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Integral Approximation of the Characteristic Function of an Interval by Trigonometric Polynomials

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Abstract—We prove that the value $E_{n-1}(\chi_h)_L$ of the best integral approximation of the characteristic function χ_h of an interval $(-h, h)$ on the period $[-\pi, \pi)$ by trigonometric polynomials of degree at most $n-1$ is expressed in terms of zeros of the Bernstein function $\cos\{nt - \arccos[(2q-(1+q^2)\cos t)/(1+q^2-2q\cos t)]\}$, $t \in [0, \pi]$, $q \in (-1, 1)$. Here, the parameters q, h, and n are connected in a special way; in particular, $q = \sec h - \tan h$ for $h = \pi/n$.

Keywords: integral and uniform approximation of functions by polynomials, canonical sets.

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INTRODUCTION

This paper and the previous paper by the authors [2] are initiated by a simple, at the first glance, question: *"What is the value of the best approximation in L of the characteristic function of the interval $(-\pi/n, \pi/n)$ by trigonometric polynomials of degree $n-1$ on the period*[3] $\mathbb{T} = [-\pi, \pi)$?*"* The answer turned out to be nontrivial; it is formulated in Section 5 (see formula (5.9) and also formula (5.1) and assertion (3) of Theorem 6) in terms of the first two zeros of the Bernstein function $\mathcal{B}(t) = \mathcal{B}(t, q, 0)$ defined by formulas (3.6)–(3.8) with $q = \sec(\pi/n) - \tan(\pi/n)$.

In [2], we obtained upper estimates for $E_{n-1}(\chi_h)_L$ for $0 < h \le \pi$ and proved their sharpness, if h is any zero of the function $\cos nt$ from $(0, \pi)$. The case of an arbitrary h brought us to the question about sign-functions orthogonal to the subspace $\mathcal{T}_{n-1}$ of trigonometric polynomials of degree at most $n-1$. The answer to this question is a key for solving the problem about the sharp value of the quantity $E_{n-1}(\chi_h)_L$ for $0 < h < \pi$. After the main result (Theorem 5) was obtained, we learned that the trigonometric polynomials whose zeros provide the description of the required class of sign-functions (see Theorem 4 and Remark 1) arose in the investigations by Geronimus [6, 7, 23, 24] and Peherstorfer [26, 27]. In Section 4, we give our (elementary) proof of Theorem 4, which, in our opinion, is of independent interest and close ideologically to Korkin and Zolotarev's proof (1873, [12, Paper 8]) of the result on the best integral approximation on $[-1, 1]$ of the function x^n by the subspace $\mathcal{P}_{n-1}$ of polynomials of degree at most $n-1$. Korkin and

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[3]It is convenient to interpret the period $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ as the unit circle or as the half-interval $[\alpha, \alpha + 2\pi)$ with the identical endpoints; usually, $\alpha = 0$ or $\alpha = -\pi$.

Zolotarev's methods were developed in works by Geit (see the bibliography in his paper [5]). In Section 5, Theorem 4 is used as an instrument for calculating $E_{n-1}(\chi_h)_L$ for all $h \in (0, \pi)$.

A brief history of the problem of the approximation of specific functions in the integral metric is contained in [2]. In addition, we shall present several known facts having the direct relation to the theme of this paper and concerning the approximation of step functions.

In 1853, Chebyshev [18] solved the problem of the best uniform approximation on a segment of the function x^n by the subspace $\mathcal{P}_{n-1}$. Zolotarev (1868, see [10, Problem I]) investigated the problem of the uniform approximation on $[-1,1]$ of the function $x^n - qx^{n-1}$, $q \in \mathbb{R}$, by the subspace $\mathcal{P}_{n-2}$. He also solved the problem (1877, [10, Problem IV]) that, as claimed in [1, Appendices and Problems, Problem 35], coincides in essence with the following problem: *among all fractions of the form* φ/ψ, *where* $\varphi, \psi \in \mathcal{P}_n$, *find a fraction least deviating from the function* $\operatorname{sgn} x$ *on the set* $[-1, -a] \cup [a, 1]$ $(0 < a < 1)$ *in the uniform metric.* Recently, a polynomial analog of this problems ($\psi \equiv 1$) was considered in [22].

In paper [28] by Vaaler, sharp results by Beurling of the 1930s, Selberg (1974), and Vaaler himself are presented on the integral (including one-sided) approximation on $\mathbb{R}$ of the step function $\operatorname{sgn} x$ and the characteristic function of an interval by entire functions, as well as their applications in the form of short proofs of known results from number theory established earlier by Montgomery, Vaughan, Erdös, Turán, and others.

Sharp results on the L-approximation of the characteristic function of a spherical cap on a multidimensional Euclidean sphere by polynomials of total degree not higher than a given one were obtained recently by Deikalova [9].

1. NOTATION AND DEFINITIONS

We use the following notation:

$L = L(\mathbb{T})$ is the space of 2π-periodic Lebesgue integrable functions $f\colon\ \mathbb{T} \mapsto \mathbb{R}$ with the norm $\|f\|_L = \int_{-\pi}^{\pi} |f(t)|\,dt$;

$C_{2\pi} = C(\mathbb{T})$ is the space of continuous 2π-periodic functions $f\colon\ \mathbb{T} \to \mathbb{R}$ with the uniform norm $\|f\|_{C_{2\pi}} = \max\{|f(t)| :\ t \in \mathbb{T}\}$;

$L_\infty = L_\infty(\mathbb{T})$ is the space of 2π-periodic functions $f\colon\ \mathbb{T} \to \mathbb{R}$ measurable and essentially bounded on $\mathbb{T}$ with the norm $\|f\|_{L_\infty} = \operatorname{ess\,sup}\{|f(t)| :\ t \in \mathbb{T}\}$;

$\mathcal{T}_n$ is the subspace of trigonometric polynomials $g(t) = a_0 + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt)$ of degree at most n with real coefficients;

$\mathcal{P}_n$ is the subspace of algebraic polynomials $p(x) = \sum_{k=0}^{n} c_k x^k$ of degree at most n $(\deg p \le n)$ with real coefficients;

$\mathcal{T}_{n-1}^{\perp}$ is the set of functions $\varphi \in L$ orthogonal to the subspace $\mathcal{T}_{n-1}$, i.e., the set of functions $\varphi \in L$ such that $\int_{-\pi}^{\pi} \varphi(t)\, g(t)\, dt = 0$ for all $g \in \mathcal{T}_{n-1}$;

$E_{n-1}(f)_L = \min\{\|f - g\|_L :\ g \in \mathcal{T}_{n-1}\}$ is the value of the best integral approximation of a function $f \in L$ by the subspace $\mathcal{T}_{n-1}$.

Below, we give definitions, which are somewhat different from those used in [27, Definition 2] and [21, Ch. 3, Sect. 10, (10.7)].

Definition 1. Let $\{t_j\}_{j=1}^{r}$ be a set of r pairwise different points $t_1 < t_2 < \ldots < t_r$ from the half-interval $[t_1, t_1 + 2\pi)$. The function $\sigma(t) = \varepsilon \sum_{j=1}^{r} (-1)^j \chi_{(t_j, t_{j+1})}(t)$, $t \in \mathbb{R}$, is called the *sign-function* corresponding to the set $\{t_j\}_{j=1}^r$; here, $\varepsilon = \pm 1$, $t_{r+1} = t_1 + 2\pi$, and $\chi_{(t_j,t_{j+1})}$ is the 2π-periodic extension of the characteristic function of the interval (t_j, t_{j+1}) to $\mathbb{R}$.

Definition 2. If a sign-function σ from $\mathcal{T}_{n-1}^{\perp}$ corresponds to the set $\{t_j\}_{j=1}^r$ of points from Definition 1, i.e., $\int_{-\pi}^{\pi} \sigma(t)\, g(t)\, dt = 0$ for all $g \in \mathcal{T}_{n-1}$, then this set of points is called *canonical* for $\mathcal{T}_{n-1}$.

2. SEVERAL GENERAL STATEMENTS FROM THEORY OF L-APPROXIMATION

In 1898, Markov proved [13, Paper 9] that, for any fixed $\theta \in \mathbb{R}$, the set of zeros of the function $\cos(nt + \theta)$ located on the period $[-\pi, \pi)$ is canonical for $\mathcal{T}_{n-1}$; i.e.,

$$\int_{-\pi}^{\pi} \operatorname{sgn}\{\cos(nt+\theta)\}\, g(t)\, dt = 0 \qquad \text{for all} \quad g \in \mathcal{T}_{n-1}. \tag{2.1}$$

In addition, in the same paper, he established that the problem of constructing canonical sets for $\mathcal{T}_{n-1}$ is closely connected with the problem of the integral approximation of functions from $C_{2\pi}$ by the subspace $\mathcal{T}_{n-1}$ on the period (see [1, Ch. 2, Sect. 50] and [21, Ch. 3, Sect. 10, Theorem 10.5]).

Theorem 1 (Markov). *Let a set $\{t_j\}_{j=1}^r$ of points $t_1 < t_2 < \ldots < t_r$, where $t_r < t_1 + 2\pi$, be canonical for $\mathcal{T}_{n-1}$. If a polynomial $g_0 \in \mathcal{T}_{n-1}$ interpolates a function f from $C_{2\pi}$ at the points t_j, $j = 1, 2, \ldots, r$, and the difference $f - g_0$ changes its sign at these points and has no other points of sign change on any half-interval $[\alpha, \alpha + 2\pi)$ containing the set $\{t_j\}_{j=1}^r$, then the polynomial g_0 is the polynomial of the best integral approximation for f. In addition,*

$$E_{n-1}(f)_L = \|f - g_0\|_L = \left| \int_{-\pi}^{\pi} f(t) \operatorname{sgn}\{f(t) - g_0(t)\}\, dt \right|.$$

Theorem 2. *In order that a polynomial $g_0 \in \mathcal{T}_{n-1}$ provide the best integral approximation for a function $f \in L$, it is sufficient that the function* $\operatorname{sgn}[f(t) - g_0(t)]$ *belong to the set $\mathcal{T}_{n-1}^{\perp}$. In addition,*

$$E_{n-1}(f)_L = \|f - g_0\|_L = \left| \int_{-\pi}^{\pi} f(t) \operatorname{sgn}\{f(t) - g_0(t)\}\, dt \right|.$$

If the set of points $t \in \mathbb{T}$ such that $f(t) = g_0(t)$ has zero measure, then this condition is also necessary.

The proof of Theorem 2 is based on the Nikol'skii duality relations [15, Corollary 2]; see also [11, Proposition 2.5.2 and Theorem 3.3.2].

Peherstorfer [26, Theorem 2] found necessary and sufficient conditions for a set $\{t_j\}_{j=1}^{2r}$ of points $t_1 < t_2 < \ldots < t_{2r}$ (where $t_{2r} < t_1 + 2\pi$) to be canonical for $\mathcal{T}_{n-1}$. Geronimus [6, 7, 23] made a substantial contribution to this range of problems.

Theorem 3 (Peherstorfer). *Let $r, n \in \mathbb{N}$, $r \geq n$, and let $g \in \mathcal{T}_r$ be a trigonometric polynomial of the form $g(t) = a_0 + \sum_{k=1}^{r-1} (a_k \cos kt + b_k \sin kt) + A \cos rt + B \sin rt$, $A^2 + B^2 > 0$. Then, the conjunction of the conditions*

(1) *the number of sign inversions of the polynomial g on the period $\mathbb{T}$ is equal to $2r$;*

(2) *the set of zeros* $t_1 < t_2 < \ldots < t_{2r}$ *of the polynomial* g *on the period* $[t_1, t_1 + 2\pi)$ *is canonical for* $\mathcal{T}_{n-1}$;
is equivalent to the condition

(3) *there exists a polynomial*[4]

$$p(z) = \prod_{j=1}^{r-n}(z - z_j) \qquad (z \in \mathbb{C}, \quad z_j \in \mathbb{C}, \quad |z_j| < 1)$$

such that[5] $g(t) = \operatorname{Re}\{(A - iB)z^{2n-r}p^2(z)\}$ *for* $z = e^{it}$ *and* $t \in \mathbb{T}$.

Theorem 3 implies that two families of sets canonical for $\mathcal{T}_{n-1}$ with the number of points $2n$ and $2n+2$ are characterized by zeros of the functions $\cos n(t+\theta)$, $\theta \in \mathbb{R}$, and (see the second part of formula (3.8), the first paragraph of Section 4, and Theorem 4) by zeros of polynomials of the form $aR_{q,0}(t+\theta) + bR_{q,\pi/2}(t+\theta)$, respectively, where $a, b, \theta \in \mathbb{R}$, $a^2 + b^2 > 0$, $q \in (-1, 1)$, and the polynomial $R_{q,\xi}$ is defined by the formula $R_{q,\xi}(t) = \cos[(n+1)t+\xi] - 2q\cos(nt+\xi) + q^2\cos[(n-1)t+\xi]$.

In Section 6, we find a representation of the polynomial $R_{q,\xi}(t)$ (see (6.16)) in terms of the Bernstein function $\mathcal{B}(t, q, \xi)$, which contains useful information about zeros of the polynomial $R_{q,\xi}(t)$.

3. THE BERNSTEIN FUNCTION

In 1935, Geronimus ([23], see [24, formula (105)]) found a polynomial $p_q \in \mathcal{P}_{n-2}$ of the best integral approximation on $[-1, 1]$ for the function $x^n - qx^{n-1}$ for $q \in \mathbb{R}$ and, thereby, solved the integral variant of the Zolotarev problem[6]. He found an explicit formula for the difference $G_q(x) = x^n - qx^{n-1} - p_q(x)$; in particular, for $-1 \le q \le 1$, the following formula is valid:

$$2^n G_q(\cos t)\sin t = \sin(n+1)t - 2q\sin nt + q^2\sin(n-1)t. \tag{3.1}$$

This result is connected (see footnote 8 on p. S23) with Bernstein's results of 1912–1913 [3, Papers 7–9] about the approximation of the simplest fraction $f_a(x) = 1/(x - a)$, $a > 1$, by the subspace $\mathcal{P}_n$ on $[-1, 1]$ in the uniform metric. This problem was stated by Chebyshev (1892) [19]. He called the value of the best uniform approximation

$$E_n(f_a)_{C[-1,1]} = \inf_{p \in \mathcal{P}_n} \|f_a - p\|_{C[-1,1]} \tag{3.2}$$

of a function f_a by the subspace $\mathcal{P}_n$ the *absolute error* [19]. Here, $C[-1, 1]$ is the space of continuous functions $f\colon [-1, 1] \mapsto \mathbb{R}$ with the norm $\|f\|_{C[-1,1]} = \max\{|f(x)|\colon x \in [-1, 1]\}$. Chebyshev, in the paper mentioned, found a *relative error*, i.e., the value of the best uniform approximation of the function f_a by the subspace $\mathcal{P}_n$ on the segment $[-1, 1]$ with the weight function ? (avtor)[7] $1/f_a$

$$\inf_{p \in \mathcal{P}_n} \|(f_a - p)/f_a\|_{C[-1,1]} = \inf_{p \in \mathcal{P}_n} \|1 - (x - a)p(x)\|_{C[-1,1]}$$

$$= \frac{2}{(a + \sqrt{a^2 - 1})^{n+1} + (a - \sqrt{a^2 - 1})^{n+1}}, \tag{3.3}$$

[4]In a product, if the lower index is greater than the upper one, then the product is considered to be equal to 1.

[5]$\operatorname{Re} z = x$ means the real part of the complex number $z = x + iy$, and $\operatorname{Im} z = y$ means its imaginary part.

[6]This result was rediscovered by Galeev (1975) [4] with the application of other approaches.

[7]As Chebyshev actually noted [19], problem (3.3) is equivalent to the problem about the extremal extrapolation of a polynomial, which he investigated in paper [20] in 1881. The problem mentioned is reduced to the problem of finding the maximal possible value $p(a)$ at a point $a > 1$ on the class of polynomials $p \in \mathcal{P}_n$ whose uniform norm is bounded by one on the segment $[-1, 1]$.

and the corresponding best polynomial.

Bernstein [3, Paper 8] evaluated absolute error (3.2):

$$E_n(f_a)_{C[-1,1]} = \frac{1}{(a^2-1)(a+\sqrt{a^2-1})^n}. \tag{3.4}$$

He found two forms for the difference $\Phi_a(x) = f_a(x) - p_a(x)$ between the function f_a and the polynomial $p_a \in \mathcal{P}_n$ of the best uniform approximation on $[-1,1]$, namely, the *algebraic* form

$$\Phi_a(x) = \frac{[ax-1+s(a)s(x)]\,[x+s(x)]^n + [ax-1-s(a)s(x)]\,[x-s(x)]^n}{2s^2(a)[a+s(a)]^n(x-a)}, \quad s(x) = \sqrt{x^2-1},$$

and the *trigonometric* form

$$\Phi_a(\cos t) = \frac{\cos\,[nt-\delta(t,a)]}{(a^2-1)(a+\sqrt{a^2-1})^n}, \qquad \delta(t,a) = \arccos\frac{1-a\cos t}{a-\cos t}.$$

The function $\delta(t,a)$ decreases with respect to t on $[0,\pi]$, $\delta(0,a)=\pi$, and $\delta(\pi,a)=0$. This implies that the function $\varphi(t,a) = nt-\delta(t,a)$ increases with respect to t on $[0,\pi]$; in addition, $\varphi(0,a) = -\pi$, $\varphi(\pi,a) = n\pi$, and the difference $\Phi_a(x) = f_a(x) - p_a(x)$ has an $(n+2)$-point alternance on $[-1,1]$. Therefore (see [14, Ch. 2, Sect. 2]), p_a is the polynomial of the best uniform approximation of the function f_a on $[-1,1]$.

Let $t \in [0,\pi]$, $n \in \mathbb{N}$, $a \in (-\infty,-1) \cup (1,+\infty)$, and $\xi \in \mathbb{R}$. We call the function

$$B(t,a,\xi) = \cos\,[nt-\delta(t,a)+\xi]\,, \qquad \delta(t,a) = \arccos\frac{1-a\cos t}{a-\cos t}, \tag{3.5}$$

the *Bernstein function*. For a number $a \in (-\infty,-1) \cup (1,+\infty)$, there is a unique number $q \in (-1,0) \cup (0,1)$ such that $a = (q+1/q)/2$. In Section 6, we show that, in terms of the parameter $q \in (-1,1)$, the Bernstein function has the form

$$B(t,a,\xi) = \mathcal{B}(t,q,\xi) = \cos\,[nt-\lambda(t,q)+\xi] = \frac{R_{q,\xi}(t)}{2q\cos t-(1+q^2)}, \qquad t \in [0,\pi], \tag{3.6}$$

where

$$\delta(t,a) = \lambda(t,q) = \arccos\frac{2q-(1+q^2)\cos t}{1+q^2-2q\cos t}, \tag{3.7}$$

$$\begin{aligned} R_{q,\xi}(t) &= \cos[(n+1)t+\xi] - 2q\cos(nt+\xi) + q^2\cos[(n-1)t+\xi] \\ &= \big\{\cos(n+1)t - 2q\cos nt + q^2\cos(n-1)t\big\}\cos\xi \\ &\quad - \big\{\sin(n+1)t - 2q\sin nt + q^2\sin(n-1)t\big\}\sin\xi. \end{aligned} \tag{3.8}$$

We can consider (3.6) as a representation of the polynomial $R_{q,\xi}$ in terms of the Bernstein function on $[0,\pi]$. The proof of this representation, as well as a similar representation on $[-\pi,0]$, is given in Section 6 (see formula (6.16)). In that section, we study properties of function (3.6) related to its zeros and alternance points. In particular, we prove Theorem 10, which implies that the zeros of Geronimus sine-polynomial (3.1) located in $(0,\pi)$ coincide [8] with the alternance points of the function $\mathcal{B}(t) = \mathcal{B}(t,q,0)$.

[8] This conclusion also follows from Lemma 3 in [27] and the results from monograph [25, Ch. 1, Subsect. 4.3].

4. A FAMILY OF SIGN-FUNCTIONS

We associate the following polynomial with a triple of numbers $n \in \mathbb{N}$, $\xi \in \mathbb{R}$, and $q \in (-1,1)$:

$$Q_{q,\xi}(z) = e^{i\xi}\left(z^{n+1} - 2qz^n + q^2 z^{n-1}\right) = e^{i\xi} z^{n-1}(z-q)^2.$$

For $z = e^{it}$, the real part of this polynomial coincides with the polynomial $R_{q,\xi}(t)$ (3.8); i.e., $\operatorname{Re}\left\{Q_{q,\xi}\left(e^{it}\right)\right\} = R_{q,\xi}(t) = \cos[(n+1)t+\xi] - 2q\cos(nt+\xi) + q^2\cos[(n-1)t+\xi]$.

For $r = n+1$, $n \in \mathbb{N}$, $A - iB = e^{i\xi}$, and $p(z) = z - q$, Theorem 3 implies the following theorem.

Theorem 4 (Markov, Geronimus, Peherstorfer)**.** *For any $n \in \mathbb{N}$, $q \in [-1,1]$, and $\xi \in \mathbb{R}$, the following relation holds*:

$$\int_{-\pi}^{\pi} \operatorname{sgn}\{R_{q,\xi}(t)\}\, g(t)\, dt = 0 \quad \textit{for any polynomial} \quad g \in \mathcal{T}_{n-1}; \tag{4.1}$$

consequently, $R_{q,\xi}$ cannot be approximated by the subspace $\mathcal{T}_{n-1}$ in the metric of the space L; i.e., $E_{n-1}(R_{q,\xi})_L = \|R_{q,\xi}\|_L$.

Remark 1. For $q = 0$ and $q = \pm 1$, Theorem 4 is reduced to the statement about the orthogonality of $\operatorname{sgn}\{\cos(n+1)t\}$ and $\operatorname{sgn}\{\cos nt\}$, respectively, to the subspace $\mathcal{T}_{n-1}$(Markov, see formula (2.1) above). For $-1 < q < 1$, Theorem 4 was proved by Geronimus in the case $\xi = -\pi/2$ and by Peherstorfer in the general case.

Below, we give our proof of Theorem 4 based on the Viéte theorem, Newton recurrence formulas, and Euler formula.

We denote by $\mathbb{S} = \left\{z = e^{it} : t \in \mathbb{T}\right\}$ the unit circle in the complex plane $\mathbb{C}$.

Lemma 1. *For $q \in (-1,1)$ and $\xi \in \mathbb{R}$, the polynomial $R_{q,\xi}$ has $2n+2$ different zeros on $\mathbb{T}$.*

Proof. The polynomial $R_{q,\xi}(t)$ coincides with $\operatorname{Re}\left\{Q_{q,\xi}\left(e^{it}\right)\right\}$. All zeros of the polynomial $Q_{q,\xi}(z)$ are inside the unit circle. According to the argument principle [17, Ch. 1, Theorem 1.91.1], if z ranges over $\mathbb{S}$, then $Q_{q,\xi}(z)$ makes $n+1$ turns around the origin and, hence, intersects the imaginary axis $2n+2$ times, and the polynomial $R_{q,\xi}(t)$ has $2n+2$ zeros on $\mathbb{T}$, respectively. $\square$

Proof of Theorem 4. We can assume that $0 \le q \le 1$, since the case $-1 \le q \le 0$ is reduced to the case $0 \le q \le 1$ by replacing t by $\pi - t$. For $q = 0$ and $q = 1$, assertion (4.1) is valid (see Remark 1), because $R_{0,\xi}(t) = \cos[(n+1)t+\xi]$ and $R_{1,\xi}(t) = 2(\cos t - 1)\cos(nt+\xi)$. It remains to consider the case $0 < q < 1$.

By Lemma 1, the polynomial $R_{q,\xi}(t)$ has exactly $2n+2$ pairwise different zeros $t_1 < t_2 < \ldots < t_{2n+2}$ on $[t_1, t_1 + 2\pi)$. Assertion (4.1) of Theorem 4 is equivalent to each of the following four assertions:

$$\int_{t_1}^{t_{2n+3}} \operatorname{sgn}\{R_{q,\xi}(t)\}\, e^{ikt}\, dt = 0 \quad \text{for} \quad k = 0,1,2,\ldots,n-1, \quad \text{where} \quad t_{2n+3} = t_1 + 2\pi;$$

$$\sum_{\nu=1}^{2n+2} (-1)^{\nu} \int_{t_\nu}^{t_{\nu+1}} e^{ikt}\, dt = 0 \quad \text{for} \quad k = 0,1,2,\ldots,n-1;$$

$$\sum_{\nu=1}^{2n+2}(-1)^\nu\{t_{\nu+1}-t_\nu\}=0,\quad \sum_{\nu=1}^{2n+2}(-1)^\nu\left\{e^{ikt_{\nu+1}}-e^{ikt_\nu}\right\}=0 \quad \text{for} \quad k=1,2,\dots,n-1;$$

$$\pi+\sum_{j=1}^{n+1}t_{2j-1}-\sum_{j=1}^{n+1}t_{2j}=0,\quad \sum_{j=1}^{n+1}e^{ikt_{2j}}-\sum_{j=1}^{n+1}e^{ikt_{2j-1}}=0 \quad \text{for} \quad k=1,2,\dots,n-1. \tag{4.2}$$

To prove Theorem 4, it is sufficient to establish the validity of equalities (4.2).

First, note that the following relations hold:

$$\frac{1}{2}\left|\int_{t_1}^{t_{2n+3}}\operatorname{sgn}\{R_{q,\xi}(t)\}dt\right|=\frac{1}{2}\left|\sum_{\nu=1}^{2n+2}(-1)^\nu\{t_{\nu+1}-t_\nu\}\right|=\left|\pi+\sum_{j=1}^{n+1}t_{2j-1}-\sum_{j=1}^{n+1}t_{2j}\right|<\pi. \tag{4.3}$$

Now, let us consider the polynomial $P(z)=P_{q,\xi}(z)$ of degree $2n+2$ all zeros $z_1,z_2,\dots,z_{2n+2}$ of which are on the unit circle $\mathbb{S}$ and are related to the zeros $t_1<t_2<\dots<t_{2n+2}$ of the polynomial $R_{q,\xi}(t)$ as follows:

$$z_1=e^{it_1},\quad z_2=e^{it_2},\quad \dots,\quad z_{2n+2}=e^{it_{2n+2}}. \tag{4.4}$$

With the help of the Euler formula, we conclude that, for $z=e^{it}$, the polynomial $P(z)$ has the form

$$\begin{aligned}P(z)=P_{q,\xi}(z)&=2\varepsilon z^{n+1}R_{q,\xi}(t)=z^{2n}(z-q)^2+\varepsilon^2(qz-1)^2\\&=[z^n(z-q)]^2-[i\varepsilon(qz-1)]^2=P^+(z)P^-(z),\qquad \varepsilon=e^{-i\xi},\end{aligned} \tag{4.5}$$

where

$$P^+(z)=P^+_{q,\xi}(z)=z^n(z-q)+i\varepsilon(qz-1)=z^{n+1}-qz^n+i\varepsilon qz-i\varepsilon, \tag{4.6}$$

$$P^-(z)=P^-_{q,\xi}(z)=z^n(z-q)-i\varepsilon(qz-1)=z^{n+1}-qz^n-i\varepsilon qz+i\varepsilon. \tag{4.7}$$

Since the factor $2\varepsilon z^{n+1}$ does not vanish on $\mathbb{S}$, the polynomial $P(z)$, by (4.5), has property (4.4).

The equalities $P=P^+P^-$ and $\deg P^+=\deg P^-=n+1$ imply that each of the polynomials P^+ and P^- has exactly $n+1$ zeros $z_1^+=e^{it_1^+},\dots,\ z_{n+1}^+=e^{it_{n+1}^+}$ and $z_1^-=e^{it_1^-},\dots,\ z_{n+1}^-=e^{it_{n+1}^-}$ on $\mathbb{S}$ (all zeros are pairwise different) and has no other zeros in $\mathbb{C}$. In addition, the following relations hold: $\{z_1,z_2,\dots,z_{2n+2}\}=\{z_1^+,z_2^+,\dots,z_{n+1}^+\}\cup\{z_1^-,z_2^-,\dots,z_{n+1}^-\}$ and $\{t_1,t_2,\dots,t_{2n+2}\}=\{t_1^+,t_2^+,\dots,t_{n+1}^+\}\cup\{t_1^-,t_2^-,\dots,t_{n+1}^-\}$.

In order to understand the structure of the positional relationship of zeros of the polynomials P, P^+, and P^-, we consider the case $q=0$. In this case, we have $P(z)=z^{2n+2}+\varepsilon^2=z^{2n+2}+e^{-i2\xi}$. It follows that $z_\nu^{2(n+1)}=e^{i2(n+1)t_\nu}=-e^{i2\nu\pi}e^{-i2\xi}=e^{i(2\nu-1)\pi}e^{-i2\xi}=e^{i[(2\nu-1)\pi-2\xi]}$. Consequently, zeros of the polynomial $P(z)=P_{0,\xi}(z)$ have the form

$$z_\nu=e^{it_\nu},\qquad t_\nu=\frac{(2\nu-1)\pi}{2(n+1)}-\frac{\xi}{n+1},\qquad \nu=1,2,\dots,2n+2.$$

Let us find zeros of the polynomial $P^+_{0,\xi}(z)=z^{n+1}-i\varepsilon=z^{n+1}-ie^{-i\xi}=z^{n+1}-e^{i\pi/2}e^{-i\xi}=z^{n+1}-e^{i(\pi/2-\xi)}$. We have $(z_\nu^+)^{n+1}=e^{i(n+1)t_\nu^+}=e^{i2(\nu-1)\pi}e^{i(\pi/2-\xi)}=e^{i[2(\nu-1)\pi+\pi/2-\xi]}=e^{i[(4\nu-3)/2\pi-\xi]}$; i.e.,

$$z_\nu^+=e^{it_\nu^+},\qquad t_\nu^+=\frac{(4\nu-3)\pi}{2(n+1)}-\frac{\xi}{n+1}=t_{2\nu-1},\qquad \nu=1,2,\dots,n+1.$$

Since the set of zeros of the polynomial $P = P_{0,\xi}$ with odd indices coincides with the set of zeros of the polynomial $P^+ = P^+_{0,\xi}$, the set of zeros of the polynomial $P = P_{0,\xi}$ with even indices also coincides with the set of zeros of the polynomial $P^- = P^-_{0,\xi}$, i.e., we have

$$z^+_\nu = e^{it^+_\nu} = z_{2\nu-1}, \qquad t^+_\nu = t_{2\nu-1}, \quad \nu = 1, 2, \dots, n+1; \tag{4.8}$$

$$z^-_\nu = e^{it^-_\nu} = z_{2\nu}, \qquad t^-_\nu = t_{2\nu}, \qquad \nu = 1, 2, \dots, n+1. \tag{4.9}$$

Thus, for $q = 0$, the structure of the positional relationship of zeros of the polynomials P, P^+, and P^- is clear. In particular, for zeros of the polynomials P^+ and P^-, the alternation property holds.

Properties (4.8) and (4.9) remain valid for an arbitrary $q \in (0, 1)$. Indeed, let us consider, for example, the alternation property. If it is violated for a value $\widehat{q} \in (0, 1)$, then, by the continuous dependence of the zeros on the parameter q, there exists $q^* \in (0, \widehat{q}\,]$ for which a certain zero of the polynomial $P^-_{q^*,\xi}$ coincides with a certain zero of the polynomial $P^+_{q^*,\xi}$. But then, the number of different zeros of the polynomial $P_{q^*,\xi}$ is at most $2n+1$, which contradicts Lemma 1, by which the polynomial $R_{q^*,\xi}(t)$ and, therefore, $P_{q^*,\xi}$ have $2n+2$ different zeros. Consequently, for zeros of the polynomials $P = P_{q,\xi}$, $P^+ = P^+_{q,\xi}$, and $P^- = P^-_{q,\xi}$, equalities (4.8) and (4.9) remain valid for all $0 < q < 1$.

Applying the Viéte theorem for $P^+(z) = z^{n+1} - qz^n + i\varepsilon qz - i\varepsilon$ and taking into account (4.8), we obtain

$$- i\varepsilon = -e^{i\pi/2}e^{-i\xi} = e^{-i\pi/2}e^{-i\xi} = e^{-i(\pi/2+\xi)} = (-1)^{n+1}\prod_{\nu=1}^{n+1} z^+_\nu = e^{-i(n+1)\pi}\prod_{\nu=1}^{n+1} z_{2\nu-1}$$

$$= e^{-i(n+1)\pi}\prod_{\nu=1}^{n+1} e^{it_{2\nu-1}} = \exp\left\{ i\left[-(n+1)\pi + \sum_{\nu=1}^{n+1} t_{2\nu-1}\right]\right\}.$$

This implies the equality

$$-\frac{\pi}{2} - \xi = -(n+1)\pi + \sum_{\nu=1}^{n+1} t_{2\nu-1} + 2\pi\ell_1, \tag{4.10}$$

where ℓ_1 is integer.

Similarly, applying the Viéte theorem for $P^-(z) = z^{n+1} - qz^n - i\varepsilon qz + i\varepsilon$ and taking into account (4.9), we arrive at the equality

$$\frac{\pi}{2} - \xi = -(n+1)\pi + \sum_{\nu=1}^{n+1} t_{2\nu} + 2\pi\ell_2, \qquad \ell_2 \in \mathbb{Z}. \tag{4.11}$$

Subtracting equality (4.11) from (4.10), we obtain the equality

$$\pi + \sum_{\nu=1}^{n+1} t_{2\nu-1} - \sum_{\nu=1}^{n+1} t_{2\nu} = 2\pi(\ell_2 - \ell_1). \tag{4.12}$$

From (4.3), it follows that (4.12) holds only for $\ell_2 - \ell_1 = 0$; i.e., the first equality in (4.2) is proved.

Let us introduce the notation for the sums of the kth degrees of the zeros

$$s^+_k = \sum_{\nu=1}^{n+1} (z^+_\nu)^k = \sum_{\nu=1}^{n+1} e^{ikt_{2\nu-1}}, \qquad s^-_k = \sum_{\nu=1}^{n+1} (z^-_\nu)^k = \sum_{\nu=1}^{n+1} e^{ikt_{2\nu}}, \qquad k = 1, \dots, n-1, \tag{4.13}$$

of the polynomials

$$P^{+}(z) = z^{n+1} + \sum_{\ell=1}^{n+1} a_{\ell}^{+} z^{n+1-\ell} = z^{n+1} - qz^{n} + i\varepsilon qz - i\varepsilon,$$

$$P^{-}(z) = z^{n+1} + \sum_{\ell=1}^{n+1} a_{\ell}^{-} z^{n+1-\ell} = z^{n+1} - qz^{n} - i\varepsilon qz + i\varepsilon,$$

respectively.

To prove the remaining equalities in (4.2), we apply the Newton recurrence formulas (see [8, Sect. 90])

$$s_1^{+} + a_1^{+} = 0, \qquad s_2^{+} + a_1^{+} s_1^{+} + 2a_2^{+} = 0, \qquad s_3^{+} + a_1^{+} s_2^{+} + a_2^{+} s_1^{+} + 3a_3^{+} = 0, \ \ldots$$
$$\ldots, \ s_{n-1}^{+} + a_1^{+} s_{n-2}^{+} + a_2^{+} s_{n-3}^{+} + \ldots + a_{n-2}^{+} s_1^{+} + (n-1)a_{n-1}^{+} = 0,$$

which allow us to express s_k^{+}, for $k = 1, 2, \ldots, n-1$, in terms of the coefficients of the polynomial P_q^{+}: $a_1^{+} = -q$, $a_2^{+} = 0$, $\ldots$, $a_{n-1}^{+} = 0$. As a result, we conclude that $s_k^{+} = q^k$, $k = 1, 2, \ldots, n-1$.

In the same way, we conclude that $s_k^{-} = q^k$ for $k = 1, 2, \ldots, n-1$. Consequently, $s_k^{-} - s_k^{+} = 0$ for $k = 1, 2, \ldots, n-1$, which, in view of notation (4.13), are equalities (4.2) required. □

Remark 2. Formulas (4.6) and (4.7) imply the identity

$$\left|P_{q,\xi}^{+}(e^{it})\right|^2 + \left|P_{q,\xi}^{-}(e^{it})\right|^2 = 4\left(1 + q^2 - 2q\cos t\right) \qquad (t \in \mathbb{T}, \quad \xi \in \mathbb{R}, \quad q \in (-1, 1), \quad n \in \mathbb{N}),$$

which is the periodic analog of the corresponding statements established in [12, Paper 8, Sect. 9] and developed in [5, Lemma 1].

5. L-APPROXIMATION OF THE FUNCTION χ_h BY TRIGONOMETRIC POLYNOMIALS

For a number $h \in (0, \pi]$, we consider the characteristic function

$$\chi_h(t) = \chi_{(-h,h)}(t) = \begin{cases} 1, & |t| < h, \\ 0, & h \le |t| \le \pi, \end{cases}$$

of the interval $(-h, h)$ extended periodically to $\mathbb{R}$ with period 2π.

In this section, we consider the problem of calculating the value $E_{n-1}(\chi_h)_L$ of the best integral approximation on the period $\mathbb{T} = [-\pi, \pi)$ of the function χ_h by the subspace $\mathcal{T}_{n-1}$. Sometimes, we will use certain statements proved in the Appendix (Section 6).

For brevity, we denote the value $E_{n-1}(\chi_h)_L$ by $\mathfrak{I}_n(h)$; i.e., we put

$$\mathfrak{I}_n(h) = E_{n-1}(\chi_h)_L \qquad \text{for} \qquad h \in (0, \pi). \tag{5.1}$$

The equality $1 - \chi_h(t) = \chi_{\pi-h}(\pi - t)$ implies that

$$\mathfrak{I}_n(\pi - h) = \mathfrak{I}_n(h) \qquad \text{for} \qquad h \in (0, \pi). \tag{5.2}$$

In [2, Theorem 1.3.1] we proved statements equivalent to the following ones:

$$\mathfrak{I}_n(h) = 2h \qquad \text{for} \qquad 0 < h \le \frac{\pi}{2n}, \qquad n \ge 1, \tag{5.3}$$

$$\mathfrak{I}_n(h) \le \frac{\pi}{n} \qquad \text{for} \qquad 0 < h \le \pi, \qquad n \ge 1; \tag{5.4}$$

here, inequality (5.4) turns into equality if h coincides with any zero of the polynomial $\cos nt$ from the interval $(0, \pi)$, i.e., for $h = h_j = (2j-1)\pi/(2n)$, $j \in \{1, 2, \dots, n\}$.

Let $-1 < q < 1$ and $t_j = t_j(q) \in (0, \pi)$ $(j = 1, 2, \dots, n+1)$ be zeros of the polynomial[9]

$$R_q(t) = R_{q,0}(t) = \cos(n+1)t - 2q\cos nt + q^2\cos(n-1)t \tag{5.5}$$

enumerated in ascending order. According to formula (6.16), for $\xi = 0$, the following representation of the polynomial $R_q(t)$ is valid for $t \in [0, \pi]$:

$$R_q(t) = \{2q\cos t - (1+q^2)\}\cos[nt - \lambda(t,q)], \qquad \lambda(t,q) = \arccos\frac{2q - (1+q^2)\cos t}{1 + q^2 - 2q\cos t}, \tag{5.6}$$

which provides an intensional information about its zeros $t_1, t_2, \dots, t_{n+1}$ in the interval $(0, \pi)$. In particular (see assertion (b) of Theorem 10 in Section 6), each of these zeros $t_j = t_j(q)$ is a continuous decreasing function of the parameter $q \in (-1, 1)$. Therefore, taking into account the explicit form of the polynomials $R_{-1}(t) = (\cos t + 1)\cos nt$ and $R_1(t) = (\cos t - 1)\cos nt$, we arrive at the following statement about zeros of R_q.

Lemma 2. *Let $n \in \mathbb{N}$, $n \ge 2$. Then zeros $t_1 < t_2 \ldots < t_{n+1}$ of the polynomial R_q in the interval $(0, \pi)$ continuously depend on the parameter q and decrease when $q \in (-1, 1)$ increases: the first zero $t_1 = t_1(q)$ decreases in the interval $\left(0, \frac{\pi}{2n}\right)$, the jth zero $t_j = t_j(q)$ $(2 \le j \le n)$ decreases in the interval $\left(\frac{(2j-3)\pi}{2n}, \frac{(2j-1)\pi}{2n}\right)$, and the $(n+1)$th zero $t_{n+1} = t_{n+1}(q)$ decreases in the interval $\left(\pi - \frac{\pi}{2n}, \pi\right)$.*

For $-1 < q < 1$ and $2 \le \ell \le n$, we denote by $\tau_{q,\ell}$ the cosine-polynomial of degree at most $n-1$ that interpolates the characteristic function χ_{t_ℓ} of the interval $(-t_\ell, t_\ell)$ at the points t_j, $1 \le j \le n+1$, $j \ne \ell$. The existence and uniqueness of the polynomial $\tau_{q,\ell}$ (see [21, Ch. 3, Sect. 3, Property 3]) follow from the fact that $\{1, \cos t, \cos 2t, \dots, \cos(n-1)t\}$ is a Chebyshev system on $[0, \pi]$.

Theorem 5. *For $-1 < q < 1$ and $2 \le \ell \le n$, the polynomial $\tau_{q,\ell} \in \mathcal{T}_{n-1}$ is the polynomial of the best integral approximation for the function χ_{t_ℓ}; moreover, the following equalities hold:*

$$E_{n-1}(\chi_{t_\ell})_L = \|\chi_{t_\ell} - \tau_{q,\ell}\|_L = \left|\int_{-\pi}^{\pi} \chi_{t_\ell}(t)\,\mathrm{sgn}\{R_q(t)\}\,dt\right|; \tag{5.7}$$

$$E_{n-1}(\chi_{t_\ell})_L = \begin{cases} 2t_\ell - 4\sum\limits_{j=1}^{\ell/2}(t_{2j-1} - t_{2j-2}) & \text{for even } \ell, \\ 2t_\ell - 4\sum\limits_{j=1}^{[\ell/2]}(t_{2j} - t_{2j-1}) & \text{for odd } \ell; \end{cases} \tag{5.8}$$

[9] The polynomial R_q has $n+1$ zeros in the interval $(0, \pi)$, and all of them are different; this follows from Lemma 1, the evenness of this polynomial, and the formulas $R_q(0) = (1-q)^2 > 0$ and $R_q(\pi) = (-1)^{n+1}(1+q)^2 \ne 0$.

here, $t_0 = 0$ *and* $[\ell/2]$ *is the integer part of the number* $\ell/2$*.*

The proof is close to the proof of Theorem 2.1.2 from [2], which corresponds to the case $q = 1$. Let us repeat briefly the reasoning adapting it to the case $|q| < 1$.

The polynomial $\tau_{q,\ell}$ and the function χ_{t_ℓ} are even functions; hence, $\tau_{q,\ell}$ interpolates χ_{t_ℓ} at the points $\pm t_j,\quad 1 \le j \le n+1,\quad j \ne \ell$. These points divide the period $\mathbb{T}$ (it is convenient to interpret the period as the unit circle) into $2n$ parts located symmetrically with respect to the origin. We will call the two symmetric parts $[-t_{\ell+1}, -t_{\ell-1}]$ and $[t_{\ell-1}, t_{\ell+1}]$ (containing the points $-t_\ell$ and t_ℓ, respectively) *large*, and all the remaining parts will be called *small*.

At the endpoints of an arbitrary small part, the polynomial $\tau_{q,\ell}$ takes equal values (either zero or equal to 1). Therefore, the derivative $\tau'_{q,\ell}$ has at least one zero inside each of these parts. However, the number of small parts is $2(n-1)$; hence, the polynomial $\tau'_{q,\ell}$ has no other zeros, since its degree is $n-1$. Consequently, there is exactly one zero inside each small part. From this, we conclude that the derivative does not vanish on both large parts. Therefore, the polynomial $\tau_{q,\ell}$ decreases on the segment $[t_{\ell-1}, t_{\ell+1}]$ from the value $\tau_{q,\ell}(t_{\ell-1}) = \chi_{t_\ell}(t_{\ell-1}) = 1$ to the value $\tau_{q,\ell}(t_{\ell+1}) = \chi_{t_\ell}(t_{\ell+1}) = 0$ and increases on the segment $[-t_{\ell+1}, -t_{\ell-1}]$ from 0 to 1. The polynomial $\tau_{q,\ell}$ takes equal positive values at the discontinuity points of the function χ_{t_ℓ}, i.e., at the points t_ℓ and $-t_\ell$. Consequently, the equality $\operatorname{sgn}\{\chi_{t_\ell}(t) - \tau_{q,\ell}(t)\} = (-1)^{\ell+1}\operatorname{sgn}\{R_q(t)\}$ holds.

By assertion (4.1) of Theorem 4, the function $\operatorname{sgn}\{R_q(t)\}$ belongs to $\mathcal{T}_{n-1}^{\perp}$. From this and Theorem 2, it follows that $\tau_{q,\ell} \in \mathcal{T}_{n-1}$ is the polynomial of the best integral approximation for χ_{t_ℓ}; moreover, the following equalities hold:

$$E_{n-1}(\chi_{t_\ell})_L = 2\left|\int_0^\pi \chi_{t_\ell}(t)\operatorname{sgn}\{R_q(t)\}\,dt\right| = 2\left|\int_0^{t_\ell} \chi_{t_\ell}(t)\operatorname{sgn}\{R_q(t)\}\,dt\right|,$$

which are equivalent to assertions (5.7) and (5.8). □

The case $\ell = 2$ is an important particular case of Theorem 5. In this case $\tau_{q,2} \in \mathcal{T}_{n-1}$ is the polynomial of the best integral approximation for χ_{t_2}; moreover,

$$E_{n-1}(\chi_{t_2})_L = \|\chi_{t_2} - \tau_{q,2}\|_L = 2t_2 - 4t_1 \qquad (-1 < q < 1,\quad 2 \le n). \tag{5.9}$$

Taking into account Lemma 2, we can use $h = t_2(q) \in \left(\dfrac{\pi}{2n}, \dfrac{3\pi}{2n}\right)$ instead of the parameter $q \in (-1, 1)$. In addition, the following equality holds:

$$q = q(h) = \frac{\sin h + \cos nh}{\cos(n-1)h}, \qquad \text{where} \quad h = t_2(q) \in \left(\frac{\pi}{2n}, \frac{3\pi}{2n}\right). \tag{5.10}$$

Indeed, from formula (5.5), we obtain the equation

$$R_q(h) \equiv \cos(n+1)h - 2q\cos nh + q^2\cos(n-1)h = 0. \tag{5.11}$$

Equation (5.11) has two solutions with respect to the parameter q. One of these solutions is greater than one in absolute value. The other solution $q = q(h)$ coincides with $q(h)$ from (5.10); the solution $q(h)$ ranges over the interval $(-1, 1)$ monotonically decreasing when h increases in the interval $\left(\dfrac{\pi}{2n}, \dfrac{3\pi}{2n}\right)$.

We denote by $t_1 = t_{1,h}$ the first positive zero of the polynomial $R_{q(h)}(t)$. To derive more information about this zero, we use representation (5.6). By (6.13) and the last two equalities in (6.11), the function $\widetilde{\varphi}(t) = nt - \lambda(t, q(h))$ increases with respect to t on $[0, \pi]$; moreover, $\widetilde{\varphi}(0) = -\pi$ and $\widetilde{\varphi}(\pi) = n\pi$. From this, with the help of representation (5.6), we conclude that $t_1 = t_{1,h}$ coincides with a root of the equation $nt - \lambda(t, q(h)) = -\pi/2$. Writing this equation in the equivalent form $\cos nt = \sin \lambda(t, q(h))$ and applying formula (6.12) from Section 6, we arrive at the equation

$$\cos nt = \frac{\{1 - q^2(h)\} \sin t}{1 + q^2(h) - 2q(h) \cos t}, \tag{5.12}$$

whose unique root on the interval $\left(0, \frac{\pi}{2n}\right)$ is $t_{1,h}$.

Theorem 6. *The following assertions are valid*:

(1) *the value* $v_{1,n} = nt_{1,\pi/n}$ *decreases with respect to* $n \geq 2$;

(2) *the limit* $v_1 = \lim\limits_{n\to\infty} v_{1,n} = 0.97116830789\ldots$ *coincides with the unique root of the equation*[10]

$$\cos v = \frac{2v\pi}{v^2 + \pi^2} \tag{5.13}$$

on $[0, \pi/2]$;

(3) $\lim\limits_{n\to\infty} n\Im_n\left(\frac{\pi}{n}\right) = 2\pi - 4v_1 = 2.398512075618\ldots$

Proof. For $h = \pi/n$, Eq. (5.12) takes the form

$$\cos nt = \frac{\left\{1 - q^2\left(\frac{\pi}{n}\right)\right\} \sin t}{1 + q^2\left(\frac{\pi}{n}\right) - 2q\left(\frac{\pi}{n}\right) \cos t}. \tag{5.14}$$

As mentioned above, $t_1 = t_{1,\pi/n}$ coincides with the unique root of this equation located in the interval $\left(0, \frac{\pi}{2n}\right)$. The change of variable $t = v/n$ transforms Eq. (5.14) to the form

$$\cos v = \frac{\left\{1 - q^2\left(\frac{\pi}{n}\right)\right\} \sin \frac{v}{n}}{1 + q^2\left(\frac{\pi}{n}\right) - 2q\left(\frac{\pi}{n}\right) \cos \frac{v}{n}}. \tag{5.15}$$

It is clear that the root of this equation located in the interval $(0, \pi/2)$ coincides with $v_{1,n}$; in addition, it is the unique root in this interval.

Let us show that $v_{1,n}$ decreases with respect to $n \geq 2$. Indeed, the left-hand side of Eq. (5.15) $\cos v$ is a decreasing function in the interval $(0, \pi/2)$. Therefore, it is sufficient to prove that the right-hand side of the equation increases with respect to $n \geq 2$ for any fixed value v from $(0, \pi/2)$. To prove this assertion, we use one more change: $\alpha = 1/n$. Taking into account (5.10), after transformations, we come to the following formulas? e? (avtor):

$$q(\alpha\pi) = \frac{1 - \sin \alpha\pi}{\cos \alpha\pi} = \sec \alpha\pi - \tan \alpha\pi, \quad q^2(\alpha\pi) = \frac{1 - \sin \alpha\pi}{1 + \sin \alpha\pi},$$

$$1 - q^2(\alpha\pi) = \frac{2 \sin \alpha\pi}{1 + \sin \alpha\pi}, \quad 1 + q^2(\alpha\pi) = \frac{2}{1 + \sin \alpha\pi}.$$

[10]Note that the root of Eq. (5.13) coincides with the root of the equation $\sec v - \tan v = v/\pi$ on $[0, \pi/2]$.

Using these formulas? e? (avtor), we rewrite Eq. (5.15) as

$$\cos v = \frac{\sin \alpha\pi \sin \alpha v}{1 - \cos \alpha\pi \cos \alpha v}. \tag{5.16}$$

In order to prove that the right-hand side of Eq. (5.15) increases with respect to $n \geq 2$ for any fixed $v \in (0, \pi/2)$, it is sufficient to show that the right-hand side of Eq. (5.16)

$$r(\alpha, v) = \frac{\sin \alpha\pi \sin \alpha v}{1 - \cos \alpha\pi \cos \alpha v} = \frac{\cos(\pi - v)\alpha - \cos(\pi + v)\alpha}{2 - \cos(\pi - v)\alpha - \cos(\pi + v)\alpha}$$

is a decreasing function with respect to $\alpha \in (0, 1/2)$ for a fixed $v \in (0, \pi/2)$.

Let us find the partial derivative of the function $r(\alpha, v)$ with respect to α and multiply it by the positive function $\{2 - \cos(\pi - v)\alpha - \cos(\pi + v)\alpha\}^2/2$. As a result, we obtain

$$\frac{\big\{2 - \cos(\pi - v)\alpha - \cos(\pi + v)\alpha\big\}^2}{2} \frac{\partial r(\alpha, v)}{\partial \alpha} = 2\pi v\alpha\,(\cos v\alpha - \cos \pi\alpha)\Big(\frac{\sin \pi\alpha}{\pi\alpha} - \frac{\sin v\alpha}{v\alpha}\Big). \tag{5.17}$$

The right-hand side of equality (5.17) is negative for $\alpha \in (0, 1/2)$ and $v \in (0, \pi/2)$, since the function $\sin x/x$ decreases with respect to $x \in (0, \pi)$. Thus, the value $v_{1,n} = n\, t_{1,\pi/n}$ decreases with respect to $n \geq 2$.

In order to determine the limit of the quantity $v_{1,n}$ as $n \to \infty$, we assume that the parameter α in the right-hand side of Eq. (5.16) tends to zero and obtain the "limiting" equation $\cos v = 2v\pi/(v^2 + \pi^2)$. By the continuity, root v of this equation located on $[0, \pi/2]$ coincides with the limit $v_1 = \lim\limits_{n\to\infty} v_{1,n} = \lim\limits_{n\to\infty} n\, t_{1,\pi/n}$ required. Solving the "limiting" equation numerically on $[0, \pi/2]$, we find that $v_1 = 0.97116830789\ldots$

Thus, we have

$$\begin{aligned}\lim_{n\to\infty} n\, \mathfrak{I}_n\left(\frac{\pi}{n}\right) &= \lim_{n\to\infty} 2n\left(\frac{\pi}{n} - 2t_{1,\pi/n}\right) = 2\lim_{n\to\infty}\left(\pi - 2\,n\,t_{1,\pi/n}\right) \\ &= 2(\pi - 2v_1) = 2.398512075618\ldots\end{aligned}$$

Theorem 6 is proved. □

In some problems of approximation theory, it is convenient to use the function $\mathcal{X}_h(t) = \chi_h(t)/(2h)$, $0 < h \leq \pi$, instead of the classical characteristic function χ_h of the interval $(-h, h)$. We call the function $\mathcal{X}_h(t)$ the *L-normalized* characteristic function of the interval $(-h, h)$, since $\|\mathcal{X}_h\|_L = 1$. It is clear that $E_{n-1}(\mathcal{X}_h)_L = E_{n-1}(\chi_h)_L/(2h) = \mathfrak{I}_n(h)/(2h)$. In the case $h = t_2$ (see formula (5.9)), the following equality holds:

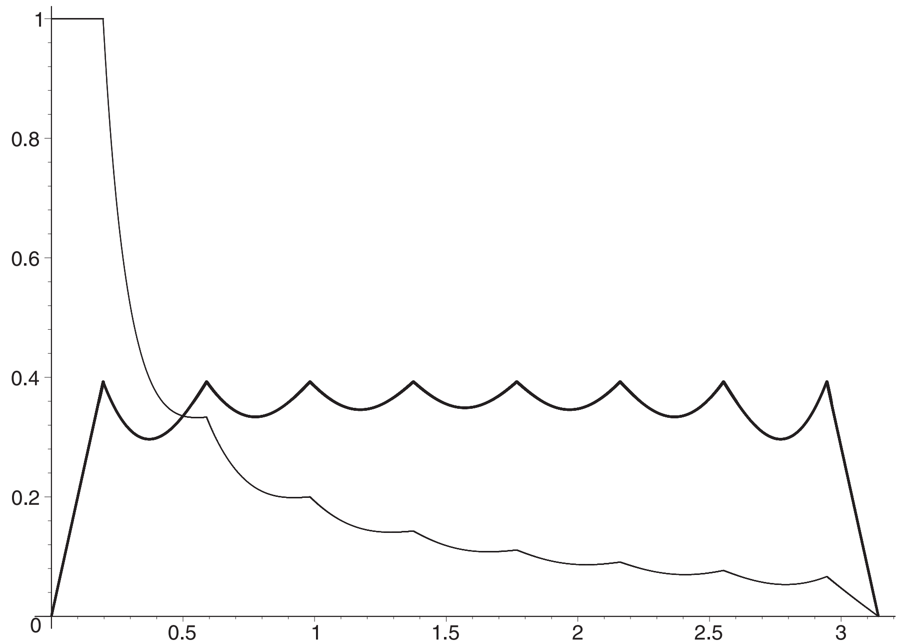
$$E_{n-1}(\mathcal{X}_{t_2})_L = 1 - \frac{2t_1}{t_2}.$$

With the help of Theorem 6, we conclude that the sequence $\mathcal{E}_{k-1} = E_{k-1}\left(\mathcal{X}_{\pi/k}\right)_L$, $k = 2, 3, \ldots$, tends to the number $1 - 2v_1/\pi$ monotonically increasing; i.e.,

$$\mathcal{E}_1 < \mathcal{E}_2 < \ldots < \mathcal{E}_k < \mathcal{E}_{k+1} < \ldots < \lim_{n\to\infty} \mathcal{E}_n = 1 - \frac{2v_1}{\pi} = 0.3817350529\ldots$$

Let us return to studying the behavior of the value $\mathfrak{I}_n(h) = E_{n-1}(\chi_h)_L$ with respect to h for a fixed $n \geq 2$. First, we consider the case $n = 2$, when, by properties (5.2) and (5.3), it is sufficient to find $\mathfrak{I}_2(h)$ for $\pi/4 \leq h \leq \pi/2$. Acting as above (see formulas (5.10) and (5.11)), we find the first zero $t_1 = t_{1,h} = \pi/4 + h/2 - \arccos\Bigl(\dfrac{\sin h}{2\sin(\pi/4 + h/2)}\Bigr)$ of the corresponding Bernstein function. Then, from (5.1), (5.9), and (5.10), we obtain $\mathfrak{J}_2(h) = -\pi + 4\arccos\Bigl(\dfrac{\sin h}{2\sin(\pi/4 + h/2)}\Bigr)$ for $\pi/4 \le h \le \pi/2$. In particular, $2\,\mathfrak{J}_2\,(\pi/2) = 2\pi/3 = 2.094395102393\ldots$

For $n \ge 3$, we failed to find explicit formulas for $\mathfrak{J}_n(h)$ similar to the case $n = 2$. However, with the help of Theorem 5, we can find this function on a fine grid with any level of accuracy. In such a way, we have constructed the graph of the function $\mathfrak{J}_8(h)$ represented in the figure by the thick line. Let us recall (see (5.3) and (5.2)) that $\mathfrak{J}_8(h) = 2h$ for $h \in (0, \pi/16)$ and $\mathfrak{J}_8(h) = 2(\pi - h)$ for $h \in (15\pi/16, \pi)$.



The graphs of the function $\mathfrak{J}_8(h)$ (the thick line) and the function $\mathfrak{J}_8(h)/(2h)$ on the segment $[0, \pi]$.

In the figure, the thin line represents the graph of the function $\mathfrak{J}_8(h)/(2h) = E_7(\mathcal{X}_h)_L$. Note that the function $\mathfrak{J}_8(h)/(2h)$ is constant and equal to one on the half-interval $(0, \pi/16]$, which follows from (5.3).

6. APPENDIX. PROPERTIES OF THE BERNSTEIN FUNCTION

In this section, we study properties of the Bernstein function defined in (3.5):

$$B(t, a, \xi) = \cos\,[nt - \delta(t, a) + \xi] \qquad \Bigl(\delta(t, a) = \arccos u(t, a), \qquad u(t, a) = \frac{1 - a\cos t}{a - \cos t}\Bigr), \tag{6.1}$$

where $n \in \mathbb{N}$, $a \in (-\infty, -1) \cup (1, +\infty)$, $\xi \in \mathbb{R}$, and $t \in [0, \pi]$. In addition, we shall also construct an extension of this function to $[-\pi, 0]$.

It is easy to verify the following relations:

$$1 - u^2(t, a) = 1 - \Bigl(\frac{1 - a\cos t}{a - \cos t}\Bigr)^2 = \frac{(a^2 - 1)\sin^2 t}{(a - \cos t)^2}; \tag{6.2}$$

$$\sin\delta(t,a)=\sin\arccos u(t,a)=\sqrt{1-u^2(t,a)}=\frac{\sqrt{a^2-1}\sin t}{|a-\cos t|}; \tag{6.3}$$

$$\frac{\partial u(t,a)}{\partial t}=\frac{(a^2-1)\sin t}{(a-\cos t)^2};\qquad \frac{\partial\delta(t,a)}{\partial t}=\frac{-\sqrt{a^2-1}}{|a-\cos t|};\qquad \delta(0,a)=\pi,\qquad \delta(\pi,a)=0. \tag{6.4}$$

It follows from (6.4) that, for $|a|>1$, the function $\varphi(t,a,\xi)=nt-\delta_a(t)+\xi$ increases with respect to t on $[0,\pi]$; moreover, $\varphi(0,a,\xi)=-\pi+\xi$ and $\varphi(\pi,a,\xi)=n\pi+\xi$. From this, taking into account (6.1), we obtain the following statement.

Lemma 3. *For $\xi,a\in\mathbb{R}$, $|a|>1$, the function $B(t)=B(t,a,\xi)$ defined by formula* (6.1) *has an $(n+1)$-point alternance on the half-interval* $[0,\pi)$.

The function $B(t,a,\xi)$, for $\xi=0$, arose in Bernstein's investigations in connection with problem (3.2). He proved [3, Paper 8, Sect. 3] Lemma 3 for $\xi=0$, which allowed him to find solution (3.4) of problem (3.2). In the case of an arbitrary $\xi\in\mathbb{R}$, the proof of Lemma 3 is similar.

For $a,\xi\in\mathbb{R}$, $|a|>1$, let us consider the function

$$\beta(t,a,\xi)=2(\cos t-a)B(t,a,\xi)=2(\cos t-a)\cos\left[nt-\delta(t,a)+\xi\right],\qquad t\in[0,\pi].$$

Bernstein's remark [3, Paper 9, Sect. 2, footnote 3] implies that the function $\beta(t,a,\xi)$ is a trigonometric polynomial for $\xi=0$. In order to verify the validity of a similar assertion in the case of an arbitrary $\xi\in\mathbb{R}$, let us use formulas (6.2), (6.3), and standard trigonometric formulas.

For a number $a\in(-\infty,-1)\cup(1,+\infty)$, there exists a unique number $q\in(-1,0)\cup(0,1)$ related to a as follows: $a=\big(q+1/q\big)/2$. Moreover, a parameter $q\in(-1,0)$ corresponds to the parameter $a\in(-\infty,-1)$ and a parameter $q\in(0,1)$ corresponds to the parameter $a\in(1,+\infty)$. Namely, the following relations hold:

$$a+\sqrt{a^2-1}=1/q,\qquad a-\sqrt{a^2-1}=q\qquad\text{for}\qquad a>1,\qquad q\in(0,1); \tag{6.5}$$

$$a-\sqrt{a^2-1}=1/q,\qquad a+\sqrt{a^2-1}=q\qquad\text{for}\qquad a<-1,\qquad q\in(-1,0). \tag{6.6}$$

First, let us consider the case $a>1$:

$$\beta(t,a,\xi)=2(\cos t-a)\Big[\cos(nt+\xi)\cos\delta(t,a)+\sin(nt+\xi)\sin\delta(t,a)\Big]$$

$$=2(\cos t-a)\left[\frac{1-a\cos t}{a-\cos t}\cos(nt+\xi)+\sqrt{1-\left(\frac{1-a\cos t}{a-\cos t}\right)^2}\sin(nt+\xi)\right]$$

$$=\Big\{a+\sqrt{a^2-1}\Big\}\cos[(n+1)t+\xi]-2\cos(nt+\xi)+\Big\{a-\sqrt{a^2-1}\Big\}\cos[(n-1)t+\xi]. \tag{6.7}$$

From formulas (6.7) and (6.5), we obtain the equality

$$q\,\beta(t,a,\xi)=\cos[(n+1)t+\xi]-2q\cos(nt+\xi)+q^2\cos[(n-1)t+\xi]. \tag{6.8}$$

Now, let us consider the case $a<-1$:

$$\beta(t,a,\xi)=2(a\cos t-1)\cos(nt+\xi)+2\sqrt{a^2-1}\sin t\sin(nt+\xi)$$

$$=\Big\{a-\sqrt{a^2-1}\Big\}\cos[(n+1)t+\xi]-2\cos(nt+\xi)+\Big\{a+\sqrt{a^2-1}\Big\}\cos[(n-1)t+\xi].$$

From this, subject to (6.6), it follows that

$$q\,\beta(t,a,\xi) = \cos[(n+1)t+\xi] - 2q\cos(nt+\xi) + q^2\cos[(n-1)t+\xi]. \tag{6.9}$$

Comparing formulas (6.8) and (6.9), we arrive at the conclusion that the one family of polynomials

$$R_{q,\xi}(t) = q\,\beta(t,a,\xi) = 2q(\cos t - a)B(t,a,\xi)$$
$$= \cos[(n+1)t+\xi] - 2q\cos(nt+\xi) + q^2\cos[(n-1)t+\xi], \qquad -1 < q < 1, \tag{6.10}$$

corresponds to the two families of polynomials $\{\beta(t,a,\xi),\ a > 1\}$ and $\{\beta(t,a,\xi),\ a < -1\}$. In addition, the parameters a and q are connected by formulas (6.5) and (6.6), and the two limit values $a = \pm\infty$ corresponds to the value $q = 0$ simultaneously.

It will be easier to study the Bernstein function in terms of the parameter q. First, we rewrite the function $\delta(t,a)$ in terms of the parameter $q \in (-1,1)$:

$$\delta(t,a) = \lambda(t,q) = \arccos\widetilde{u}(t,q), \qquad \widetilde{u}(t,q) = \frac{2q-(1+q^2)\cos t}{1+q^2-2q\cos t}.$$

Further, we need the following formulas similar to formulas (6.2)–(6.4):

$$1-\widetilde{u}^2(t,q) = \frac{(1-q^2)^2\sin^2 t}{(1+q^2-2q\cos t)^2}, \qquad \lambda(0,q)=\pi, \qquad \lambda(\pi,q)=0, \tag{6.11}$$

$$\sin\lambda(t,q) = \sin\arccos\widetilde{u}(t,q) = \sqrt{1-\widetilde{u}^2(t,q)} = \frac{(1-q^2)\sin t}{1+q^2-2q\cos t}, \tag{6.12}$$

$$\frac{\partial\widetilde{u}(t,q)}{\partial t} = \frac{(1-q^2)^2\sin t}{(1+q^2-2q\cos t)^2} = \frac{1-\widetilde{u}^2(t,q)}{\sin t}, \qquad \frac{\partial\widetilde{u}(t,q)}{\partial q} = \frac{2(1-q^2)\sin^2 t}{(1+q^2-2q\cos t)^2} = 2\left\{1-\widetilde{u}^2(t,q)\right\},$$

$$\frac{\partial\lambda(t,q)}{\partial t} = \frac{-1}{\sqrt{1-\widetilde{u}^2(t,q)}}\frac{\partial\widetilde{u}(t,q)}{\partial t} = \frac{-\sqrt{1-\widetilde{u}^2(t,q)}}{\sin t} = \frac{q^2-1}{1+q^2-2q\cos t}, \tag{6.13}$$

$$\frac{\partial\lambda(t,q)}{\partial q} = \frac{-1}{\sqrt{1-\widetilde{u}^2(t,q)}}\frac{\partial\widetilde{u}(t,q)}{\partial q} = -2\sqrt{1-\widetilde{u}^2(t,q)} = -2\sin\lambda(t,q) = \frac{2(q^2-1)\sin t}{1+q^2-2q\cos t}. \tag{6.14}$$

Changing the variable $a = (1+q^2)/(2q)$ in $B(t,a,\xi)$, we obtain a new function depending on q, which we denote by $\mathcal{B}(t,q,\xi)$.

Finally, taking into account (6.10), in the case $t \in [0,\pi]$ $(\xi\in\mathbb{R},\ n\in\mathbb{N})$, we have

$$B(t,a,\xi) = \mathcal{B}(t,q,\xi) = \cos\left[nt-\lambda(t,q)+\xi\right] = \frac{R_{q,\xi}(t)}{2q\cos t-(1+q^2)}, \qquad -1<q<1. \tag{6.15}$$

Now, let us pass to studying the case $t \in [-\pi,0]$. We express the polynomial $R_{q,\xi}(t)$ for $t \in [-\pi,0]$ in terms of the Bernstein function with the help of the change $t = \theta-\pi$, $\theta\in[0,\pi]$:

$$R_{q,\xi}(t) = R_{q,\xi}(\theta-\pi) = (-1)^{n+1}\Big\{\cos\big[(n+1)\theta+\xi\big] + 2q\cos(n\theta+\xi) + q^2\cos\big[(n-1)\theta+\xi\big]\Big\}$$
$$= (-1)^{n+1}R_{-q,\xi}(\theta) = (-1)^{n+1}\big\{2q\cos t-(1+q^2)\big\}\mathcal{B}_{-q,\xi}(t+\pi), \qquad t\in[-\pi,0].$$

From this, we obtain the representation

$$\frac{R_{q,\xi}(t)}{2q\cos t-(1+q^2)} = \begin{cases} (-1)^{n+1}\mathcal{B}(t+\pi,-q,\xi) & \text{for} \quad -\pi\le t\le 0, \\ \mathcal{B}(t,q,\xi) & \text{for} \quad 0\le t\le\pi. \end{cases} \tag{6.16}$$

This representation can be rewritten in other equivalent form:

$$\frac{R_{q,\xi}(t)}{2q\cos t-(1+q^2)}=\cos\left[nt+\xi-\mu(t,q)\right],\qquad t\in[-\pi,\pi],\tag{6.17}$$

where $\mu(t,q)=\pi+\lambda(t+\pi,-q)$ for $t\in[-\pi,0]$ and $\mu(t,q)=\lambda(t,q)$ for $t\in[0,\pi]$. It is easy to verify that, for any $q\in(-1,1)$, the function $\mu(t,q)$, as a function of the variable t, is decreasing on $[-\pi,\pi]$, infinitely differentiable in the interval $(-\pi,\pi)$, and such that $\mu(-\pi,q)=2\pi$ and $\mu(\pi,q)=0$.

Representations (6.16), (6.17), and Lemma 3 imply that, for $n\in\mathbb{N}$, $q\in(-1,1)$, and $\xi\in\mathbb{R}$, the function

$$F_{q,\xi}(t)=\frac{R_{q,\xi}(t)}{2q\cos t-(1+q^2)}=\frac{\cos\big[(n+1)t+\xi\big]-2q\cos(nt+\xi)+q^2\cos\big[(n-1)t+\xi\big]}{2q\cos t-(1+q^2)}\tag{6.18}$$

has an $(n+1)$-point alternance on each of the half-intervals $[-\pi,0)$ and $[0,\pi)$; the function $F_{q,\xi}$ has a $(2n+2)$-point alternance on the whole period $\mathbb{T}=[-\pi,\pi)$; and the polynomial $R_{q,\xi}$ has $n+1$ different zeros on each of the half-intervals $[-\pi,0)$ and $[0,\pi)$.

From this, with the help of the Chebyshev alternance theorem (see [14, Ch. 3, Sect. 4, Theorem 2]) and Theorem 4, we obtain the following theorem, in which, together with the notation $E_{n-1}(f)_L$ for the value of the best integral approximation of a function $f\in L$ by the subspace $\mathcal{T}_{n-1}$ (see Section 1), we use the notation for the value $E_n(F)_{C_{2\pi}}\ \big(=\inf_{g\in\mathcal{T}_n}\|F-g\|_{C_{2\pi}}\big)$ of the best uniform approximation of a function $F\in C_{2\pi}$ by the subspace $\mathcal{T}_n$.

Theorem 7. *For $n\in\mathbb{N}$, $q\in(-1,1)$, and $\xi\in\mathbb{R}$, the function $F_{q,\xi}$ cannot be approximated neither in the uniform norm (by the subspace $\mathcal{T}_n$) nor in the integral norm (by the subspace $\mathcal{T}_{n-1}$). This means that*

$$E_n(F_{q,\xi})_{C_{2\pi}}=\|F_{q,\xi}\|_{C_{2\pi}},\qquad E_{n-1}(F_{q,\xi})_L=\|F_{q,\xi}\|_L,$$

and the identically zero function is the unique polynomial of the best approximation for $F_{q,\xi}$.

We associate with a pair of numbers $q\in(-1,1)$ and $\xi\in\mathbb{R}$ the function

$$f_{q,\xi}(t)=\frac{2q}{2q\cos t-(1+q^2)}\left\{\cos\xi+\frac{2q\sin\xi\sin t}{1-q^2}\right\}.$$

Below, we study the value $E_n(f_{q,\xi})_{C_{2\pi}}=\inf_{g\in\mathcal{T}_n}\|f_{q,\xi}-g\|_{C_{2\pi}}$.

Theorem 8. *For any $q\in(-1,1)$, $\xi\in\mathbb{R}$, and $n\in\mathbb{N}$, we have*

$$E_n(f_{q,\xi})_{C_{2\pi}}=\frac{4q^{n+2}}{(1-q^2)^2}.\tag{6.19}$$

Assertion (6.19) for $\xi=0$ is equivalent to Bernstein's result (3.4).

Proof. For $q=0$, assertion (6.19) is obvious. Let $q\in(-1,0)\cup(0,1)$. By the first assertion of Theorem 7, to prove (6.19), it is sufficient to show that the function

$$\frac{4q^{n+2}}{(1-q^2)^2}\,\frac{R_{q,\xi}(t)}{2q\cos t-(1+q^2)}-f_{q,\xi}(t)\tag{6.20}$$

is a trigonometric polynomial of degree at most n.

The second equality in (3.8) implies that $R_{q,\xi}(t) = R_{q,0}(t)\cos\xi + R_{q,\pi/2}(t)\sin\xi$. Therefore, to prove (6.20), it is sufficient to show that the function

$$\frac{2q^{n+1}}{(1-q^2)^2}\,\frac{R_{q,0}(t)}{2q\cos t-(1+q^2)} - \frac{1}{2q\cos t-(1+q^2)}$$

and the function

$$\frac{q^n}{1-q^2}\,\frac{R_{q,\pi/2}(t)}{2q\cos t-(1+q^2)} + \frac{\sin t}{2q\cos t-(1+q^2)} \tag{6.21}$$

are even and odd trigonometric polynomials of degree at most n, respectively. These assertions are equivalent to the following ones:

$$\frac{2q^{n+1}\left\{T_{n+1}(x)-2qT_n(x)+q^2T_{n-1}(x)\right\}-(1-q^2)^2}{2qx-(1+q^2)} \in \mathcal{P}_n, \tag{6.22}$$

$$\frac{q^{n}\left\{U_{n}(x)-2qU_{n-1}(x)+q^2U_{n-2}(x)\right\}-(1-q^2)}{2qx-(1+q^2)} \in \mathcal{P}_{n-1}, \tag{6.23}$$

where T_k and U_k the Chebyshev polynomials of the first and second kind, respectively. Inclusions (6.22) and (6.23) hold if and only if the polynomials in the numerators of the fractions vanish for $x = \frac{1}{2}\Big(q+\frac{1}{q}\Big)$, which holds by the known formulas (see [16, Ch. 1, Sect. 1, formulas (20), (21)])

$$2T_k\left(\frac{1}{2}\Big(q+\frac{1}{q}\Big)\right) = q^k + \frac{1}{q^k}, \qquad \Big(q-\frac{1}{q}\Big)U_k\left(\frac{1}{2}\Big(q+\frac{1}{q}\Big)\right) = q^{k+1} - \frac{1}{q^{k+1}}.$$

The theorem is proved. □

Application of Theorem 8 in the case $\xi = \pi/2$ and the change $x = \cos t$ yield the following statement.

Theorem 9. *For any $q \in (-1,1)$ and $n \in \mathbb{N}$, the following equality holds:*

$$\inf_{p\in\mathcal{P}_{n-1}} \left\| \left\{ \frac{2q}{2qx-(1+q^2)} - p(x) \right\} \sqrt{1-x^2} \right\|_{C[-1,1]} = \frac{2q^{n+1}}{1-q^2}.$$

Representation (6.16) allows us to conclude that the $(n+1)$-point alternance of the function $\frac{R_{q,\xi}(t)}{2q\cos t-(1+q^2)}$ on $[0,\pi)$ coincides with the set of zeros of the partial derivative

$$\frac{\partial\mathcal{B}(t,q,\xi)}{\partial t} = \frac{\partial}{\partial t}\cos\big[nt-\lambda(t,q)+\xi\big] = -\left(n+\frac{1-q^2}{1+q^2-2q\cos t}\right)\sin\big[nt-\lambda(t,q)+\xi\big]$$

$$= \left(n+\frac{1-q^2}{1+q^2-2q\cos t}\right)\cos\big[nt-\lambda(t,q)+\xi+\pi/2\big] = \left(n+\frac{1-q^2}{1+q^2-2q\cos t}\right)\mathcal{B}\left(t,q,\xi+\pi/2\right).$$

From this, (6.15), and (6.18), we obtain

$$\frac{\partial\mathcal{B}(t,q,\xi)}{\partial t} = \left(n+\frac{1-q^2}{1+q^2-2q\cos t}\right)\mathcal{B}\left(t,q,\xi+\pi/2\right)$$

$$= -\left(n+\frac{1-q^2}{1+q^2-2q\cos t}\right)\frac{\sin[(n+1)t+\xi]-2q\sin(nt+\xi)+q^2\sin[(n-1)t+\xi]}{2q\cos t-(1+q^2)}.$$

Thus, the following theorem is valid, in which we denote by $\{t_j(q,\xi)\}_{j=1}^{n+1}$ the set of zeros of the function $\mathcal{B}(t) = \mathcal{B}(t,q,\xi)$ on $[0,\pi)$ in ascending order.

Theorem 10. *For $n \in \mathbb{N}$, $q \in (-1,1)$, and $\xi \in \mathbb{R}$, the following assertions are valid*:

(a) *the $(n+1)$-point alternance of the function $\mathcal{B}(t,q,\xi)$ on the half-interval $[0,\pi)$ coincides with the set of zeros of the function $\mathcal{B}\Big(t,q,\xi+\pi/2\Big)$ on $[0,\pi)$; namely,*

$$\mathcal{B}\left(t_j\left(q,\xi+\pi/2\right),q,\xi\right) = \varepsilon_1(-1)^j, \qquad j=1,2,\dots,n+1, \qquad \varepsilon_1 = \pm 1;$$

$$\mathcal{B}\left(t_j(q,\xi),q,\xi+\pi/2\right) = \varepsilon_2(-1)^j, \qquad j=1,2,\dots,n+1, \qquad \varepsilon_2 = \pm 1;$$

(b) *the zeros $t_j(q,\xi)$ of the function $\mathcal{B}(t) = \mathcal{B}(t,q,\xi)$, located in the interval $(0,\pi)$, are decreasing functions of the parameter $q \in (-1,1)$.*

Assertion (b) of Theorem 10 is a consequence of equalities (6.14), from which it is seen that the function $\lambda(t,q)$ decreases with respect to $q \in (-1,1)$ for every fixed $t \in (0,\pi)$. Now, it remains to observe that each zero of the function $\mathcal{B}(t) = \mathcal{B}(t,q,\xi)$ located in the interval $(0,\pi)$ coincides with a root of the equation $nt + \xi - (2k-1)\pi/2 = \lambda(t,q)$, where k is integer.

Meinardus [25, Ch. 1, Sect. 4, Subsect. 4.3] (see also [27, Lemma 3]), using another method, obtained a result containing assertion (b) of Theorem 10 in the case $\xi = -\pi/2$.

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