Bernstein's Polynomial Inequalities and Functional Analysis

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1. Introduction

This expository article shows how classical inequalities for the derivative of polynomials can be proved in real and complex Hilbert spaces using only elementary arguments from functional analysis. As we shall see, there is a surprising interconnection between an equality of norms for symmetric multilinear mappings due to Banach and an inequality for the derivative of trigonometric polynomials due to van der Corput and Schaake. We encounter little extra difficulty in establishing our inequalities in several or infinite dimensions.

After giving the definitions of polynomials and derivatives in normed linear spaces, we establish a lemma of Hörmander, which is an extension of a theorem of Laguerre to complex vector spaces. This powerful lemma is the key to the proofs of the polynomial inequalities we discuss; however, its proof is a simple argument relying only on the fundamental theorem of algebra. Following de Bruijn (who considered only the case of the complex plane), we deduce a theorem which obtains discs inside the range of a complex-valued polynomial on the closed unit ball of a complex Hilbert space. Here the size of the disc is determined by the value of the derivative.

An easy consequence is an extension to complex Hilbert spaces of an estimate of Malik on the derivative of polynomials whose roots lie outside a given disc. (Malik's estimate generalized a conjecture of Erdös that was proved by Lax.) Another consequence is an extension to complex Hilbert spaces of the classical complex form of Bernstein's inequality. Still another consequence is an inequality for the derivative of a polynomial on a complex Hilbert space whose real part has a known bound on the closed unit ball. When the Hilbert space is the complex plane, this inequality contains an inequality of Szegö and leads to an inequality of van der Corput and Schaake for trigonometric polynomials, which is a strengthened form of the Bernstein inequality.

Using methods of van der Corput and Schaake, we deduce an inequality for the derivative of homogeneous polynomials on real Hilbert spaces that extends a result of O. D. Kellogg for $\mathbb{R}^n$. A slight extension of a result of Banach is an immediate consequence. Specifically, the norm of a continuous symmetric multilinear mapping is the same as the norm of the associated homogeneous polynomial on any real Hilbert space. From this, we deduce an estimate on the derivative of polynomials which satisfy an ℓ^2-growth condition on real Hilbert spaces. Finally, we give an argument which shows how to derive the inequality of van der Corput and Schaake for trigonometric polynomials from the two dimensional case of Banach's result.

A source for this approach to polynomial inequalities is [8].

2. Definitions and notation

The reader who wishes may take all vector spaces below to be finite dimensional so that the definition of polynomials is already familiar. To give the general definition, let X and Y be any real or complex normed linear spaces and let $F : X \times \cdots X \to Y$ be a continuous symmetric m-linear mapping with respect to the chosen scalar field, where m is a positive integer. Define $\hat{F}(x) = F(x, \ldots, x)$ for $x \in X$. We say that a mapping $P : X \to Y$ is a *homogeneous polynomial of degree* m if $P = \hat{F}$ for some continuous symmetric m-linear mapping F as above. Define a mapping $P : X \to Y$ to be a *polynomial of degree* $\leq m$ if

$$P = P_0 + P_1 + \cdots + P_m,$$

where $P_k : X \to Y$ is a homogeneous polynomial of degree k for $k = 1, \cdots, m$ and a constant function when $k = 0$. (Note that a constant polynomial is not a homogeneous polynomial by our definition unless it is the zero polynomial.)

This definition of polynomials agrees with the classical definition when $X = \mathbb{R}^n$ and $Y = \mathbb{R}$ and when $X = \mathbb{C}^n$ and $Y = \mathbb{C}$. In either case,

$$P(x_1, \ldots, x_n) = \sum_{k=0}^{m} \sum_{k_1 + \cdots + k_n = k} a_{k_1 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n},$$

where $k_1, \ldots, k_n$ are restricted to the non-negative integers and the coefficients $a_{k_1 \cdots k_n}$ are in the appropriate scalar field, i.e., Y. As expected, with our definitions, if a polynomial P satisfies $P(tx) = t^m P(x)$ for all $x \in X$ and $t \in \mathbb{R}$, then P is a homogeneous polynomial of degree m. When F is as above, for convenience we will write $F(x^j y^k)$ for $F(\underbrace{x, \ldots, x}_{j}, \underbrace{y, \ldots, y}_{k})$. Thus, the binomial theorem for F can be written as

$$\hat{F}(x + y) = \sum_{k=0}^{m} \binom{m}{k} F(x^{m-k} y^k). \tag{1}$$

It is not difficult to show [10, §26.2] that a weaker definition suffices. Specifically, a continuous mapping $P : X \to Y$ is a polynomial of degree $\le m$ if and only if

$$Q(\lambda) = \ell(P(x + \lambda y)), \ \lambda \text{ scalar},$$

is a polynomial of degree $\le m$ (in the classical sense) for every $x, y \in X$ and every $\ell \in Y^*$, where Y^* denotes the space of all continuous linear functionals on Y.

Let $\mathcal{L}(X, Y)$ denote the space of all continuous linear mappings $L : X \to Y$ with the operator norm, i.e., $\|L\| = \sup_{\|x\|\le 1} \|Lx\|$. If P is a mapping of a domain $\mathcal{D}$ in X into Y and if $x \in \mathcal{D}$, we say that an $L \in \mathcal{L}(X, Y)$ is the Fréchet derivative of P at x if

$$\lim_{y\to 0} \frac{\|P(x + y) - P(x) - L(y)\|}{\|y\|} = 0.$$

We denote the Fréchet derivative of P at x by $DP(x)$. Clearly

$$DP(x)y = \frac{d}{dt} P(x + ty)\Big|_{t=0} \tag{2}$$

when $DP(x)$ exists. If P is a polynomial of degree $\le m$, then $DP(x)$ exists for all $x \in X$ and $x \to DP(x)$ is a polynomial mapping of X into $\mathcal{L}(X, Y)$ of degree $\le m-1$. Indeed, it suffices to show this for homogeneous polynomials $\hat{F}$ of degree m and here the Fréchet differentiability of $\hat{F}$ follows easily from (1) with

$$D\hat{F}(x)y = mF(x^{m-1}y) \tag{3}$$

For example, if $P(x) = (P_1(x), \ldots, P_m(x))$ is a polynomial mapping of $\mathbb{R}^n$ into $\mathbb{R}^m$, then the matrix of $DP(x)$ is the $m \times n$ Jacobian matrix $[\partial P_i(x)/\partial x_j]$. The same formula also holds when $\mathbb{R}$ is replaced by $\mathbb{C}$ except that $\partial P_i(x)/\partial x_j$ now denotes a complex derivative. (The proof in both cases follows easily from (2) and the chain rule.) See [10] and [7] for further discussion of polynomials and Fréchet differentiability.

3. Polynomials on complex spaces

The lemma below is the key to the proofs of the polynomial inequalities we wish to give. Let X be a complex normed linear space. Recall that a function $f : X \times X \to \mathbb{C}$ is called a *Hermitian form on* X if $f(x, y)$ is linear in x for each $y \in X$ and $\overline{f(x, y)} = f(y, x)$ for all $x, y \in X$. For example, when $X = \mathbb{C}^n$, the Hermitian forms f on X are given by $f(x, y) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i \overline{y_j}$, where $\overline{a_{ij}} = a_{ji}$ for all $1 \le i, j \le n$.

Lemma 1 (Hörmander [11].) *Put* $A = \{x \in X : \; f(x,x) \geq 0, x \neq 0\}$. *If* $P : X \to \mathbb{C}$ *is a (non-constant) homogeneous polynomial with* $P(x) \neq 0$ *for all* $x \in A$, *then* $DP(x)y \neq 0$ *for all* $x, y \in A$.

Proof. By definition, $P = \hat{F}$ for some continuous symmetric m-linear mapping F on X. If the lemma is false, there exist $x, y \in A$ with $DP(y)x = 0$, so $F(y^{m-1}x) = 0$ by (3). Then by the binomial theorem (1), the coefficient of λ^{m-1} in the polynomial $\lambda \to P(x + \lambda y)$ is 0. By the fundamental theorem of algebra,

$$P(x + \lambda y) = c(\lambda - \lambda_1) \cdots (\lambda - \lambda_m),$$

where $c \neq 0$, and hence $\sum_{k=1}^{m} \lambda_k = 0$. None of the roots λ_k is 0 since $P(x) \neq 0$. Then $x + \lambda_k y \neq 0$ since otherwise $y = \alpha x$, where $\alpha = -1/\lambda_k$, and this gives

$$P(y) = F(y^{m-1}(\alpha x)) = \alpha F(y^{m-1}x) = 0.$$

Hence by hypothesis, $f(x+\lambda_k y, x+\lambda_k y) < 0$ since $P(x+\lambda_k y) = 0$. This inequality expands to

$$f(x,x) + 2 \operatorname{Re} \lambda_k f(y,x) + |\lambda_k|^2 f(y,y) < 0,$$

so $\operatorname{Re} \lambda_k f(y,x) < 0$. Therefore,

$$0 = \operatorname{Re} \left[\left(\sum_{k=1}^{m} \lambda_k\right) f(y,x) \right] = \sum_{k=1}^{m} \operatorname{Re} \lambda_k f(y,x) < 0,$$

the desired contradiction. □

Note that the above lemma holds for any complex vector space and without any continuity assumptions if formula (3) is taken as a definition. See [15] for further discussion of Hörmander's results and related references.

We now apply the above lemma to obtain an extension of a theorem of de Bruijn [5], who considered the case $X = \mathbb{C}$ and deduced the Erdös-Lax theorem [13]. We carry his argument further to obtain an extension (Corollary 4 below) of Malik's generalization [14] of the Erdös-Lax theorem. We also deduce an extension (Corollary 3 below) of an inequality of Szegö [18], which we apply to trigonometric polynomials in the next section.

Theorem 2 *Let* X *be a complex Hilbert space and let* $P : X \to \mathbb{C}$ *be a polynomial of degree* $\leq m$. *Define* $S(x) = mP(x) - DP(x)x$ *for* $x \in X$ *and let* $X_1 = \{x \in X : \; \|x\| \leq 1\}$. *Then*

$$DP(x)y + S(x) \in mP(X_1)$$

for all $x, y \in X_1$.

Corollary 3 *If* $|\mathrm{Re}\, P(x)| \le 1$ *for all* $x \in X_1$*, then*

$$|DP(x)y| + |\mathrm{Re}\, S(x)| \le m$$

for all $x, y \in X_1$*.*

Corollary 4 *Let* $r \ge 1$*. If* $|P(x)| \le 1$ *for all* $x \in X_1$ *and if* P *has no zeros in the closed ball in* X *about 0 with radius* r*, then* $||DP(x)|| \le m/(1+r)$ *for all* $x \in X_1$*.*

Proofs. Our approach to the proof of Theorem 2 is to add an additional dimension to X and use the extra variable to make P into a homogeneous polynomial. Let $X' = X \times \mathbb{C}$ and write the elements of X' as ordered pairs $\langle x, \lambda \rangle$. Define a Hermitian form f on X by $f(\langle x, \lambda \rangle, \langle y, \mu \rangle) = \lambda \overline{\mu} - (x, y)$ and note that

$$A = \{\langle x, \lambda \rangle \in X' :\ \|x\| \le |\lambda|,\ \lambda \ne 0\}.$$

Suppose $\alpha \in \mathbb{C}$ with $\alpha \notin P(X_1)$. Define

$$Q(\langle x, \lambda \rangle) = \lambda^m [\alpha - P(x/\lambda)]$$

for $\lambda \ne 0$ and note that Q extends to all of X'. Then $Q : X' \to \mathbb{C}$ is a homogeneous polynomial of degree m (by the equivalent weaker definition mentioned in the introduction) and $Q(\langle x, \lambda \rangle) \ne 0$ for all $\langle x, \lambda \rangle \in A$. Hence by Lemma 1, $DQ(\langle x, 1 \rangle)\langle y, 1 \rangle \ne 0$ for all $x, y \in X_1$. Now by (2) and the rules of differentiation,

$$\begin{aligned} DQ(\langle x, 1 \rangle)\langle y, 1 \rangle &= \frac{d}{dt} Q(\langle x + ty, 1 + t \rangle)\Big|_{t=0} \\ &= \frac{d}{dt}(1+t)^m \left[\alpha - P\left(\frac{x + ty}{1 + t}\right)\right]\Bigg|_{t=0} \\ &= m\alpha - [DP(x)y + S(x)]. \end{aligned}$$

Thus $DP(x)y + S(x) \ne m\alpha$, which proves Theorem 2.

To deduce Corollary 3, observe that $|\mathrm{Re}\,[DP(x)y + S(x)]| \le m$ for all $x, y \in X_1$ by Theorem 2. Here y can be replaced by λy where λ is a complex number with $|\lambda| = 1$ and λ can be chosen so that the left-hand side of the above inequality is the required expression.

To prove Corollary 4, note that by Theorem 2, for each $x, y \in X_1$, the closed (possibly degenerate) disc with center $S(x)$ and radius $|DP(x)y|$ is contained in the closed disc about 0 with radius m. Hence

$$|DP(x)y| + |S(x)| \le m. \tag{4}$$

Define $P_r(x) = P(rx)$ and put $S_r(x) = mP_r(x) - DP_r(x)x$. By hypothesis and Theorem 2, for each $x, y \in X_1$, the closed disc with center $S_r(x)$ and radius $|DP_r(x)y|$ does not contain 0 so $|DP_r(x)y| \leq |S_r(x)|$. Since $DP_r(x)y = rDP(rx)y$, $S_r(x) = S(rx)$ and $r \geq 1$, it follows that

$$r|DP(x)y| \leq |S(x)| \tag{5}$$

for all $x, y \in X_1$. Combining (4) and (5), we have $(1 + r)|DP(x)y| \leq m$, which gives Corollary 4. □

The extended Erdös-Lax theorem is the case $r = 1$ of Corollary 4. Note that this follows immediately from Theorem 2 since for $x, y \in X_1$, the closed disc Δ with center $S(x)$ and radius $|DP(x)y|$ is contained in the closed disc about 0 with radius m but Δ does not contain the point 0. The largest possible diameter of Δ is m and hence $|DP(x)y| \leq m/2$.

To state our extensions of Bernstein's theorem, define the norm of a polynomial $P : X \to Y$ by

$$\|P\| = \sup\{\|P(x)\| : \; x \in X_1\} \tag{6}$$

and define the norm of a continuous m-linear mapping $F : X \times \cdots X \to Y$ by

$$\|F\| = \sup\{\|F(x_1, \ldots, x_m)\| : \; x_1, \ldots, x_m \in X_1\}.$$

Obviously, $\|\hat{F}\| \leq \|F\|$. (If X is a complex normed linear space, by the maximum principle [10, Th. 3.18.4], the value of $\|P\|$ does not change when the supremum in (6) is taken over only the unit vectors in X.) Suppose Y is any complex normed linear space.

Theorem 5 *If X is a complex Hilbert space and if $P : X \to Y$ is a polynomial of degree $\leq m$, then $\|DP\| \leq m\|P\|$.*

Corollary 6 *If X is a complex Hilbert space and if $F : X \times \cdots \times X \to Y$ is a continuous symmetric m-linear mapping, then $\|F\| = \|\hat{F}\|$.*

The above corollary will be generalized later (Theorem 9) to real Hilbert spaces. See [9] for inequalities between $\|F\|$ and $\|\hat{F}\|$ for other spaces.

Proofs. Without loss of generality, we may assume that $\|P\| = 1$. We first apply linear functionals to reduce to the case where $Y = \mathbb{C}$. Specifically, let $\ell \in Y^*$ with $\|\ell\| = 1$ and define $Q(x) = \ell(P(x))$ for $x \in X$. Then $Q : X \to \mathbb{C}$ is a polynomial of degree $\leq m$ satisfying $|Q(x)| \leq 1$ for all $x \in X_1$ and $DQ(x)y = \ell(DP(x)y)$. Let $x, y \in X_1$. Then $|\ell(DP(x)y)| \leq m$ by Corollary 3 and by the

Hahn-Banach theorem [10, Th. 2.7.4], we may choose ℓ so that $\ell(DP(x)y) = \|DP(x)y\|$. Hence $\|DP(x)y\| \le m$ for all $x, y \in X_1$ and Theorem 5 follows.

We deduce Corollary 6 from the above theorem by induction. The equality is obviously true when $m = 1$. Suppose it is true for $m - 1$. Then holding x_m fixed, we have that

$$\|F(x_1, \dots, x_m)\| \le \sup\{\|F(x^{m-1}x_m)\| : \ x \in X_1\}$$

for all $x_1, \dots, x_{m-1} \in X_1$. Since $D\hat{F}(x)x_m = mF(x^{m-1}x_m)$, by Theorem 5, $\|F(x^{m-1}x_m)\| \le \|\hat{F}\|$ for all $x, x_m \in X_1$. Therefore, $\|F\| \le \|\hat{F}\|$, as required. □

4. Polynomials on real spaces

The results of this section depend on an inequality for trigonometric polynomials which we will deduce from our previous results. By definition, a *trigonometric polynomial* $T(\theta)$ *of degree* $\le m$ is given by

$$T(\theta) = \sum_{k=0}^{m} (a_k \cos k\theta + b_k \sin k\theta), \tag{7}$$

where the coefficients $a_0, \dots, a_m$ and $b_0, \dots, b_m$ are complex numbers. If all the coefficients are real numbers, we say that $T(\theta)$ is a *real trigonometric polynomial*. It is not difficult to show using the addition formulae for the sine and cosine functions that the product of a trigonometric polynomial of degree $\le m$ with a trigonometric polynomial of degree $\le n$ is a trigonometric polynomial of degree $\le m + n$. Hence any sum

$$\sum_{j+k \le m} c_{jk} \cos^j \theta \sin^k \theta, \tag{8}$$

where j and k are non-negative integers and each c_{jk} is a real number, is a real trigonometric polynomial of degree $\le m$.

Theorem 7 (van der Corput and Schaake [6]). *If* $T(\theta)$ *is a real trigonometric polynomial of degree* $\le m$ *satisfying* $|T(\theta)| \le 1$ *for all real* θ*, then*

$$T'(\theta)^2 + m^2 T(\theta)^2 \le m^2 \tag{9}$$

for all real θ*.*

Corollary 8 (Bernstein [2, p. 39].) *If* $T(\theta)$ *is a trigonometric polynomial of degree* $\le m$ *satisfying* $|T(\theta)| \le 1$ *for all real* θ*, then* $|T'(\theta)| \le m$ *for all real* θ*.*

Note that (9) holds with equality for all real θ when $T(\theta) = \cos m\theta$ and when $T(\theta) = \sin m\theta$. Bernstein's original statement of Corollary 8 had the bound of $2m$ in place of m. (See [17, p. 569] for a discussion of priorities.)

Proofs. Our method of proof is to express $T(\theta)$ as the real part of a polynomial on the unit circle and apply Corollary 3 in the case $X = \mathbb{C}$. Let T be given by (7) and define the conjugate $\tilde{T}$ of T by

$$\tilde{T}(\theta) = \sum_{k=0}^{m}(-b_k \cos k\theta + a_k \sin k\theta).$$

Define a polynomial $P : \mathbb{C} \to \mathbb{C}$ by $P(z) = \sum_{k=0}^{m} c_k z^k$, where $c_k = a_k - ib_k$ for $k = 0, \ldots, m$. Then

$$P(e^{i\theta}) = T(\theta) + i\tilde{T}(\theta) \tag{10}$$

for all real θ. By hypothesis and the maximum principle for harmonic functions, $|\mathrm{Re}\, P(z)| \leq 1$ for all $|z| \leq 1$ and hence $|P'(z)| + |\mathrm{Re}\, S(z)| \leq m$ for all $|z| \leq 1$ by Corollary 3.

Let $z = e^{i\theta}$. Differentiating (10) with respect to θ, we see that $izP'(z) = T'(\theta) + i\tilde{T}'(\theta)$ and hence $\mathrm{Re}\, S(z) = mT(\theta) - \tilde{T}'(\theta)$. Now if t_1 and t_2 are any real numbers with $t_1^2 + t_2^2 = 1$, then by the Cauchy-Schwarz inequality,

$$\begin{aligned} |T'(\theta)t_1 + mT(\theta)t_2| &= |T'(\theta)t_1 + \tilde{T}'(\theta)t_2 + \mathrm{Re}\, S(z)t_2| \\ &\leq |T'(\theta)t_1 + \tilde{T}'(\theta)t_2| + |\mathrm{Re}\, S(z)| \\ &\leq \sqrt{T'(\theta)^2 + \tilde{T}'(\theta)^2} + |\mathrm{Re}\, S(z)| \\ &= |P'(z)| + |\mathrm{Re}\, S(z)| \leq m. \end{aligned}$$

The maximum of the left-hand side of the above is $r = \sqrt{T'(\theta)^2 + m^2T(\theta)^2}$ and it is attained when $t_1 = T'(\theta)/r$ and $t_2 = mT(\theta)/r$ if $r \neq 0$. Thus (9) holds.

One can deduce Corollary 8 easily by letting λ be a complex number with $|\lambda| = 1$ and applying Theorem 7 to $S(\theta) = \mathrm{Re}\,[\lambda T(\theta)]$. □

Let Y be any real normed linear space.

Theorem 9 (Banach [1].) *If X is a real Hilbert space and if $F : X \times \cdots \times X \to Y$ is a continuous symmetric m-linear mapping, then $\|F\| = \|\hat{F}\|$.*

Lemma 10 *If $P : X \to Y$ is a homogeneous polynomial of degree m, then $\|DP\| \leq m\|P\|$.*

Note that Lemma 10 is an analogue of Theorem 5 for the case of real scalars. It was proved for the case $X = \mathbb{R}^n$ by Kellogg [12]. See [3, p. 62] for a direct proof of Theorem 9 using only Hilbert space techniques.

Proofs. To prove the lemma, we may suppose that $\|P\| = 1$. As in the proof of Theorem 5, we may apply linear functionals to reduce to the case $Y = \mathbb{R}$. Let x and y be unit vectors in X and let $\{x, w\}$ be an orthonormal basis for the space spanned by x and y. Then $y = t_1 x + t_2 w$, where $t_1^2 + t_2^2 = 1$. Define

$$T(\theta) = P((\cos\theta)x + (\sin\theta)w)$$

and note that $T(\theta)$ is a real trigonometric polynomial of degree $\leq m$ since it is of the form (8) by the binomial theorem (1). Clearly $T(0) = P(x)$ and $T'(0) = DP(x)w$. By (3), we have $DP(x)x = mP(x)$, and hence

$$DP(x)y = t_1 DP(x)x + t_2 DP(x)w = t_1 mT(0) + t_2 T'(0).$$

Since $\|P\| = 1$, it follows that $|T(\theta)| \leq 1$ for all real θ and hence $|DP(x)y)| \leq m$ by the Cauchy-Schwarz inequality and (9). In fact, this inequality holds for all $x, y \in X_1$ since these vectors can be written as scalar multiples of unit vectors.

Theorem 9 follows easily from the above lemma by induction as in the proof of Corollary 6. □

The case $X = \mathbb{R}$ of our next theorem is a sharpening given in [6] of a theorem of Bernstein.

Theorem 11 *If X is a real Hilbert space and if $P : X \to \mathbb{R}$ is a polynomial of degree $\leq m$ satisfying*

$$|P(x)|^2 \leq (1 + \|x\|^2)^m$$

for all $x \in X$, then

$$\|DP(x)\|^2 + S(x)^2 \leq m^2(1 + \|x\|^2)^{m-1}$$

for all $x \in X$.

Proof. Our approach is similar to that of the proof of Theorem 2. Let $X' = X \times \mathbb{R}$ and note that X' is a real Hilbert space in the norm $\|\langle x, t\rangle\| = (\|x\|^2 + t^2)^{1/2}$. Define a homogeneous polynomial $Q : X' \to \mathbb{R}$ of degree m by $Q(\langle x, t\rangle) = t^m P(x/t)$ for $t \neq 0$. By hypothesis,

$$|Q(\langle x, t\rangle)|^2 \leq |t|^{2m} \left(1 + \left\|\frac{x}{t}\right\|^2\right)^m = \|\langle x, t\rangle\|^{2m}$$

so $\|Q\| \le 1$. By a differentiation as in the proof of Theorem 2,

$$DQ(\langle x,1\rangle)\langle y,t\rangle = DP(x)y + tS(x).$$

Hence by Lemma 10,

$$|DP(x)y + tS(x)| \le m\|\langle x,1\rangle\|^{m-1}\|\langle y,t\rangle\|.$$

By replacing y by ry in the above, where $y \in X_1$, and maximizing the left-hand side over all r and t satisfying $r^2 + t^2 = 1$, we obtain

$$\sqrt{[DP(x)y]^2 + S(x)^2} \le m(1 + \|x\|^2)^{\frac{m-1}{2}}$$

for all $y \in X_1$, and Theorem 11 follows. □

We will show that Theorem 7 can be derived from the preceding theorem using only the case $X = \mathbb{R}$. Thus any of the results of this section can be derived from any of the others (except Corollary 8) by arguments given here.

Suppose $T(\theta)$ is a real trigonometric polynomial of degree $\le m$ satisfying $|T(\theta)| \le 1$ for all real θ. It suffices to prove (9) for the case $\theta = 0$ since this case can be applied to the trigonometric polynomial $S(\phi) = T(\theta + \phi)$ for fixed θ. Let P be the polynomial defined in the proof of Theorem 7 and define

$$Q(t) = (1 + t^2)^m \operatorname{Re} P(z(t)),$$

where $z(t) = (1 + it)^2/(1 + t^2)$. Then $Q(t)$ is a polynomial of degree $\le 2m$ on $\mathbb{R}$. If $t = \tan\theta$, then $z(t) = e^{2i\theta}$ so $Q(t) = (1 + t^2)^m T(2\theta)$ by (10). Hence, $|Q(t)| \le (1 + t^2)^m$ for all $t \in \mathbb{R}$ and therefore

$$Q'(0)^2 + [2mQ(0)]^2 \le (2m)^2$$

by Theorem 11. Clearly $Q(0) = T(0)$ and by differentiating $Q(\tan\theta)$ at $\theta = 0$, we obtain $Q'(0) = 2T'(0)$. Thus (9) holds at $\theta = 0$, proving Theorem 7.

Bernstein theorems for arbitrary normed linear spaces are given in [8] and [16]. In fact, an elementary argument is given in [16] to show that Markov's theorem for the first derivative holds in any normed linear space. For a discussion of connections between Bernstein's inequality for entire functions and functional analysis, see [4].

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