{\pi}{n} \right) < 1.$$

For $r = 2$, the conjecture gives the estimate $K_{n,2}(\frac{\pi}{n}) = \frac{1}{2}$ which is (to a certain extent) stronger than Korneichuk’s estimate (because $\omega_2(f, \delta) \leq 2\omega_1(f, \delta)$), so it would be interesting to prove (or to disprove) it in this particular case. Meanwhile, according to Theorems 2.1 and 2.4, we have

$$\frac{1}{2} \leq K_{n,2} \left(\frac{\pi}{n} \right) \leq \frac{5}{8}, \quad \frac{1}{2} \leq K_{n,2} \left(\frac{2\pi}{n} \right) \leq \frac{17}{32}.$$

For arbitrary r , it seems unlikely that the value of the Stechkin constant will ever be precisely determined, but it would be a good achievement to narrow the interval

for $K_{n,r}(\frac{2\pi}{n})$, say, to $[\gamma_r^*, 2\gamma_r^*]$, and to settle the correct order of $K_{n,r}(\frac{\pi}{n})$ with respect to r .

(7) We finish this section with the remark that if, with some constant $c(\delta)$, the inequality

$$E_{n-1}(f) \leq c(\delta)\gamma_r^*\omega_r(f, \delta)$$

is true for an even $r = 2k$, then it is true for the odd $r = 2k - 1$ too, with the same constant $c(\delta)$. Indeed, since $\gamma_{2k-1}^* = 2\gamma_{2k}^*$ and $\omega_{2k}(f, \delta) \leq 2\omega_{2k-1}(f, \delta)$, we have

$$\begin{aligned} E_{n-1}(f) &\leq c(\delta)\gamma_{2k}^*\omega_{2k}(f, \delta) \\ &\leq c(\delta)\gamma_{2k}^* \cdot 2\omega_{2k-1}(f, \delta) = c(\delta)\gamma_{2k-1}^*\omega_{2k-1}(f, \delta). \end{aligned}$$

Therefore, it is sufficient to prove upper estimates only for even $r = 2k$.

(8) The paper is organized as follows. Preliminaries are given in Sects. 3–4. In Sects. 5–6, the upper estimates of Theorems 2.1–2.3 are proved (as Theorems 5.1, 5.2, and 6.1, respectively). In Sect. 7 we discuss a lower estimate for $\delta = \frac{\pi}{n}$, and the lower estimate in Theorem 2.1 is the subject of Sect. 8 (Theorem 8.2). Finally, in Sect. 9 we prove Theorem 2.4 (as Theorem 9.2).

3 Smoothing Operators

Here we present the general idea of our method.

(1) For a fixed k , with

$$\widehat{\Delta}_t^{2k}(f, x) := \sum_{i=-k}^k (-1)^i \binom{2k}{k+i} f(x + it)$$

being the central difference of order $2k$ with the step t , and with ϕ_h being an integrable function which satisfies conditions

$$(a) \quad \phi_h(t) = \phi_h(-t), \quad (b) \quad \text{supp } \phi_h = [-h, h], \quad (c) \quad \int_{\mathbb{R}} \phi_h(t) dt = 1, \tag{3.1}$$

consider the following operator

$$W_h(f, x) := \frac{1}{\binom{2k}{k}} \int_{\mathbb{R}} \widehat{\Delta}_t^{2k}(f, x)\phi_h(t) dt. \tag{3.2}$$

If a given subspace $\mathcal{S} \in C[-\pi, \pi]$ is invariant under the operator W_h , and if the restriction W_h to \mathcal{S} has a bounded inverse, then, for any $f \in \mathcal{S}$, we have a trivial estimate

$$\|f\| \leq \|W_h^{-1}\|_{\mathcal{S}} \|W_h(f)\|, \quad f \in \mathcal{S}. \tag{3.3}$$

It follows immediately from the definition that

$$\|W_h(f)\| \leq \|\phi_h\|_1 \gamma_{2k}^* \omega_{2k}(f, h), \quad \gamma_{2k}^* = \frac{1}{\binom{2k}{k}}, \tag{3.4}$$

and we arrive at the following inequality:

$$\|f\| \leq c_{2k}(h)\gamma_{2k}^*\omega_{2k}(f, h), \quad c_{2k}(h) = \|\phi_h\|_1 \|W_h^{-1}\|_{\mathcal{S}}, \tag{3.5}$$

valid for all functions f from a given subspace \mathcal{S} .

(2) Next, we present W_h as $W_h = I - U_h$, which allows us to get some bounds for $\|W_h^{-1}\|$ in (3.5) in terms of U_h .

To this end, for integer i (and, in fact, for any $i \neq 0$), define the dilations ϕ_{ih} and the convolution operators I_{ih} by the rule

$$\phi_{ih}(t) := \frac{1}{|i|}\phi_h\left(\frac{t}{i}\right), \quad I_{ih}(f) := f * \phi_{ih} := \int_{\mathbb{R}} f(\cdot - t)\phi_{ih}(t) dt, \quad I_0(f) := f.$$

Then, taking into account that

$$\int_{\mathbb{R}} f(x - it)\phi_h(t) dt = \int_{\mathbb{R}} f(x - \tau)\frac{1}{|i|}\phi_h\left(\frac{\tau}{i}\right) d\tau = I_{ih}(f),$$

and that also $I_{ih} = I_{-ih}$ (because ϕ_{ih} is even), we may put W_h in the following form:

$$W_h = \frac{1}{\binom{2k}{k}} \sum_{i=-k}^k (-1)^i \binom{2k}{k+i} I_{ih} = I - 2 \sum_{i=1}^k (-1)^{i+1} a_i I_{ih}, \quad a_i := \frac{\binom{2k}{k+i}}{\binom{2k}{k}}.$$

So, with the further notations

$$U_h := 2 \sum_{i=1}^k (-1)^{i+1} a_i I_{ih}, \quad \psi_{kh} := 2 \sum_{i=1}^k (-1)^{i+1} a_i \phi_{ih},$$

we obtain

$$W_h = I - U_h, \quad U_h(f) = f * \psi_{kh}.$$

Respectively, we may rewrite the inequality (3.5) in the following way:

Lemma 3.1 *If the operator $(I - U_h)^{-1}$ is bounded on a given subspace \mathcal{S} , then, for all $f \in \mathcal{S}$, we have*

$$\|f\| \leq c_{2k}(h)\gamma_{2k}^*\omega_{2k}(f, h), \quad c_{2k}(h) = \|\phi_h\|_1 \|(I - U_h)^{-1}\|_{\mathcal{S}}.$$

(3) Now, we call upon elementary properties of Banach algebras (see, e.g., Kantorovich, Akilov [8, Chap. 5, Sect. 4]) for the claim that if an operator $U : \mathcal{S} \rightarrow \mathcal{S}$ satisfies $\sum_{m=0}^{\infty} \|U^m\| < \infty$, then the operator $I - U$ is invertible, and the norm of its inverse admits the estimate

$$\|(I - U)^{-1}\|_{\mathcal{S}} \leq \sum_{m=0}^{\infty} \|U^m\|_{\mathcal{S}}. \tag{3.6}$$

Lemma 3.1 implies then the following:

Proposition 3.2 *If ϕ_h is such that $\sum_{m=0}^\infty \|U_h^m\|_{\mathcal{S}} = A_h < \infty$, then, for any $f \in \mathcal{S}$, we have*

$$\|f\| \leq c_{2k}(h)\gamma_{2k}^*\omega_{2k}(f, h), \quad c_{2k}(h) = A_h\|\phi_h\|_1.$$

(4) Finally, let us make a short remark about the structure of the subspaces \mathcal{S} that may be taken into consideration. It is clear that, if \mathcal{S} is shift-invariant, i.e., together with f it also contains $f(\cdot + t)$ for any t , then \mathcal{S} is invariant under the action of W_h for any h . A typical example is a subspace \mathcal{S} that contains (or does not contain) certain monomials $(\cos kx, \sin kx)$.

We will consider $\mathcal{S} = T_{n-1}^\perp$, the subspace of functions spanned by $(\cos kx, \sin kx)_{k \geq n}$, which are orthogonal to trigonometric polynomials of degree $\leq n - 1$.

4 A Difference Analogue of the Bohr–Favard Inequality

Denote by T_{n-1}^\perp the set of functions f which are orthogonal to T_{n-1} , i.e., such that

$$\int_{-\pi}^\pi f(x)\tau(x) dx = 0, \quad \forall \tau \in T_{n-1}.$$

The Bohr–Favard inequality for such functions reads

$$\|f\| \leq \frac{F_r}{n^r} \|f^{(r)}\|, \quad f \in T_{n-1}^\perp, \tag{4.1}$$

where F_r are the Favard constants, which are usually defined by the formula

$$F_r := \frac{4}{\pi} \sum_{i=0}^\infty \frac{(-1)^{i(r+1)}}{(2i + 1)^{r+1}},$$

and which satisfy the following relations:

$$F_0 = 1 < F_2 = \frac{\pi^2}{8} < \dots < \frac{4}{\pi} < \dots < F_3 = \frac{\pi^3}{24} < F_1 = \frac{\pi}{2}.$$

In this section we obtain a difference analogue of the Bohr–Favard inequality in the form

$$\|f\| \leq c_{n,2k}(h)\gamma_{2k}^*\omega_{2k}(f, h), \quad f \in T_{n-1}^\perp,$$

using the approach from the previous section (Proposition 3.2). Namely, we consider the operator

$$U_h = 2 \sum_{i=1}^k (-1)^{i+1} a_i I_{ih}, \quad a_i = \binom{2k}{k+i} / \binom{2k}{k},$$

with the following specific choice of I_h (and respectively of ϕ_h):

$$I_h(f, x) := \frac{1}{h^2} \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} f(x - t_1 - t_2) dt_1 dt_2,$$

i.e., taking $I_h(f)$ as the Steklov function of order 2. It is known that $I_h(f, x) = f * \phi_h$, where

$$\phi_h(t) = \begin{cases} \frac{1}{h}(1 - \frac{|t|}{h}), & t \in [-h, h], \\ 0, & \text{otherwise,} \end{cases}$$

i.e., ϕ_h is the L_1 -normalized B-spline of order 2 (the hat-function) with the step-size h supported on $[-h, h]$. One can easily verify that

$$I''_{ih}(f, x) = -\frac{1}{(ih)^2} \widehat{\Delta}_{ih}^2 f(x) = \frac{1}{(ih)^2} [f(x - ih) - 2f(x) + f(x + ih)]. \tag{4.2}$$

We denote by $\|U_h\|_{T_{n-1}^\perp}$ the norm of the operator U_h on the space T_{n-1}^\perp .

Lemma 4.1 *We have*

$$\|U''_h\| \leq \frac{\pi^2 \mu^2}{h^2}, \quad \mu^2 := \mu_{2k}^2 := \frac{8}{\pi^2} \sum_{\text{odd } i}^k \frac{a_i}{i^2} < 1. \tag{4.3}$$

Proof (1) We have

$$\begin{aligned} U''_h(f, x) &= 2 \sum_{i=1}^k (-1)^{i+1} a_i I''_{ih}(f, x) \\ &= 2 \sum_{i=1}^k (-1)^{i+1} \frac{a_i}{(ih)^2} [f(x - ih) - 2f(x) + f(x + ih)] \\ &= \frac{2}{h^2} \sum_{i=1}^k (-1)^{i+1} \frac{a_i}{i^2} [f(x - ih) - 2f(x) + f(x + ih)] \\ &= \frac{2}{h^2} \sum_{i=1}^k (-1)^{i+1} a'_i [-2f(x)] \\ &\quad + \frac{2}{h^2} \sum_{i=1}^k (-1)^{i+1} a'_i [f(x - ih) + f(x + ih)], \end{aligned} \tag{4.4}$$

where in the last line we put $a'_i = \frac{a_i}{i^2}$. Hence,

$$\begin{aligned} \frac{h^2}{4} \frac{\|U''_h(f)\|}{\|f\|} &\leq \left| \sum_{i=1}^k (-1)^{i+1} a'_i \right| + \sum_{i=1}^k |a'_i| = \sum_{i=1}^k (-1)^{i+1} a'_i + \sum_{i=1}^k a'_i \\ &= 2 \sum_{\text{odd } i}^k a'_i = 2 \sum_{\text{odd } i}^k \frac{a_i}{i^2} =: \frac{\pi^2}{4} \mu^2, \end{aligned}$$

i.e.,

$$\|U_h''\| \leq \frac{\pi^2 \mu^2}{h^2}.$$

(2) The estimate for μ^2 follows from the fact that $a_i = \binom{2k}{2k+i} / \binom{2k}{k} < 1$, and that $\sum_{i=1}^\infty \frac{1}{i^2} = \frac{\pi^2}{6}$:

$$\frac{\pi^2}{8} \mu^2 = \sum_{\text{odd } i}^k \frac{a_i}{i^2} < \sum_{\text{odd } i}^\infty \frac{1}{i^2} = \sum_{i=1}^\infty \frac{1}{i^2} - \sum_{\text{even } i}^\infty \frac{1}{i^2} = \left(1 - \frac{1}{4}\right) \sum_{i=1}^\infty \frac{1}{i^2} = \frac{\pi^2}{8}.$$

We will prove in Sect. 6 that $1 - \mu_{2k}^2 \asymp \frac{1}{\sqrt{2k}}$. □

Lemma 4.2 *We have*

$$\|U_h^m\|_{T_{n-1}^\perp} \leq F_{2m} \left(\frac{\pi^2 \mu^2}{n^2 h^2}\right)^m. \tag{4.5}$$

Proof If f is orthogonal to T_{n-1} , then so are its Steklov functions $I_{ih}(f)$, hence $U_h(f)$ and the iterates $U_h^m(f)$ as well. Also, the operators D^2 (of double differentiation) and U_h commute on $C^2[-\pi, \pi]$ (since D^2 and I_{ih} clearly commute). Therefore, using the Bohr–Favard inequality with the $(2m)$ -th derivative, we obtain

$$\begin{aligned} \|U_h^m(f)\|_{T_{n-1}^\perp} &\leq \frac{F_{2m}}{n^{2m}} \|D^{2m} U_h^m(f)\| = \frac{F_{2m}}{n^{2m}} \|(D^2 U_h)^m(f)\| \\ &\leq \frac{F_{2m}}{n^{2m}} \|D^2 U_h\|^m \|f\|. \end{aligned} \tag{4.6}$$

By (4.3), we get $\|D^2 U_h\| \leq \frac{\pi^2 \mu^2}{h^2}$, hence the conclusion. (It follows from the definition that the U_h operator takes functions in $C^k[-\pi, \pi]$ to functions in $C^{k+2}[-\pi, \pi]$, therefore both $D^{2m} U_h^m$ and $(D^2 U_h)^m$ are defined on $C[-\pi, \pi]$.) □

Remark 4.3 For $h = \frac{\pi}{n}$, we have equality in (4.5), i.e.,

$$\|U_h^m\|_{T_{n-1}^\perp} = F_{2m} \mu^{2m}, \quad h = \frac{\pi}{n},$$

which is attained on the Favard function $\varphi_n(x) = \text{sgn} \sin nx$. Indeed, for $h = \frac{\pi}{n}$, we have

$$\varphi_n(x - ih) - 2\varphi_n(x) + \varphi_n(x + ih) = \begin{cases} -4\varphi_n(x), & \text{odd } i, \\ 0, & \text{even } i, \end{cases}$$

and it follows from (4.4) that $U_h''(\varphi_n) = -\frac{\pi^2 \mu^2}{h^2} \varphi_n$, and, respectively,

$$D^{2m} U_h^m(\varphi_n) = (-1)^m \left(\frac{\pi^2 \mu^2}{h^2}\right)^m \varphi_n.$$

On the other hand, the Bohr–Favard inequality turns into equality on the functions $f \in T_{n-1}^\perp$ such that $f^{(2m)}(x) = a\varphi_n(x - b)$, hence on $U_h^m(\varphi_n)$. Therefore, in (4.6), we have equalities all the way through.

Proposition 4.4 *Let $f \in T_{n-1}^\perp$, and let $h > \frac{\pi}{n}\mu$. Then*

$$\|f\| \leq c_{n,2k}(h)\gamma_{2k}^*\omega_{2k}(f, h), \tag{4.7}$$

where

$$c_{n,2k}(h) = \left(\cos \frac{\pi}{2}\rho\right)^{-1}, \quad \rho = \frac{\pi\mu}{nh} < 1. \tag{4.8}$$

Proof From Proposition 3.2, using the estimate (4.5), we obtain

$$c_{n,2k}(h) = \sum_{m=0}^\infty \|U_h^m\|_{T_{n-1}^\perp} \leq \sum_{m=0}^\infty F_{2m}\rho^{2m} = \left(\cos \frac{\pi}{2}\rho\right)^{-1},$$

the last equality (provided $\rho < 1$) being the Taylor expansion of $\sec \frac{\pi}{2}x = 1/\cos \frac{\pi}{2}x$. (The latter is usually given in terms of the Euler numbers E_{2m} as $\sec x = \sum_{m=0}^\infty \frac{|E_{2m}|}{(2m)!}x^{2m}$, see, e.g., Gradshteyn, Ryzhik [6, Sect. 1.411.9], so we have $\sec \frac{\pi}{2}x = \sum_{m=0}^\infty \frac{|E_{2m}|\pi^{2m}}{2^{2m}(2m)!}x^{2m}$, and we use the fact that $F_{2m} = \frac{|E_{2m}|\pi^{2m}}{2^{2m}(2m)!}$, see [6, Sect. 0.233.6].) \square

Theorem 4.5 *If $f \in T_{n-1}^\perp$, then, for any $\alpha > 1$, we have*

$$\|f\| \leq c_\alpha\gamma_{2k}^*\omega_{2k}\left(f, \frac{\alpha\pi}{n}\right), \quad c_\alpha = \left(\cos \frac{\pi}{2\alpha}\right)^{-1}. \tag{4.9}$$

Proof Just set $h = \frac{\alpha\pi}{n}$ in (4.7), and use the fact that $\mu < 1$. \square

Let us give some particular cases of Theorem 4.5:

- (1) $\alpha = 2, \quad c_\alpha = \left(\cos \frac{\pi}{4}\right)^{-1} = \sqrt{2}, \quad \|f\| \leq 1\frac{1}{2}\gamma_{2k}^*\omega_{2k}\left(f, \frac{2\pi}{n}\right);$
- (2) $\alpha = \frac{3}{2}, \quad c_\alpha = \left(\cos \frac{\pi}{3}\right)^{-1} = 2, \quad \|f\| \leq 2\gamma_{2k}^*\omega_{2k}\left(f, \frac{3\pi}{2n}\right);$
- (3) $\alpha = \frac{4}{3}, \quad c_\alpha = \left(\cos \frac{3\pi}{8}\right)^{-1} = 2.61, \quad \|f\| \leq 2\frac{2}{3}\gamma_{2k}^*\omega_{2k}\left(f, \frac{4\pi}{3n}\right);$
- (4) $\alpha = \frac{5}{4}, \quad c_\alpha = \left(\cos \frac{2\pi}{5}\right)^{-1} = 3.23, \quad \|f\| \leq 3\frac{1}{4}\gamma_{2k}^*\omega_{2k}\left(f, \frac{5\pi}{4n}\right).$

From the relations $\cos \frac{\pi}{2}x = \sin \frac{\pi}{2}(1 - x) \geq \frac{\pi}{4}(1 - x^2)$, it follows that, in (4.9),

$$c_\alpha < \frac{4}{\pi}\left(1 - \frac{1}{\alpha^2}\right)^{-1},$$

i.e., c_α behaves like $\frac{2}{\pi} \frac{1}{\alpha-1}$ as $\alpha \searrow 1$.

Theorem 4.6 *If $f \in T_{n-1}^\perp$, then, for $\delta = \frac{\pi}{n}$, we have*

$$\|f\| \leq c_{2k} \gamma_{2k}^* \omega_{2k}\left(f, \frac{\pi}{n}\right), \quad c_{2k} = \mathcal{O}(\sqrt{2k}). \tag{4.11}$$

Proof Setting $h = \frac{\pi}{n}$ in (4.7), we obtain the inequality (4.11) with the constant

$$c_{2k} = \left(\cos \frac{\pi}{2} \mu_{2k}\right)^{-1} < \frac{4}{\pi} (1 - \mu_{2k}^2)^{-1}, \tag{4.12}$$

and we prove in Sect. 6 that $1 - \mu_{2k}^2 \asymp \frac{1}{\sqrt{2k}}$. □

5 Stechkin Inequality for $\frac{\pi}{n} < \delta \leq \frac{2\pi}{n}$

(1) Consider the de la Vallée Poussin sum (operator)

$$v_{m,n} = \frac{1}{n-m} \sum_{i=m}^{n-1} s_i, \tag{5.1}$$

which is an average of $(n - m)$ Fourier sums s_i of degree i . For $m = n - 1$ and for $m = 0$, it becomes the Fourier sum s_{n-1} and the Fejer sum $\sigma_n = \frac{1}{n} \sum_{i=0}^{n-1} s_i$, respectively.

Since $v_{m,n}(f)$ is the convolution of f with the de la Vallée Poussin kernel $V_{m,n}$, we clearly have

$$\omega_k(v_{m,n}(f), \delta) \leq \|v_{m,n}\| \omega_k(f, \delta),$$

where $\|v_{m,n}\|$ is the norm, or the Lebesgue constant, of the operator $v_{m,n}$.

Stechkin [11] made detailed studies of behaviour of the value $\|v_{m,n}\|$ as a function of m and n . We will need just two facts from his work, one of them combined with a later result of Galkin [5].

(a) The norm $\|v_{m,n}\|$ depends only on ratio m/n , and in a monotone way. Precisely, with

$$\ell(x) := \frac{2}{\pi} \int_0^\infty \frac{|\sin xt \cdot \sin t|}{t^2} dt,$$

which is (non-trivially) a monotonely increasing function of x , we have

$$\|v_{m,n}\| = \ell(x_{m/n}), \quad x_{m/n} := \frac{1 + m/n}{1 - m/n}.$$

(b) The values of ℓ at integer points can be related to the so-called Watson constants $L_{M/2}$ (for $M = 2N$, they turn into the Lebesgue constants $L_N := \|s_N\|$ of the Fourier operator s_N). Namely,

$$\ell(M + 1) = L_{M/2},$$

and from the result of Galkin [5] that $L_{M/2} < \frac{4}{\pi^2} \ln(M + 1) + 1$, we conclude that

$$\ell(p) < \frac{4}{\pi^2} \ln p + 1 \quad \text{for integer } p, \tag{5.2}$$

therefore (rather roughly)

$$\ell(x) < \frac{4}{\pi^2} \ln(x + 1) + 1 \quad \text{for all } x. \tag{5.3}$$

(2) Now, from definition (5.1), we see firstly that $v_{m,n}(f)$ is a trigonometric polynomial of degree $\leq n - 1$, hence

$$E_{n-1}(f) \leq \|f - v_{m,n}(f)\|,$$

and secondly that $v_{m,n}$ acts as an identity on T_m , therefore

$$f - v_{m,n}(f) \perp T_m.$$

So, we may apply Proposition 4.4 to the difference $f - v_{m,n}(f)$ to obtain

$$\begin{aligned} E_{n-1}(f) &\leq \|f - v_{m,n}(f)\| \\ &\leq c_{m+1,2k}(h) \gamma_{2k}^* \omega_{2k}(f - v_{m,n}(f), h) \\ &\leq c_{m+1,2k}(h) (1 + \|v_{m,n}\|) \gamma_{2k}^* \omega_{2k}(f, h) \\ &= \left[\cos\left(\frac{\pi}{2} \frac{\pi\mu}{(m+1)h}\right) \right]^{-1} \left[1 + \ell\left(\frac{1+m/n}{1-m/n}\right) \right] \gamma_{2k}^* \omega_{2k}(f, h). \end{aligned}$$

Now, with some parameter $s \in [0, 1)$ which may well depend on n and h , we put in the last line

$$m = \lfloor sn \rfloor.$$

With such an m , we have $m + 1 > sn$ and $m/n \leq s$; therefore,

$$E_{n-1}(f) \leq \left[\cos\left(\frac{\pi}{2} \frac{\mu}{s} \frac{\pi}{nh}\right) \right]^{-1} \left[1 + \ell\left(\frac{1+s}{1-s}\right) \right] \gamma_{2k}^* \omega_{2k}(f, h). \tag{5.4}$$

Finally, taking $h = \frac{\alpha\pi}{n}$, and evaluating the factor $1 + \ell(x_s)$ with the help of (5.3), we obtain

$$E_{n-1}(f) \leq \left(\cos \frac{\pi\mu}{2\alpha s} \right)^{-1} \left[2 + \frac{4}{\pi^2} \ln\left(\frac{2}{1-s}\right) \right] \gamma_{2k}^* \omega_{2k}\left(f, \frac{\alpha\pi}{n}\right), \tag{5.5}$$

where we can minimize the right-hand side with respect to $s \in (\frac{\mu}{\alpha}, 1)$.

(3) Now, using the last estimate, we establish Stechkin inequalities for particular α 's.

Theorem 5.1 For all $n \geq 1$, we have

$$E_{n-1}(f) \leq c\gamma_{2k}^* \omega_{2k}\left(f, \frac{2\pi}{n}\right), \quad c = 5.$$

Proof In (5.5), take $\alpha = 2$ and majorize μ by 1. Then the constant for $\delta = \frac{2\pi}{n}$ takes the form

$$c = \left(\cos \frac{\pi}{4s}\right)^{-1} \left[2 + \frac{4}{\pi^2} \ln\left(\frac{2}{1-s}\right)\right].$$

It turns out that the value $s = 8/9$ is almost optimal, and we obtain a Stechkin inequality with the constant

$$c = \left(\cos \frac{9\pi}{32}\right)^{-1} \left[2 + \frac{4}{\pi^2} \ln 18\right] = 4.999144 < 5. \tag{5.6}$$

To ensure that our obtained numerical estimate less than 5 is free from a round-off error, we notice that, for $s = \frac{8}{9}$, we have in (5.4)

$$\ell\left(\frac{1+s}{1-s}\right) = \ell(17) = L_8.$$

Therefore, in passing from (5.4) to (5.5), we can use the estimate (5.2) instead of (5.3), thus changing in (5.6) the value $\ln 18$ to $\ln 17$, and that will give the constant $c = 4.962628$. We can lower the estimate a bit further by computing directly the Lebesgue constant $L_8 = 2.137730$, hence getting

$$c = \left(\cos \frac{9\pi}{32}\right)^{-1} [1 + L_8] = 4.946034,$$

so that $c < 5$ is secured. □

Theorem 5.2 For any $\alpha > 1$, there exists a constant c_α that depends only on α such that

$$E_{n-1}(f) \leq c_\alpha \gamma_{2k}^* \omega_{2k}\left(f, \frac{\alpha\pi}{n}\right), \quad n \geq 1. \tag{5.7}$$

Proof Putting (a non-optimal) $s = \frac{1}{\sqrt{\alpha}}$ in (5.5), and again majorizing μ by 1, we obtain (5.7) with

$$\begin{aligned} c_\alpha &= \left(\cos \frac{\pi}{2\sqrt{\alpha}}\right)^{-1} \left(\frac{4}{\pi^2} \ln\left(\frac{2\sqrt{\alpha}}{\sqrt{\alpha}-1}\right) + 2\right) \\ &\leq \frac{4}{\pi} \frac{\alpha}{\alpha-1} \left(\frac{4}{\pi^2} \ln\left(\frac{2\sqrt{\alpha}}{\sqrt{\alpha}-1}\right) + 2\right), \end{aligned}$$

where we have used the inequality $\cos \frac{\pi}{2}x \geq \frac{\pi}{4}(1-x^2)$ for $|x| \leq 1$. □

6 Stechkin Inequality for $\delta = \frac{\pi}{n}$

Theorem 6.1 For $\delta = \frac{\pi}{n}$, and $r = 2k$, we have

$$E_{n-1}(f) \leq c_r \left(\frac{\pi}{n}\right) \gamma_r^* \omega_r\left(f, \frac{\pi}{n}\right), \quad n \geq 1, \tag{6.1}$$

where

$$c_r \left(\frac{\pi}{n}\right) = \mathcal{O}(\sqrt{r} \ln r). \tag{6.2}$$

Proof From the estimate (5.5), with $h = \frac{\pi}{n}$ and $s = \sqrt{\mu}$, we obtain the inequality (6.1) with the constant

$$\begin{aligned} c_{2k} \left(\frac{\pi}{n}\right) &= \left(\cos \frac{\pi}{2} \sqrt{\mu}\right)^{-1} \left(\frac{4}{\pi^2} \ln\left(\frac{2}{1-\sqrt{\mu}}\right) + 2\right) \\ &< \frac{4}{\pi} \frac{1}{1-\mu} \left(\frac{4}{\pi^2} \ln\left(\frac{2}{1-\sqrt{\mu}}\right) + 2\right). \end{aligned}$$

The estimate (6.2) now follows from the fact that

$$1 - \mu_{2k}^2 > \frac{c_1}{\sqrt{2k}}, \quad c_1 = \frac{2}{3},$$

which we prove in the next lemma. With the value $c_1 = \frac{2}{3}$ at hand, we can give the explicit estimate $c_r \left(\frac{\pi}{n}\right) < 2\sqrt{r} \ln r + 12\sqrt{r}$. □

Lemma 6.2 For $\mu_{2k}^2 := \frac{8}{\pi^2} \sum_{\text{odd } i}^k \frac{a_i}{i^2}$, where $a_i := \binom{2k}{k+i} / \binom{2k}{k}$, we have

$$\frac{c_1}{\sqrt{2k}} < 1 - \mu_{2k}^2 < \frac{c_2}{\sqrt{2k}}, \quad c_1 = \frac{2}{3}, \quad c_2 = \frac{5}{4}. \tag{6.3}$$

Proof Let us compute the value $\widehat{\Delta}_t^{2k}(f_0, x)$ for $f_0(x) = \cos x$ at $x = 0$. Since

$$\begin{aligned} \widehat{\Delta}_t^{2k}(f_0, x) &= -\cos(x-t) + 2\cos x - \cos(x+t) = 2(1 - \cos t) \cos x \\ &= 4 \sin^2 \frac{t}{2} \cos x, \end{aligned}$$

we have

$$\widehat{\Delta}_t^{2k}(f_0, x)|_{x=0} = 4^k \sin^{2k} \frac{t}{2}.$$

On the other hand, by definition,

$$\begin{aligned} \widehat{\Delta}_t^{2k}(f_0, x)|_{x=0} &= \sum_{i=-k}^k (-1)^i \binom{2k}{k+i} \cos(x+it)|_{x=0} \\ &= \binom{2k}{k} \left[1 - 2 \sum_{i=1}^k (-1)^{i+1} a_i \cos it \right]. \end{aligned}$$

So, we have

$$1 - 2 \sum_{i=1}^k (-1)^{i+1} a_i \cos it = \lambda_k \sin^{2k} \frac{t}{2}, \quad \lambda_k := \frac{4^k}{\binom{2k}{k}}.$$

Integrating both parts twice, first between 0 and u , and then between 0 and π , we obtain for the left-hand side:

$$\left[\frac{u^2}{2} + 2 \sum_{i=1}^k (-1)^{i+1} \frac{a_i}{i^2} \cos iu \right]_0^\pi = \frac{\pi^2}{2} - 4 \sum_{\text{odd } i}^k \frac{a_i}{i^2} = \frac{\pi^2}{2} (1 - \mu_{2k}^2),$$

and for the right-hand side:

$$\lambda_k \int_0^\pi \int_0^u \sin^{2k} \left(\frac{t}{2} \right) dt du = \lambda_k \int_0^\pi (\pi - t) \sin^{2k} \left(\frac{t}{2} \right) dt = 4\lambda_k \int_0^{\pi/2} \tau \cos^{2k}(\tau) d\tau$$

(we first changed the order of integration and then put $\tau = \frac{\pi}{2} - \frac{t}{2}$). So, equating the rightmost values in the last two lines, we obtain

$$1 - \mu_{2k}^2 = \frac{8}{\pi^2} \frac{4^k}{\binom{2k}{k}} \int_0^{\pi/2} t \cos^{2k} t dt. \tag{6.4}$$

Now, by the Wallis inequality, we have

$$\sqrt{\frac{\pi}{2}} \sqrt{2k} \leq \frac{4^k}{\binom{2k}{k}} \leq \sqrt{\frac{\pi}{2}} \sqrt{2k+1},$$

while the integral admits the two-sided estimate

$$\frac{1}{2k+1} \leq \int_0^{\pi/2} t \cos^{2k}(t) dt \leq \frac{1}{2k},$$

because $\sin t \leq t \leq \frac{\sin t}{\cos t}$ on $[0, \frac{\pi}{2}]$, and $\int_0^{\pi/2} \sin(t) \cos^m(t) dt = \frac{1}{m+1}$. Hence

$$\frac{8}{\pi^2} \sqrt{\frac{\pi}{2}} \frac{\sqrt{2k}}{2k+1} \leq 1 - \mu_{2k}^2 \leq \frac{8}{\pi^2} \sqrt{\frac{\pi}{2}} \frac{\sqrt{2k+1}}{2k},$$

and (6.3) follows with $c_1 = \frac{8}{\pi^2} \sqrt{\frac{\pi}{2}} \frac{2k}{2k+1} > \frac{2}{3}$ and $c_2 = \frac{8}{\pi^2} \sqrt{\frac{\pi}{2}} \sqrt{\frac{2k+1}{2k}} < \frac{5}{4}$. □

7 On the Factor \sqrt{r} at $\delta = \frac{\pi}{n}$

For $\delta = \frac{\pi}{n}$, our estimates for the Stechkin constant (with the lower bound yet to be proved) look as follows:

$$c' \gamma_r^* \leq K_{n,r} \left(\frac{\pi}{n} \right) \leq c \sqrt{r} \ln r \gamma_r^*,$$

i.e., the upper and lower bounds do not match. In Sect. 2 we already expressed our belief that additional factors on the right are redundant. However, as we show in this section, appearance of the factor \sqrt{r} within our method is unavoidable. (The factor $\ln r$ originates from the use of the de la Vallée Poussin sums, and perhaps can be removed by some more sophisticated technique.)

From our initial steps (3.2)–(3.4), it is easy to see that our upper estimates in all Stechkin inequalities are valid not only for the standard modulus of smoothness $\omega_{2k}(f, h)$, but also for the modulus

$$\omega_{2k}^*(f, h) := \left\| \int_{\mathbb{R}} \widehat{\Delta}_t^{2k}(f, \cdot) \phi_h(t) dt \right\|, \tag{7.1}$$

which has a smaller value at every h . It is clear that the Stechkin constant defined with respect to a smaller modulus takes larger values, and now we show that, for the modulus $\omega_{2k}^*(f, h)$, the increase at $h = \frac{\pi}{n}$ is exactly by the factor $\sqrt{2k}$.

Theorem 7.1 *For $r = 2k$, we have*

$$\frac{\gamma_r^*}{1 - \mu_r^2} \leq \sup_{f \in T_{n-1}^\perp} \frac{\|f\|}{\omega_r^*(f, \frac{\pi}{n})} \leq \frac{4}{\pi} \frac{\gamma_r^*}{1 - \mu_r^2},$$

where

$$\frac{\gamma_r^*}{1 - \mu_r^2} \asymp \sqrt{r} \gamma_r^* \asymp \frac{r}{2^r}.$$

Proof The upper bound was established in (4.11)–(4.12). For the lower bound, take $f_0(x) = \cos nx$. Then

$$\begin{aligned} \widehat{\Delta}_t^{2k}(f_0, x) &= 4^k \sin^{2k} \left(\frac{nt}{2} \right) \cos nx, \\ \phi_{\pi/n}(t) &= \frac{n}{\pi} \left(1 - \frac{n}{\pi} |t| \right), \quad |t| \leq \frac{\pi}{n}, \end{aligned} \tag{7.2}$$

hence

$$\begin{aligned} \omega_{2k}^* \left(f_0, \frac{\pi}{n} \right) &= \left\| \int_{-\pi/n}^{\pi/n} \widehat{\Delta}_t^{2k}(f_0, \cdot) \phi_{\pi/n}(t) dt \right\| \\ &= 2 \cdot 4^k \int_0^{\pi/n} \sin^{2k} \left(\frac{nt}{2} \right) \frac{n}{\pi} \left(1 - \frac{n}{\pi} t \right) dt \\ &= \frac{8}{\pi^2} 4^k \int_0^{\pi/2} \tau \cos^{2k}(\tau) d\tau \quad \left(\tau = \frac{\pi}{2} - \frac{nt}{2} \right) \\ &\stackrel{(6.4)}{=} \frac{1 - \mu_{2k}^2}{\gamma_{2k}^*}, \end{aligned}$$

while $\|f_0\| = 1$. □

Since also $E_{n-1}(f_0) = 1$, we have the same estimate for the ratio $E_{n-1}(f_0)/\omega_{2k}^*(f_0, \frac{\pi}{n})$, therefore, for the Stechkin constant $K_{n,r}^*(\delta)$ defined with respect to the modulus $\omega_{2k}^*(f, \delta)$, we obtain at $\delta = \frac{\pi}{n}$

$$c' \sqrt{r} \gamma_r^* \leq K_{n,r}^*\left(\frac{\pi}{n}\right) := \sup_{f \in C} \frac{E_{n-1}(f)}{\omega_r^*(f, \frac{\pi}{n})} \leq c \sqrt{r} \ln r \gamma_r^*.$$

8 Lower Estimate

Lemma 8.1 *For any n, r and ϵ , and for any $\delta < \frac{\pi}{r}$, there exists an $f \in C[-\pi, \pi]$ such that,*

$$E_{n-1}(f) \geq \frac{1}{2} \gamma_{r-1}^* \omega_r(f, \delta) - \epsilon.$$

Proof Take the step periodic function

$$f_0(x) = \begin{cases} 1, & x \in (-\pi, 0]; \\ 0, & x \in (0, \pi]. \end{cases}$$

For any $x \in [-\pi, \pi]$, and for any $h < \frac{\pi}{r}$, consider the values of this function at the points $x_i = x + ih$, where $0 \leq i \leq r$. It is clear that, for some $m \leq r$, we have either

$$f_0(x_i) = 1, \quad 0 \leq i \leq m, \quad f_0(x_i) = 0, \quad m < i \leq r,$$

or the other way round. Therefore, for the modulus of smoothness $\omega_r(f_0, \delta)$, we have the following relations:

$$\begin{aligned} \omega_r(f_0, \delta) &= \max_{0 < h \leq \delta} \max_x |\Delta_h^r f_0(x)| = \max_{0 < h \leq \delta} \max_x \left| \sum_{i=0}^r (-1)^i \binom{r}{i} f_0(x + ih) \right| \\ &= \max_{0 \leq m \leq r} \left| \sum_{i=0}^m (-1)^i \binom{r}{i} \right| = \max_{0 \leq m \leq r} \left| (-1)^m \binom{r-1}{m} \right| = \binom{r-1}{\lfloor \frac{r-1}{2} \rfloor} = 1/\gamma_{r-1}^*, \end{aligned}$$

i.e.,

$$\omega_r(f_0, \delta) = 1/\gamma_{r-1}^*.$$

It is also clear that, for the best L_∞ -approximation of f_0 , we have

$$E_{n-1}(f_0) = \frac{1}{2},$$

which leads to the result for such an f_0 (without ϵ subtracted).

This is almost what we need except that f_0 is not continuous. But we can get a continuous f by smoothing f_0 at the points of discontinuity, say, by linearization. For a given ϵ , set

$$f(x) = \frac{1}{\epsilon} \int_{-\epsilon/2}^{\epsilon/2} f_0(x+t) dt,$$

i.e.,

$$f(x) = \begin{cases} 1, & x \in [-\pi + \epsilon, -\epsilon]; \\ 0, & x \in [\epsilon, \pi - \epsilon]; \\ \text{is linear on } [-\epsilon, \epsilon] \text{ and } [\pi - \epsilon, \pi + \epsilon]. \end{cases}$$

Then, from the definition (or, more generally, because f is the convolution of f_0 with a positive kernel), one can prove that

$$\omega_r(f, \delta) \leq \omega_r(f_0, \delta) = 1/\gamma_{r-1}^*.$$

As for the best approximation of f , we have

$$E_{n-1}(f) \geq \frac{1}{2} - \epsilon'.$$

Indeed, since $E_{n-1}(f) = \|f - t_{n-1}\| \leq \|f\| = 1$, the polynomial t_{n-1} of best approximation satisfies $\|t_{n-1}\| \leq 2$. Therefore, by the Bernstein inequality, we have $\|t'_{n-1}\| \leq 2(n-1)$; hence, on the interval $[-\epsilon, \epsilon]$ of the length 2ϵ , the range of t_{n-1} is not more than $4(n-1)\epsilon =: 2\epsilon'$, while the function f on the same interval takes the values 0 and 1. □

Theorem 8.2 For any r , and any $\delta < \frac{\pi}{r}$, we have

$$K_{n,r}(\delta) := \sup_{f \in C} \frac{E_{n-1}(f)}{\omega_r(f, \delta)} \geq c'_r \gamma_r^*,$$

where

$$c'_r = \begin{cases} \frac{r}{r+1}, & r = 2k - 1; \\ 1, & r = 2k. \end{cases}$$

In particular, for any r and any $n > 2r$ (i.e., when $\frac{2\pi}{n} < \frac{\pi}{r}$),

$$K_{n,r}\left(\frac{2\pi}{n}\right) := \sup_{f \in C} \frac{E_{n-1}(f)}{\omega_r(f, \frac{2\pi}{n})} \geq c'_r \gamma_r^*, \quad n > 2r.$$

Proof The first lower bound is just a reformulation of the previous lemma, because, for $\gamma_r^* := \left(\frac{r}{\lfloor \frac{r}{2} \rfloor}\right)^{-1}$, we have $\frac{1}{2}\gamma_{r-1}^* = c'_r \gamma_r^*$. □

Remark 8.3 The order $\mathcal{O}(r^{1/2}2^{-r})$ of the lower bound for the Stechkin constant was established earlier by Ivanov [7], but he did not pay attention to the exact value (and his extremal function was different from ours).

9 Stechkin Constants for Small r

For small $r = 2k$, when μ_r is noticeably smaller than 1, our method in Sect. 5 will give for the Stechkin constant the upper estimates which are better than $5\gamma_r^*$, but they will never be smaller than $2\gamma_r^*$ because of the factor $1 + \|v_{m,n}\|$.

Surprisingly, better values (for small r) which stand quite close to the lower bound $1 \cdot \gamma_r^*$ could be obtained through the technique of intermediate approximation with Steklov-type functions. (For general r , this technique provides the same overblown estimate $c_r < r^{ar}$ as Stechkin’s original proof, therefore a surprise.)

Such a technique is of course well known (it was introduced probably by Brudnyi [1] and Freud–Popov [4]), and it was exploited repeatedly for proving Stechkin inequalities of various types (e.g., for spline and one-sided approximations). Our only innovation (if any) is the use of the central differences instead of the forward ones, which reduces the constants by the factor $\binom{2k}{k}$, and the determination to take a closer look at their actual values.

Lemma 9.1 *We have*

$$E_{n-1}(f) \leq c_{2k} \left(\frac{\alpha\pi}{n} \right) \gamma_{2k}^* \omega_{2k} \left(f, \frac{\alpha\pi}{n} \right),$$

where

$$c_{2k} \left(\frac{\alpha\pi}{n} \right) = 1 + F_{2k} \frac{k^{2k}}{(\alpha\pi)^{2k}} \sum_{i=1}^k \frac{2b_i}{i^{2k}}, \quad b_i = \binom{2k}{k+i}, \tag{9.1}$$

and F_{2k} are the Favard constants.

Proof Given f , with any $2k$ times differentiable function f_h , we have

$$E_{n-1}(f) \leq E_{n-1}(f - f_h) + E_{n-1}(f_h) \leq \|f - f_h\| + \frac{F_{2k}}{n^{2k}} \|f_h^{(2k)}\|, \tag{9.2}$$

where we used the Favard inequality for the best approximations of f_h . A typical choice of f_h is via the Steklov functions of order $2k$:

$$I_{ih}(f, x) := \frac{1}{(h/k)^{2k}} \underbrace{\int_{-h/2k}^{h/2k} \cdots \int_{-h/2k}^{h/2k}}_{2k} f(x - i(t_1 + \cdots + t_{2k})) dt_1 \cdots dt_{2k},$$

for which, similar to (4.2),

$$I_{ih}^{(2k)}(f, x) = \frac{(-1)^k}{(ih/k)^{2k}} \widehat{\Delta}_{ih/k}^{2k} f(x).$$

Then, with

$$f_h := \frac{1}{\binom{2k}{k}} \sum_{\substack{i=-k \\ i \neq 0}}^k (-1)^{i+1} \binom{2k}{k+i} I_{ih}(f) = \gamma_{2k}^* \sum_{i=1}^k (-1)^{i+1} 2b_i I_{ih}(f),$$

we obtain

$$f(x) - f_h(x) = \frac{\gamma_{2k}^*}{(h/k)^{2k}} \int_{-h/2k}^{h/2k} \cdots \int_{-h/2k}^{h/2k} \widehat{\Delta}_{t_1+\cdots+t_{2k}}^{2k}(f, x) dt_1 \cdots dt_{2k},$$

and, respectively,

$$\|f - f_h\| \leq \gamma_{2k}^* \omega_{2k}(f, h),$$

$$\|f_h^{(2k)}\| \leq \gamma_{2k}^* \sum_{i=1}^k \frac{2b_i}{(ih/k)^{2k}} \omega_{2k}(f, ih/k) \leq \gamma_{2k}^* \omega_{2k}(f, h) \frac{k^{2k}}{h^{2k}} \sum_{i=1}^k \frac{2b_i}{i^{2k}}.$$

Applying (9.2), we arrive at

$$E_{n-1}(f) \leq c_{2k}(h) \gamma_{2k}^* \omega_{2k}(f, h), \quad c_{2k}(h) = 1 + F_{2k} \frac{k^{2k}}{(nh)^{2k}} \sum_{i=1}^k \frac{2b_i}{i^{2k}},$$

and we take $h = \frac{\alpha\pi}{n}$. □

In (9.1), we can obtain a small value only if $\frac{k}{\alpha\pi} < 1$, i.e., we may try $k = (1, 2, 3)$ for $\alpha = 1$, and $k = (1, 2, 3, 4, 5, 6)$ for $\alpha = 2$. So we did (dropping those values for which the resulting constants in (9.1) were not close to 1).

Theorem 9.2 For $\delta = \frac{\pi}{n}$ and $\delta = \frac{2\pi}{n}$, we have

$$E_{n-1}(f) \leq c_r(\delta) \gamma_r^* \omega_r(f, \delta),$$

where $c_{2k-1}(\delta) = c_{2k}(\delta)$, and the values of $c_{2k}(\delta)$ are given below:

$$\begin{array}{c|c} c_2(\frac{\pi}{n}) & c_4(\frac{\pi}{n}) \\ \hline 1\frac{1}{4} & 2\frac{7}{10} \end{array}, \quad \begin{array}{c|c|c} c_2(\frac{2\pi}{n}) & c_4(\frac{2\pi}{n}) & c_6(\frac{2\pi}{n}) \\ \hline 1\frac{1}{16} & 1\frac{1}{9} & 1\frac{1}{2} \end{array}.$$

Proof We will use the following values: $F_2 = \frac{\pi^2}{8}$, $F_4 = \frac{5\pi^4}{384}$, $F_6 = \frac{61\pi^6}{46080}$.

(1) For $2k = 2$, we have

$$c_2\left(\frac{\alpha\pi}{n}\right) = 1 + \frac{\pi^2}{8} \frac{2}{(\alpha\pi)^2} = 1 + \frac{1}{4\alpha^2}.$$

With $\alpha = 1$ and $\alpha = 2$, we obtain $c_2(\frac{\pi}{n}) = \frac{5}{4}$ and $c_2(\frac{2\pi}{n}) = \frac{17}{16}$. Also, with $\alpha = \frac{1}{2}$, we obtain the remarkable inequality

$$E_{n-1}(f) \leq 1 \cdot \omega_2\left(f, \frac{\pi}{2n}\right).$$

(2) For $2k = 4$,

$$c_4\left(\frac{\alpha\pi}{n}\right) = 1 + \frac{5\pi^4}{384} \frac{2^4}{(\alpha\pi)^4} \cdot 2 \left[\frac{4}{1^4} + \frac{1}{2^4} \right] = 1 + \frac{325}{192} \frac{1}{\alpha^4}.$$

With $\alpha = 1$ and $\alpha = 2$, we obtain $c_4(\frac{\pi}{n}) = \frac{517}{192} = 2.6927$, and $c_4(\frac{2\pi}{n}) = \frac{3397}{3072} = 1.1058$.

(3) For $2k = 6$, with $\alpha = 2$, we have

$$c_6\left(\frac{2\pi}{n}\right) = 1 + \frac{61\pi^6}{46080} \frac{3^6}{(2\pi)^6} \cdot 2 \left[\frac{15}{1^4} + \frac{6}{2^6} + \frac{1}{3^6} \right] = 1.4552 < 1\frac{1}{2}. \quad \square$$

Theorem 9.2 provides a certain support to our Conjecture 2.6, which says, in particular, that, for even $r = 2k$, and for $\delta \geq \frac{\pi}{n}$, the best constant in the Stechkin inequality has the value $K_{n,r}(\delta) = 1 \cdot \gamma_r^*$.

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