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ON A THEOREM OF GRACE

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In this paper we give a generalization of a classical theorem of Grace (Grace [5], compare also Szegö [10] and Pólya–Szegö [8, Fünfter Abschnitt, 2. Kapitel]). We generalize it in two directions: where Grace considers polynomials in one complex variable, we consider polynomials in several variables ranging over an arbitrary algebraically closed field of characteristic 0. At the end of the paper we confine ourselves to the case treated by Grace and show how some known inequalities can be proved simply by means of Grace's theorem. That section is entirely elementary and can be read independently of the first one by those familiar with Grace's theorem.

We shall not suppose anything known about previous work on Grace's theorem, but the proofs, though simplified by our invariant approach, are strongly influenced by those of Szegö [10].

The author's study of the topic of this paper originated from a new formulation of Grace's theorem (Theorem 3′, p. 62, for $(x+b)^n \equiv 1$). This formulation, and also the abstract characterization of Lorentz signature used here (Conditions 1° and 2°, p. 58), are due to Professor Marcel Riesz. The author wishes to express his gratitude to Professor Riesz for communicating to him these results as well as for pointing out to him the usefulness of the symbolic notation applied in the last section.

Preliminaries. Let E be a vector space over a field K of characteristic 0.

Definition. *A function $P(x)$ defined in E with values in K is called an (abstract) homogeneous polynomial of degree n if, for any $x, y \in E$, $P(sx+ty)$ is a homogeneous polynomial of degree n in $s, t \in K$ in the algebraic sense.*

From the definition it follows that $P(t_1x_1+t_2x_2+\ldots+t_kx_k)$ is also a polynomial (in the algebraic sense) in $t_1, t_2, \ldots, t_k \in K$ for arbitrary fixed $x_1, x_2, \ldots, x_k \in E$. Hence, if the dimension of E is finite, the abstract and the algebraic definitions of a polynomial coincide so that the distinction between these two concepts can be dropped.

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LEMMA 1. *To each homogeneous polynomial $P(x)$ of degree n in E there exists one and only one function $P(x_1, x_2, \ldots, x_n)$ with values in K which is defined for $x_i \in E$ and has the properties*:

a) *$P(x_1, x_2, \ldots, x_n)$ is linear in x_i if the other arguments are kept fixed, $i = 1, 2, \ldots, n$,*
b) *$P(x_1, x_2, \ldots, x_n)$ is symmetric in its arguments,*
c) *$P(x, x, \ldots, x) = P(x)$.*

PROOF. In order to prove that $P(x_1, x_2, \ldots, x_n)$ is uniquely determined, we have to show that, if $P(x_1, x_2, \ldots, x_n)$ satisfies a) and b) and if $P(x, x, \ldots, x) \equiv 0$, then $P(x_1, x_2, \ldots, x_n) \equiv 0$. But this is obvious since according to a) and b) the coefficient of $t_1 t_2 \ldots t_n$ in the identically vanishing polynomial $P(t_1 x_1 + \ldots + t_n x_n, \ldots, t_1 x_1 + \ldots + t_n x_n)$ equals $n! P(x_1, x_2, \ldots, x_n)$.

To prove the existence we set $P(x_1, x_2, \ldots, x_n)$ equal to $1/n!$ times the coefficient of $t_1 t_2 \ldots t_n$ in the expansion of $P(t_1 x_1 + t_2 x_2 + \ldots + t_n x_n)$ as a polynomial in $t_i \in K$. The verification of a), b) and c) is then easy and may be omitted.

DEFINITION. *The function $P(x_1, x_2, \ldots, x_n)$ defined by Lemma 1 is called the (n^{th}) polar form of $P(x)$.*

In case E is the finite dimensional vector space K^N of all N-tuples $(x', x'', \ldots, x^{(N)})$ where $x^{(\nu)} \in K$, this definition agrees with the usual one involving derivatives. For in this case we have the representation

$$P(x_1, x_2, \ldots, x_n) = \frac{1}{n!}\left(\prod_{k=1}^{n} \sum_{\nu=1}^{N} x_k^{(\nu)} \frac{\partial}{\partial x^{(\nu)}}\right) P(x) .$$

In fact, the right-hand side is independent of x and obviously satisfies a) and b) above, since the differentiations commute with each other. Finally, c) is a consequence of Euler's identity for homogeneous polynomials.

Beside our assumption that K is of characteristic 0 we shall suppose from now on that K is algebraically closed. As is well known (cf. [3], [11]), our hypotheses imply the existence of a maximal ordered field $K_0 \subset K$ such that $K = K_0(i)$, where $-i^2$ is the unity element of K. (Our terminology is that of Bourbaki [3]; van der Waerden [11, §§ 70–71] calls such a field "reell-abgeschlossen".) Generally K_0 is not uniquely determined. In the sequel K_0 will always denote a fixed field with these properties and the signs $>$ and $\geqq$ will always refer to the order relation in K_0.

Having fixed K_0 we have a conjugation defined in K which we shall denote in the usual manner. The definition of a hermitian symmetric form in a complex vector space can thus be extended to forms in a vector space E over K.

DEFINITION. *A function $H(x, y)$ defined for $x, y \in E$ with values in K is called a hermitian symmetric form if $H(x, y)$ is linear in x for every fixed y, and $H(x, y) = \overline{H(y, x)}$.*

We remark that for positive definite or semi-definite forms the inequality of Schwarz, $|H(x, y)|^2 \leqq H(x, x)H(y, y)$, is still valid.

A generalization of Grace's theorem. We first give a lemma, which generalizes a theorem of Laguerre [6, p. 48–50], usually employed in the proof of Grace's theorem.

LEMMA 2. *Let $P(x)$ be a homogeneous polynomial of degree n and $H(x, y)$ a hermitian symmetric form defined in E. Suppose that $P(x) \neq 0$ for every $x \neq 0$ such that $H(x, x) \geqq 0$. Then it follows that $P(x_1, x, \ldots, x)$, the first polar, cannot vanish if $x \neq 0$, $x_1 \neq 0$, and $H(x, x) \geqq 0$, $H(x_1, x_1) \geqq 0$.*

PROOF. $P(sx + tx_1)$ is a homogeneous polynomial in s and t. Since K is algebraically closed, we can write it as the product of linear factors

$$P(sx + tx_1) = \prod_{i=1}^{n} (s\tau_i - t\sigma_i) .$$

Identification of the coefficients of $s^{n-1}t$ on both sides of this identity gives $nP(x_1, x, \ldots, x) = \sum_1^n \tau_1 \ldots \tau_{i-1}(-\sigma_i)\tau_{i+1} \ldots \tau_n$, and since $P(x) = P(x, x, \ldots, x) = \tau_1 \ldots \tau_n$, we obtain

$$nP(x_1, x, \ldots, x)/P(x) = -\sum_{i=1}^{n} \sigma_i/\tau_i .$$

To prove that this sum does not vanish, we first observe that, since P vanishes at $\sigma_i x + \tau_i x_1$, we must have

$$\begin{aligned} &H(\sigma_i x + \tau_i x_1, \sigma_i x + \tau_i x_1) \\ &\quad = \sigma_i \bar{\sigma}_i H(x, x) + 2\operatorname{Re}\big(\sigma_i \bar{\tau}_i H(x, x_1)\big) + \tau_i \bar{\tau}_i H(x_1, x_1) < 0 . \end{aligned}$$

From the assumptions of the lemma it follows that the first and last terms are non-negative so that we must have $\operatorname{Re}\big(\sigma_i \bar{\tau}_i H(x, x_1)\big) < 0$. After dividing this inequality by $\tau_i \bar{\tau}_i$, which is positive, we obtain $\operatorname{Re}\big(H(x, x_1)\sigma_i/\tau_i\big) < 0$. If we add these inequalities for $i = 1, \ldots, n$, we find that $\operatorname{Re}\big(H(x, x_1) \sum_1^n \sigma_i/\tau_i\big) < 0$; and this inequality immediately implies that $\sum_1^n \sigma_i/\tau_i \neq 0$.

It is now very easy to prove a generalization of Grace's theorem:

THEOREM 1. *Let $P(x)$ be a homogeneous polynomial of degree n and $H(x, y)$ a hermitian symmetric form defined in a vector space E over K. Suppose that $P(x) \neq 0$ for every $x \neq 0$ such that $H(x, x) \geqq 0$. Then it follows that the polar $P(x_1, x_2, \ldots, x_n) \neq 0$ if $x_i \neq 0$ and $H(x_i, x_i) \geqq 0$ for $i = 1, 2, \ldots, n$.*

PROOF. From the lemma it follows that $P(x_1, x, \ldots, x) \neq 0$ for every $x \neq 0$ such that $H(x, x) \geqq 0$. Hence the lemma can be applied again, and after $n-1$ applications of it the theorem is proved.

We observe, that Lemma 2, and consequently Theorem 1, would also hold if $\geqq$ is replaced by $>$ everywhere in the statements.

Theorem 1 has been proved for an arbitrary $H(x, y)$, but it is void except for certain types of hermitian forms. For in the case of a general $H(x, y)$ there may be no polynomial $P(x)$ such that $P(x) \neq 0$ when $H(x, x) \geqq 0$, $x \neq 0$. The reason is that every homogeneous polynomial $P(x)$ vanishes for some $x \neq 0$ in any two-dimensional subspace. For if we introduce a basis in the subspace, $P(x)$ becomes a homogeneous binary polynomial in the coordinates; and since K is algebraically closed, such a polynomial has non-trivial zeros. Hence, in order that the theorem should not be void, we have to assume that

1° *there does not exist any two-dimensional subspace where $H(x, x) \geqq 0$,*

and furthermore in order to exclude uninteresting cases that

2° *there exists an x with $H(x, x) > 0$.*

For otherwise $H(x, x) \leqq 0$ for all x, and the set where $H(x, x) \geqq 0$ reduces to the set where $H(x, x) = 0$ and is thus a linear manifold since $H(x, x)$ is semi-definite. If this manifold has a dimension $\geqq 2$, it follows from 1° that the theorem is void, and, if it has dimension one, all its elements are proportional and the theorem is obvious.

Now suppose that the conditions 1° and 2° are satisfied. Take a fixed element x_0 such that $H(x_0, x_0) = 1$. Then we have the decomposition $x = tx_0 + y$, where $t = H(x, x_0)$ and $y = x - x_0 H(x, x_0)$ so that $H(y, x_0) = 0$. Hence if we put $F = \{y \mid H(y, x_0) = 0\}$, it follows that every element $x \in E$ can be written uniquely in the form $x = tx_0 + y$ with $t \in K$ and $y \in F$. $H(y, y)$ is negative definite in F. For if $0 \neq y \in F$ and $H(y, y) \geqq 0$, it would follow from the formula

$$H(tx_0 + sy, tx_0 + sy) = t\bar{t} + s\bar{s} H(y, y)\ ,$$

that $H(x, x) \geqq 0$ in the two-dimensional subspace spanned by x_0 and y, which contradicts 1°. Hence we have $H(x, x) = t\bar{t} + H(y, y)$ where $H(y, y)$, $y \in F$, is a hermitian symmetric, negative definite form defined in F. In analogy to the terminology in finite dimensional real spaces we shall say that $H(x, x)$ has Lorentz signature.

Conversely, Lorentz signature is sufficient in order that there should exist a polynomial $P(x) \neq 0$ for $H(x, x) \geqq 0$, $x \neq 0$. In fact, an example is $P(x) = t^n$. Hence:

The assumptions of Theorem 1 *can be satisfied if and only if* $H(x, y)$ *has Lorentz signature.*

We shall now give Theorem 1 a more general formulation, which is convenient in the applications. For that purpose we shall study polynomials defined in E with values in another vector space G over K. On page 55 we have only defined polynomials with values in K, but both the definition and Lemma 1 can be extended immediately to this case.

To be able to formulate a concise theorem, we need the following concept.

DEFINITION. *A set M in a vector space G over K shall be called supportable if for every $\xi \in G - M$ there exists a hyperplane through ξ and the origin which does not intersect M.*

By a hyperplane we mean a linear manifold of co-dimension 1. Hence if L is a linear form defined in G the manifold $\{\xi \mid L(\xi) = 0\}$ is a hyperplane in G. Conversely, every hyperplane in G through the origin is defined in this manner by a linear form.

THEOREM 2. *Consider a homogeneous polynomial $P(x)$ which is defined in a vector space E over K and has its values in another vector space G over K. Let $H(x, y)$ be a hermitian symmetric form in E and let M be a supportable set in G such that $P(x) \in M$ for every $x \neq 0$ such that $H(x, x) \geqq 0$. Then it follows that the polar $P(x_1, x_2, \ldots, x_n) \in M$ if $x_i \neq 0$, $H(x_i, x_i) \geqq 0$, $i = 1, 2, \ldots, n$.*

PROOF. If $\xi \notin M$, there exists a linear form L in G such that $L(\xi) = 0$ but $L(P(x)) \neq 0$ for every $x \neq 0$ satisfying $H(x, x) \geqq 0$. Hence, according to Theorem 1, we have $L(P(x_1, x_2, \ldots, x_n)) \neq 0$ if $x_i \neq 0$, $H(x_i, x_i) \geqq 0$, from which it follows that $P(x_1, x_2, \ldots, x_n) \neq \xi$.

Theorem 2 is the most general theorem in this paper. The following theorems are all special cases of it and therefore we do not always formulate them as generally as possible.

The next theorem is essentially the special case of Theorem 2 for a two-dimensional G.

THEOREM 3. *Let $P(x)$ and $Q(x)$ be two homogeneous polynomials with values in K, which are defined in a vector space E over K. Suppose that*

$Q(x) \neq 0$ if $x \neq 0$ and $H(x, x) \geqq 0$, where $H(x, y)$ is a hermitian symmetric form defined in E. Then the range of values of $P(x)/Q(x)$ when $x \neq 0$, $H(x, x) \geqq 0$ and the range of values of $P(x_1, x_2, \ldots, x_n)/Q(x_1, x_2, \ldots, x_n)$ when $x_i \neq 0$, $H(x_i, x_i) \geqq 0$, $i = 1, 2, \ldots, n$, are identical.

PROOF. We shall derive the theorem from Theorem 1, which is as easy to use as Theorem 2. We first observe that according to Theorem 1 the polar $Q(x_1, x_2, \ldots, x_n) \neq 0$ for the values of x_i in question so that the quotient $P(x_1, x_2, \ldots, x_n)/Q(x_1, x_2, \ldots, x_n)$ is well defined. Suppose that $P(x)/Q(x) \neq \alpha$, or, equivalently, that $P(x)-\alpha Q(x) \neq 0$, when $x \neq 0$, $H(x, x) \geqq 0$. Then we conclude from Theorem 1 that $P(x_1, x_2, \ldots, x_n)-$ $-\alpha Q(x_1, x_2, \ldots, x_n) \neq 0$, that is, $P(x_1, x_2, \ldots, x_n)/Q(x_1, x_2, \ldots, x_n) \neq \alpha$, when $x_i \neq 0$, $H(x_i, x_i) \geqq 0$. Hence it follows that the range of values of $P(x_1, x_2, \ldots, x_n)/Q(x_1, x_2, \ldots, x_n)$ is contained in that of

$$P(x)/Q(x) = P(x, x, \ldots, x)/Q(x, x, \ldots, x),$$

and, since the converse statement is obvious, the theorem follows.

In the rest of this paper we shall confine ourselves to the case that K is the complex field C. In this case Theorem 1 and Theorem 2 can be improved slightly by means of the following lemma.

LEMMA 3. *Let $H(x, y)$ be a hermitian symmetric form of Lorentz signature, defined in a vector space E over C of dimension > 2. Then a homogeneous complex polynomial $P(x)$ cannot vanish when $x \neq 0$, $H(x, x) \geqq 0$, if it does not vanish when $x \neq 0$, $H(x, x) = 0$.*

PROOF. We first prove the lemma for a three-dimensional space E. If we introduce a basis such that $H(x, x)$ assumes diagonal form, $H(x, x) = |x_0|^2 - |x_1|^2 - |x_2|^2$, and put $x_0 = 1$, the lemma can be given the following form:

If $P(x_1, x_2)$ is an (inhomogeneous) polynomial such that $P(x_1, x_2) \neq 0$ when $x_1, x_2 \in C$, $|x_1|^2+|x_2|^2 = 1$, then

$$P(x_1, x_2) \neq 0 \quad \text{if} \quad |x_1|^2+|x_2|^2 \leqq 1 .$$

This follows immediately from a rather elementary theorem on analytic completion (cf. Bochner and Martin [2, p. 64]). For by continuity $P(x_1, x_2) \neq 0$ when $1-\varepsilon \leqq |x_1|^2+|x_2|^2 \leqq 1$ if ε is small enough. This means that $1/P(x_1, x_2)$ is analytic in this spherical shell. Since the shell has the whole sphere $|x_1|^2+|x_2|^2 \leqq 1$ as analytic completion, $1/P(x_1, x_2)$ is analytic in $|x_1|^2+|x_2|^2 \leqq 1$, which proves the assertion.

The lemma now follows in the general case. Indeed, take a three-dimensional subspace F in E such that F contains an arbitrarily chosen vector x_0 with $H(x_0, x_0) > 0$. Then $H(x, y)$ has Lorentz signature also in F, and $P(x) \neq 0$ when $H(x, x) = 0$, $0 \neq x \in F$. Thus $P(x_0) \neq 0$ according to the three-dimensional case of the lemma.

If we apply Lemma 3 to Theorem 2, we obtain

THEOREM 2′. *Consider a homogeneous polynomial $P(x)$ which is defined in a vector space E over C with dimension > 2 and has its values in another vector space G over C. Let $H(x, y)$ be a hermitian symmetric form of Lorentz signature in E, and let M be a supportable set in G such that $P(x) \in M$ for every $x \neq 0$ satisfying $H(x, x) = 0$. Then it follows that the polar $P(x_1, x_2, \ldots, x_n) \in M$ if $x_i \neq 0$, $H(x_i, x_i) \geqq 0$, $i = 1, 2, \ldots, n$.*

Next we give an important but rather special application of Theorem 2.

THEOREM 4. *Let H be a Hilbert space over C with the scalar product (x, y) and the norm $\|x\| = (x, x)^{\frac{1}{2}}$, and let B be a Banach space over C with a norm denoted by $|\ |$. Then for any homogeneous polynomial $P(x)$ defined in H with values in B we have*

$$\sup_{x \in H} \frac{|P(x)|}{\|x\|^n} = \sup_{x_i \in H} \frac{|P(x_1, x_2, \ldots, x_n)|}{\|x_1\|\,\|x_2\| \ldots \|x_n\|}\,.$$

PROOF. Obviously the left-hand side cannot be greater than the right. Hence we have just to prove that if $|P(x)| \leqq \alpha \|x\|^n$ then also

$$|P(x_1, x_2, \ldots, x_n)| \leqq \alpha \|x_1\|\,\|x_2\| \ldots \|x_n\|\,.$$

Let E be the direct sum of H and C, that is, E is the set of all pairs $[x, t]$ where $x \in H$, $t \in C$. The hermitian symmetric form $t\bar{t} - (x, x)$ defined in E obviously has Lorentz signature. As our space G we take the direct sum of B and C, whose elements are the pairs $[\xi, \tau]$ with $\xi \in B$, $\tau \in C$. We set $M = \{[\xi, \tau] \mid \tau \neq 0, |\xi| \leqq \alpha |\tau|\}$. That M is supportable is exactly the contents of Hahn-Banach's theorem. Consider the polynomial $[P(x), t^n]$ defined in E. We have $[P(x), t^n] \in M$ when $t\bar{t} - (x, x) \geqq 0$, $t \neq 0$; for then $|P(x)| \leqq \alpha |t|^n$ since $\|x\| \leqq |t|$ and, by assumption, $|P(x)| \leqq \alpha \|x\|^n$. From Theorem 2 we can now conclude that $[P(x_1, x_2, \ldots, x_n), t_1 t_2 \ldots t_n] \in M$ if $t_i \neq 0$, $t_i \bar{t}_i - (x_i, x_i) \geqq 0$. But this means that $|P(x_1, x_2, \ldots, x_n)| \leqq \alpha |t_1|\,|t_2| \ldots |t_n|$ if $\|x_i\| \leqq |t_i|$ and hence $|P(x_1, x_2, \ldots, x_n)| \leqq \alpha \|x_1\|\,\|x_2\| \ldots \|x_n\|$, which was to be proved.

Theorem 4 is very weak compared with Theorem 2; in fact, according to Kellogg [7] it is valid also for real spaces, for which Theorem 2 does not hold. We shall return to this theorem later.

Elementary applications of Grace's theorem. In this section we shall consider complex polynomials defined in $E = C^2$, that is, the elements

x of E are the pairs $[x', x'']$ with $x', x'' \in C$. An element $x \neq 0$ in E can then be considered as the set of homogeneous coordinates of a point (also denoted by x) in the one-dimensional projective complex space, that is, the complex plane completed by a single point at infinity. A homogeneous polynomial $P(x)$ of degree n defined in C^2 can be written

$$P(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x'^{n-k} x''^k ,$$

or symbolically $P(x) = (x' + a x'')^n$. This and the following symbolic formulas should thus be interpreted by considering a as an indeterminate, next multiplying all factors together, and then replacing a^k by a_k. The polar of $P(x)$ is given by

$$P(x_1, x_2, \ldots, x_n) = (x_1' + a x_1'')(x_2' + a x_2'') \ldots (x_n' + a x_n'') ,$$

for the right-hand side satisfies the conditions a), b) and c) of Lemma 1.

A *circular region* in the complex plane is the closed interior or exterior of a circle or a closed half-plane, which may degenerate to a point, to the empty set or to the whole plane. It is immediately verified that if $H(x, y)$ is a hermitian symmetric form, then the conditions $x \neq 0$, $H(x, x) = \alpha x' \overline{x'} + \beta x' \overline{x''} + \overline{\beta} x'' \overline{x'} + \gamma x'' \overline{x''} \geqq 0$, α, γ real, mean geometrically that x belongs to a circular region. Conversely, every circular region can be defined in this way by a hermitian form.

Hence, in this special case with $x'' = 1$ Theorem 3 becomes

THEOREM 3′. *Let D be a circular region and $(x+a)^n$ and $(x+b)^n$ two polynomials such that $(x+b)^n \neq 0$ when $x \in D$. Then the range of values of $(x+a)^n/(x+b)^n$ when $x \in D$ and the range of values of*

$$\big((x_1+a) \ldots (x_n+a)\big)/\big((x_1+b) \ldots (x_n+b)\big) \quad .$$

when $x_i \in D$, $i = 1, \ldots, n$, are identical.

Observe that, if D contains the point at infinity, we must consider $x = \infty$ as a zero of $(x+b)^n$ when $b_0 = 0$.

Theorem 3′ could easily be obtained directly by reformulating Grace's original theorem, which is actually the two-dimensional case of Theorem 1.

After these preliminaries we can easily prove some inequalities. The first of them is the improvement by Szegö [9] of Bernstein's inequality for trigonometric polynomials. We refer to that paper for formulations in terms of trigonometric polynomials.

THEOREM 5. *If* $|\mathrm{Re}\,(x+a)^n| \leqq 1$ *when* $|x| \leqq 1$, *then*

$$|(x+a)^{n-1}| + \left|\mathrm{Re}\left(a(x+a)^{n-1}\right)\right| \leqq 1$$

when $|x| \leqq 1$.

PROOF. It follows from Theorem 3′ with $(x+b)^n \equiv 1$ that, if x and x_1 lie in the unit circle, we have

$$1 \geqq \left|\mathrm{Re}\left((x_1+a)(x+a)^{n-1}\right)\right| = \left|\mathrm{Re}\left(a(x+a)^{n-1} + x_1(x+a)^{n-1}\right)\right| .$$

This means that the strip $|\mathrm{Re}\, z| \leqq 1$ contains the circle with centre at $\mathrm{Re}\left(a(x+a)^{n-1}\right)$ and radius $|(x+a)^{n-1}|$. Hence

$$\left|\mathrm{Re}\left(a(x+a)^{n-1}\right) \pm |(x+a)^{n-1}|\right| \leqq 1$$

and the theorem is proved.

The proof of Theorem 5 immediately suggests a generalization.

THEOREM 6. *Let* W *be a set of points in the complex plane such that* $(x+a)^n \in W$ *when* $|x| \leqq 1$. *Then the circle with centre at* $a(x+a)^{n-1}$ *and the radius equal to* $|(x+a)^{n-1}|$ *is contained in* W.

For the case that W is convex this theorem has been given by van der Corput and Schaake [4. p. 350] with a rather difficult proof based on interpolation formulas. A number of consequences of Theorem 6 can also be found in that paper.

THEOREM 7. *Let* $(x+a)^n$ *and* $(x+b)^n$ *be two polynomials such that* $(x+a)^n$ *does not vanish in a circular region* D *with boundary* L. *Then the inequality*

$$|(x+b)^n| \leqq |(x+a)^n|$$

when $x \in L$ *implies that*

$$|(x_1+b)(x_2+b)\,\ldots\,(x_n+b)| \leqq |(x_1+a)(x_2+a)\,\ldots\,(x_n+a)|$$

when $x_1, \ldots, x_n \in D$.

PROOF. From the maximum principle it follows that the inequality $|(x+b)^n| \leqq |(x+a)^n|$ holds for $x \in D$, and hence the theorem follows immediately from Theorem 3′.

Theorem 7 is partly given by Bernstein [1, p. 56]. In particular we can let D be the upper (lower) half-plane and $(x+a)^n$ be any polynomial with all zeros coinciding, that is, $(x+a)^n = (\alpha x+\beta)^n$ with complex α and β. If we observe that $q(x) = |\alpha x+\beta|^2$ is a positive definite or semi-definite quadratic polynomial in the real variable x, and in fact the most general one, we obtain

THEOREM 8. *Let $q(x)$ be a positive definite or semi-definite quadratic polynomial. Then, if $|(x+b)^n| \leqq q(x)^{\frac{1}{2}n}$ for all real x, it follows that*

$$|(x_1+b) \ldots (x_n+b)| \leqq q(x_1)^{\frac{1}{2}} \ldots q(x_n)^{\frac{1}{2}}$$

for arbitrary real x_i.

Theorem 8 is analogous to Theorem 4. In fact, it states that Theorem 4 is valid if H is a real two-dimensional Hilbert space and B is one-dimensional. Using this result we could easily prove, by repeating arguments from the first section, that *Theorem 4 is also valid if H is a real Hilbert space and B a real Banach space of any dimension.* We shall omit, however, the details of the proof of this statement, which is essentially due to Kellogg [7].

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