Arne Beurling in memoriam

by

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Arne Karl-August Beurling was born the third of February 1905 in Gothenburg, Sweden, and died in Princeton, New Jersey, the twentieth of November 1986. He studied at Uppsala University and obtained his Ph.D. in 1933. He was Professor of Mathematics at Uppsala from 1937 to 1954, at which time he resigned to become Permanent Member and Professor at the Institute for Advanced Study in Princeton. While on leave from Uppsala he was Visiting Professor at Harvard University 1948–49.

He was a member of the Royal Swedish Academy of Sciences, the Royal Swedish Society of Sciences, the Finnish Society of Sciences, the Royal Physiographical

Society in Lund, Sweden, the Danish Society of Sciences, and the American Academy of Arts and Sciences. He was also honorary member of the Swedish Mathematical Society.

He was awarded the Swedish Academy of Sciences Prize 1937 and 1946, the Celsius Gold Medal 1961, and the University of Yeshiva Science Award 1963. In his honor a "Beurling Year" was held at the Mittag-Leffler Institute in Stockholm 1976/77.

Arne Beurling was a highly creative matematician whose legacy will influence future mathematicians for many years to come, maybe even for generations. Anybody who was close to him was influenced by his strong personality and by his intense commitment to mathematics. He published very selectively and only when all details were resolved, and a sizable part of his work has never appeared in print. There are plans to publish his collected works in the near future, and they will include much that has not been previously available to the mathematical public. Beurling's personal friends and students will never forget his unquestioning loyalty and boundless generosity. His readiness to share his ideas was unselfish in the extreme.

The work of Arne Beurling falls into three main categories: complex analysis, harmonic analysis, and potential theory. In a characteristic way he transformed all of these areas of mathematics and made them interact with each other. This unity and confluence of original ideas and methods make him unique among analysts of our time.

For convenience we shall nevertheless consider these areas separately, while endeavoring to highlight the ways in which he enriched them all.

1. Complex analysis

Beurling's thesis [1] was published in 1933, but parts of it had been written already in 1929. It was not a mere collection of interesting and important results, but also a whole program for research in function theory in the broadest sense. As such it has been one of the most influential mathematical publications in recent times.

During the 1920's and 1930's T. Carleman and R. Nevanlinna had shown the importance of harmonic majorization of the logarithm of the modulus of a holomorphic function in a plane region D. Let E be a subset of the boundary of D. They considered a harmonic function in D, later denoted by $\omega(z, E, D)$ and known as the harmonic measure at z of E with respect to D. It is defined by having boundary values 1 on E and 0 on the rest of the boundary. If it is known that $\log|f(z)| \leq \omega$ on the whole boundary, then, by the maximum principle, $\log|f(z_0)| \leq \omega(z_0, E, D)$ for every interior point z_0.

Beurling's leading idea was to find new estimates for the harmonic measure by introducing concepts, and problems, which are inherently invariant under conformal mapping. The novelty in his approach was to apply the majorization to entities, mostly of a geometric character, which are not by themselves invariant, but whose extreme values, in one sense or another, possess this property. The method may have been used before, but not in this systematic manner.

A simple example will illustrate the working of this idea. Let D be a region with finite area πR^2. Consider two points z and z_0 in D, and let $\varrho(z, z_0, D)$ be the inner distance between z and z_0 in the sense of the greatest lower bound of the lengths of arcs in D joining z and z_0. Form the ratio $l(z, z_0, D)=\varrho(z, z_0, D)/R$. There is no reason why this should be conformally invariant, but if we consider $\lambda(z, z_0, D)=\sup l(z^*, z_0^*, D^*)$ for all triples (z^*, z_0^*, D^*) obtained by applying the same conformal mapping to z, z_0 and D, then it is quite obvious that $\lambda(z, z_0, D)$ is a conformal invariant. Beurling calls it the *extremal distance* between z and z_0.

In case D is simply connected Beurling goes on to prove the identity $e^{-2G}+e^{-\lambda^2}=1$, where $G=G(z, z_0, D)$ is the Green's function of D with pole z_0. The proof is relatively easy because the conformal invariance makes it possible to replace D by the unit disk. A more difficult and also more important result is the estimate for the harmonic measure expressed through the inequality

$$\omega \leqslant \exp(-\lambda^2+1) \tag{1}$$

where $\lambda=\lambda(z_0, E, D)$ is now the extremal distance between z_0 and the boundary set E. If E is conformally equivalent to an arc on the unit circle there is also an opposite inequality

$$\omega \geqslant \exp(-\lambda^2). \tag{2}$$

These inequalities were strong enough for an independent proof of the Denjoy conjecture concerning the number of asymptotic values of an entire function of finite order.

The idea of extremal distance was a forerunner of the notion of *extremal length*. Originally, Beurling measured lengths and areas only in metrics of the form $\varrho|dz|=|\phi'(z)|\,|dz|$, where ϕ is a conformal mapping. The more general case of arbitrary ϱ was worked out later in collaboration with L. Ahlfors. Beurling's contribution to their joint papers was always substantial. This is particulary true of [26], in which the decisive idea was entirely due to Beurling. It deals with the boundary values of quasiconformal mappings. If $h(x)$ increases for $-\infty<x<\infty$ there is a quasiconformal mapping of the upper half-plane to itself with boundary values $h(x)$ on the real axis, if and only if for some constant $\varrho>1$

$$1/\varrho \leq \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leq \varrho$$

for all x and t.

The impact of extremal length on present day mathematics can hardly be overstated. It is at the basis of quasiconformal mappings, and thereby also of Teichmüller theory and the new theory of dynamical systems. Recently it has become more customary to replace extremal length by its reciprocal under the name of *modulus of curve families*. It has the advantage of carrying over more easily to several dimensions, for instance in connection with capacity and in the theory of quasiregular functions.

Except for the joint papers with Ahlfors, Beurling published less than expected on extremal length. A possible explanation is that he may have been looking in vain for a satisfactory theory of extremal metrics, which still does not exist today.

Beurling's proof of inequality (1) made use of the following fact. If $f(z)$ is holomorphic in the unit disk with Dirichlet integral $<\pi$ and $f(0)=0$, and if $|f(e^{i\theta})|>M$ on E, then

$$\operatorname{mes} E < e^{-M^2+1}. \tag{3}$$

O. Frostman's thesis appeared in 1935 and created a solid basis for potential theory. Beurling realized that the Dirichlet integral and the energy integral are dual norms, and that $\operatorname{mes} E$ in (3) should be replaced by the capacity of E. This was carried out in the influential paper [5], which became the origin of numerous studies of exceptional sets and boundary behavior of holomorphic functions.

Another central theme during the 1930's was quasi-analyticity. In a characteristic way Beurling's treatment of this topic was combined with harmonic analysis and potential theory. He published very little, but some of his results appeared in his Stanford lecture series from 1961 [31]. His collected papers will contain a complete treatment.

The following is a striking result. Let D be simply connected and γ an arc on the boundary of D. Let $f(z)$ be continuous on γ and zero on a subset of positive harmonic measure. Let $\{a_n\}$ and $\{\lambda_n\}$ be increasing sequences tending to ∞. Suppose further that the functions $f_n(z)$ are holomorphic in D and satisfy

(i) $|f_n(z)| \leq e^{\lambda_n}$ in D,

(ii) $|f_n(z)-f(z)| < e^{-a_n}$ on γ.

Under these conditions

$$\sum a_n(\lambda_n^{-1}-\lambda_{n+1}^{-1}) = \infty$$

implies $f(z)=0$ on all of γ.

In two joint papers with P. Malliavin, [32] and [43], a full description is given of the class of bounded functions on the real axis, which on that axis majorize some entire function of given exponential type. This result also solves the problem of determining the maximal length on which a set of exponentials is dense. The depth of these results is quite remarkable and the ideas behind them far-reaching. For instance, some basic ideas in the modern theory of H^1 spaces can be found in this work.

The work on complex analysis includes what may be Beurling's most famous theorem [12]. Let H^2 be the Hilbert space of holomorphic functions in the unit disk which belong to L^2 on $|z|=1$. Let E be a closed subspace, invariant under multiplication by z. Then there exists an *inner function* $\phi(z)$, i.e. $|\phi(z)|<1$, $|\phi(e^{i\theta})|=1$ a.e., such that $f\in E$ if and only if $f=\phi\cdot f_0$ with $f_0\in H^2$. This theorem introduced the notion of inner functions, and will almost certainly be considered a milestone in analysis.

2. Harmonic analysis

Beurling's first paper [3] in harmonic analysis is a beautiful extension of Wiener's proof of the prime number theorem to generalized primes. Let $1<p_1<p_2<\dots$ be the given sequence of "primes" and let $\{p_1^{a_1}\dots p_n^{a_n}\}$ be the "integers". Let $\pi(x)$ and $N(x)$ be the corresponding counting functions. Then $|N(x)-ax|<x(\log x)^{-\gamma}$, $a>0$, implies the prime-number theorem $\pi(x)\sim x/\log x$ if $\gamma>3/2$ but in a sense not if $\gamma<3/2$.

At the Scandinavian Congress in Helsinki 1938 Arne Beurling gave a talk on Fourier transforms of L^1 functions. It was also inspired by the work of N. Wiener and included some estimates of remainder terms. These estimates required new techniques and have become very important, for instance for the asymptotics of eigenvalue distributions.

Historically, the most striking result was the proof and emphasis of the formula

$$\lim \|f^n\|^{1/n} = \sup |\hat{f}(t)|$$

for the spectral radius. Beurling was clearly aware of the generality of this formula—two years before I. M. Gelfand's paper on normed rings. It was formulated so that it applies also to almost periodic functions and weighted algebras of non-quasianalytic type

$$\|f\|_p = \int |f(x)|p(x)\,dx, \quad p(x) \geq 1,$$

$$\int \frac{\log p(x)}{1+x^2}\,dx < \infty.$$

The problem of characterizing the closed ideals in L^1 spaces, and the related problem of "synthesizing" bounded functions $\varphi(x)$ from their spectrum, i.e. to approximate φ in some suitable norm by linear combinations of exponentials from the spectrum of φ remained one of Beurling's main interests. His treatment is highly original, closely tied to his approach to potential theory. If f belongs to an ideal one wishes to approximate it by functions whose Fourier transforms vanish in a neighborhood of the set $\{t | \hat{f}(t)=0\}$. A natural approach is to replace $\hat{f}$ by $T\circ\hat{f}$, where T vanishes in a neigborhood of 0. This led Beurling to introduce *contractions* T, i.e. functions with the property

$$|Tz - Tz'| \leq |z - z'| \tag{4}$$

and to study spaces on which T operates. The difficulty is that for f in L^1, $T\circ\hat{f}$ need not be a Fourier transform of a function in L^1.

When it became clear from examples by L. Schwartz and P. Malliavin that synthesis fails for L^1 algebras, Beurling worked out his beautiful theory in [34], which leads to a positive answer for the norm

$$\|f\|^* = \inf \int f^*(x)\,dx,$$

where the $f^*(x)$ are decreasing and $\geq |f(\pm x)|$. With this choice synthesis will exist because $\|f\|^*$ defines a space where T operates. In a series of papers, [35] to [39], Beurling proved spectral synthesis for various spaces of similar type (see also [14] from 1949). The leading idea was to exhibit positive measures on the dual spaces as solutions of extremal problems.

It should also be noted that all these problems, which Beurling studied throughout the 1930's and 1940's required a clear understanding of the spectrum and Fourier transform of L^∞ functions, for which no classical transform existed. His main tool was the harmonic transform

$$U_\phi(x, y) = \int_{-\infty}^{\infty} e^{-y|t| + itx}\phi(t)\,dt, \quad y > 0.$$

The spectrum was defined as the set on $y=0$ where U_ϕ does not admit harmonic continuation, and the proof that the spectrum was non-empty became a non-trivial theorem (see e.g. [11]). The theory is clearly a fore-runner of distribution theory.

In the context of harmonic analysis one should not forget his and H. Helson's beautiful result [22]: if $\hat{\sigma}(t)=\int_{-\infty}^{\infty} e^{ixt}d\sigma(x)$, $\|\hat{\sigma}\|=\int|d\sigma(x)|$, and $\|\hat{\sigma}^n\|=O(1)$, $|n|\to\infty$, then $\hat{\sigma}(t)=e^{iat+ib}$.

3. Potential theory

We have seen how Beurling's interest in complex analysis led him to the duality between capacitary measures and the Dirichlet integral in [5]. In this paper the representation

$$\int_0^\infty \frac{dt}{t^2}\int_0^{2\pi}|f(\theta+t)-f(\theta)|^2d\theta \tag{5}$$

for the Dirichlet integral of a harmonic function in a disk in terms of its boundary values f, was emphasized. The importance of contractions for spectral synthesis has also been shown. We have here all the ideas which led Beurling to the concept of Dirichlet spaces and a new foundation of potential theory. Let H be an L^2 space of functions u on a topological space. We assume that $D(u, v)$ is a bilinear form which is densely defined and that $0\leq D(Tu, Tu)\leq D(u, u)$ for all contractions T. The closure under the norm D is a Dirichlet space. With these minimal conditions a complete potential theory can be developed. $D(u, u)$ has a representation of the type (5) and a potential U_μ of a measure μ is abstractly defined by $D(U_\mu, v)=\int v\,d\mu$.

Only two short papers [28] and [29] (with J. Deny) were published, but in many later papers and books mathematicians have shown how these axiomatic ideas adapt to abstract situations. In probability and semigroup theory Dirichlet spaces are now basic tools (see e.g. the book by M. Fukushima, Dirichlet forms and Markov processes 1980). The significance for general potential theory can be seen in the Bourbaki seminar 1967 by J.-P. Kahane, where the Beurling–Deny ideas concerning the connection between the maximum principle in potential theory and the theory of negative definite functions are explained.

In the note [35] an even more general potential theory is indicated. The assumptions concerning the contractions are minimal, but they still lead to a positive extremal measure, corresponding to the equilibrium measure in classical potential theory. This result has received scarce attention but should be of great significance.

Beurling was always very interested in applications of mathematics. [27] treats a free boundary problem for the Laplace equation. An unpublished manuscript, to be included in the collected papers, contains an essentially complete existence and uniqueness proof for doubly connected regions.

As further examples of Beurling's wide interest it might be recorded that C.-G. Esseen, who obtained the optimal remainder estimate in the central limit theorem (Acta Math. 1945) and G. Borg, who proved the first inverse spectral theorems (Acta Math. 1946), were among his outstanding students in Uppsala.

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