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In memoriam: Thomas H. Wolff

FINITE TOEPLITZ MATRICES AND SHARP LITTLEWOOD CONJECTURES

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Abstract. The sharp Littlewood conjecture states that for fixed $N \geq 1$, if $D(z) = 1 + z + z^2 + \cdots + z^{N-1}$, then on the unit circle $|z| = 1$, $\|D\|_1$ is the minimum of $\|f\|_1$ for f of the form $f(z) = c_0 + c_1 z^{n_1} + \cdots + c_{N-1} z^{n_{N-1}}$ with $|c_k| = 1$; more generally, $\|D\|_p$ is the min/max of $\|f\|_p$ for fixed $p \in [0, 2]/[2, \infty]$. In the paper this is proved for the special case where $f(z) = 1 \pm z \pm z^2 \pm \cdots \pm z^{N-1}$ and $p \in [0, 4]$, by first proving stronger results for the eigenvalues of finite sections of the Toeplitz matrices of $|D|^2$ and $|f|^2$, in particular, for their Schatten p-norms. Several conjectures are also stated to the effect that these stronger results should be true for the general case of f. The approach is motivated by the uncertainty principle and two theorems of Szegő.

§1. Introduction

For $N \geq 1$, let f be a polynomial of the form

$$f(z) = c_0 + c_1 z^{n_1} + c_2 z^{n_2} + \cdots + c_{N-1} z^{n_{N-1}}, \tag{1}$$

where the complex coefficients c_k all have modulus $|c_k| = 1$ and where $0 < n_1 < n_2 < \cdots < n_{N-1}$ are integers. In [HL48] and earlier, Hardy and Littlewood considered the polynomial D defined by

$$D(z) = 1 + z + z^2 + \cdots + z^{N-1} \tag{2}$$

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(which is a special case of (1)), and they asked whether on the unit circle $z = e^{i\theta}, \theta \in [0, 2\pi]$, one has the sharp 1-norm inequality

$$||D||_1 \leq ||f||_1. \tag{3}$$

The nonsharp version $\log N \leq C||f||_1$ was known as the Littlewood conjecture and was proved in [K81] and [MPS]. Similar sharp p-norm inequalities can be conjectured for each p in the range $0 \leq p \leq \infty$ as follows:

$$||D||_p \leq ||f||_p, \quad 0 \leq p \leq 2; \tag{4}$$

$$||D||_p \geq ||f||_p, \quad 2 \leq p \leq \infty. \tag{5}$$

We shall refer to these as the sharp Littlewood conjectures. We use the definitions

$$||f||_p = \left(\int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta/2\pi \right)^{1/p}, \quad 0 < p < \infty,$$

$$||f||_0 = \lim_{p \to 0+} ||f||_p, \quad ||f||_\infty = \lim_{p \to \infty} ||f||_p,$$

which imply

$$||f||_0 = \exp \left(\int_0^{2\pi} \log |f(e^{i\theta})| d\theta/2\pi \right), \quad ||f||_\infty = \text{essential sup} \, |f(e^{i\theta})|.$$

Our purpose in this paper is to suggest that the sharp Littlewood conjectures may be simply the limiting cases of the corresponding *finite-dimensional* norm inequalities. Namely, for a positive definite Hermitian matrix A of size $n \times n$, consider the Schatten "norms" for $0 < p < \infty$, defined by $||A||_p = (\sum_{k=1}^n \lambda_k(A)^p)^{1/p}$, where $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ are the eigenvalues of A, and extend this definition to $p = 0, \infty$ in the natural way as above. For a given polynomial f of the above form and for any $n \geq 1$, let B_n be the $n \times n$ principal section of the infinite Toeplitz matrix generated by the Fourier coefficients of $|f|^2$:

$$(B_n)_{ij} = b_{j-i}, \quad 1 \leq i, j \leq n, \tag{6}$$

where

$$b_k = \int_0^{2\pi} |f(e^{i\theta})|^2 e^{-ik\theta} d\theta/2\pi = \sum_{m\in\mathbb{Z}} \hat{f}(m)\overline{\hat{f}(m-k)},$$

for all $k \in \mathbb{Z}$. For the special case of $f = D$, we denote this matrix section by A_n.

1.1. Conjecture. *For all* $n \geq 1$,

$$\|A_n\|_p \leq \|B_n\|_p, \quad 0 \leq p \leq 1; \tag{7}$$

$$\|A_n\|_p \geq \|B_n\|_p, \quad 1 \leq p \leq \infty. \tag{8}$$

A relationship between the function values $|f(e^{i\theta})|^2$ and the eigenvalues of B_n as $n \to \infty$ is provided by the following theorem.

Szegő limit theorem [GS, 5.2, p. 64–65]. *Let ψ be a measurable real-valued function on $[0, 2\pi]$ satisfying $m \leq \psi \leq M$ for some finite real constants m and M. Let $F(\lambda)$ be any real continuous function defined on the interval $m \leq \lambda \leq M$. For each integer $n \geq 1$, let T_n be the Hermitian $(n \times n)$-matrix defined by $(T_n)_{ij} = \hat{\psi}(j-i)$, $1 \leq i, j \leq n$, and let $\lambda_1(T_n), \lambda_2(T_n), \ldots, \lambda_n(T_n)$ be the eigenvalues of T_n. Then $m \leq \lambda_k(T_n) \leq M$ and*

$$\lim_{n\to\infty} \frac{F(\lambda_1(T_n)) + F(\lambda_2(T_n)) + \cdots + F(\lambda_n(T_n))}{n} = \frac{1}{2\pi}\int_0^{2\pi} F(\psi(\theta))d\theta.$$

This theorem shows that Conjecture 1.1 implies the sharp Littlewood conjectures (4) and (5). (To see this, we fix $0 < p < \infty, F(\lambda) := \lambda^p, \psi := |f|^2$ or $\psi := |D|^2$, and apply the theorem on both sides of (7) or (8). The cases of $p = 0, \infty$ then follow by continuity in p.) Next, we note that the case of $p = 0$ in Conjecture 1.1 is simply the claim that $\det A_n \leq \det B_n$. This suggests that there may be other simple matrix properties that are responsible for the whole range of p-norm inequalities (7) and (8). Conjectures along these lines are considered in §2 and §3. As an easy consequence of this approach, we obtain the following result.

1.2. Theorem. *Let $N \geq 1$, let f be a polynomial of the special form*

$$f(z) = 1 \pm z \pm z^2 \pm \cdots \pm z^{N-1}, \tag{9}$$

for any choice of $\pm$ *signs, and let* $D(z) = 1 + z + z^2 + \cdots + z^{N-1}$. *If* B_n *and* A_n *are the matrices defined by* (6) *for this* f *and* D *(respectively), and* I_n *denotes the identity* $(n \times n)$*-matrix, then for all* $n \geq 1$ *and all* $\lambda \geq 0$ *we have*

$$\det(\lambda I_n + A_n) \leq \det(\lambda I_n + B_n), \tag{10}$$

$$\|A_n\|_p \leq \|B_n\|_p, \quad 0 \leq p \leq 1, \tag{11}$$

$$\|A_n\|_p \geq \|B_n\|_p, \quad 1 \leq p \leq 2. \tag{12}$$

Consequently, by the Szegő limit theorem,

$$\int_0^{2\pi} \log(\lambda + |D(e^{i\theta})|^2)d\theta \leq \int_0^{2\pi} \log(\lambda + |f(e^{i\theta})|^2)d\theta, \quad \lambda \geq 0, \tag{13}$$

$$\|D\|_p \leq \|f\|_p, \quad 0 \leq p \leq 2, \tag{14}$$

$$\|D\|_p \geq \|f\|_p, \quad 2 \leq p \leq 4. \tag{15}$$

In particular, Theorem 1.2 gives the sharp Littlewood conjecture (3) for the special case (9). On the other hand, the p-intervals in the results (12) and (15) are surely not the best possible, since we can see directly that (12) is true for all integral values of $p \geq 1$. Indeed, fixing n and putting $A = A_n, B = B_n$, from (9) we readily deduce that $|B_{ij}| \leq A_{ij}$ for each matrix entry, so that $\|B\|_p^p = \mathrm{tr}(B^p) \leq \mathrm{tr}(|B|^p) \leq \mathrm{tr}(A^p) = \|A\|_p^p$, where $|B|$ denotes the matrix with entries $|B_{ij}|$. (See Subsection 3.2 for related remarks on Conjecture 1.1 (8) for integral p.) Another reason to believe that Theorem 1.2 can be improved is that our proof actually gives more information than stated in (10).

We prove Theorem 1.2 in §2 and state additional matrix conjectures. In §3 we list known cases of the sharp Littlewood conjectures, and we discuss the motivation for the present method. We also consider families of concave and convex functions that may be appropriate in Conjecture 1.1 in place of the pth powers x^p. Examples are the family $\log(\lambda + x)$, which appeared in Theorem 1.2, and the family $(x - \lambda)_+ - \lambda \log_+(x/\lambda)$, which seems to be a good candidate for the strongest possible family.

§2. Finite Toeplitz matrices

Fix $n \geq 1$. For a given (analytic) polynomial f expressed as

$$f(z) = \hat{f}(0) + \hat{f}(1)z + \cdots + \hat{f}(K)z^K,$$

the finite Toeplitz matrix B_n (which we defined in (6)) corresponding to $|f|^2 = f\bar{f}$ is the matrix product (Gram matrix)

$$B_n = R_n R_n^*, \tag{16}$$

where R_n^* is the conjugate transpose and R_n is the $n \times \infty$ Toeplitz matrix:

$$(R_n)_{ij} = \hat{f}(j-i), \quad 1 \leq i \leq n, \ 1 \leq j < \infty.$$

Since f is a polynomial, R_n may be truncated to become an $n \times (n+K)$ rectangular Toeplitz matrix. For example for $n = 4$ and for the polynomial $f(z) = 1 + z + z^3$, we have

$$R_4 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad B_4 = R_4 R_4^* = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

For fixed N and for the special case where $f = D$ (see (2)), let the corresponding rectangular matrices be denoted by Q_n, so that

$$A_n = Q_n Q_n^*. \tag{17}$$

To simplify notation, from this point on we sometimes omit the subscript n (having fixed it) and thus write $A_n = A, Q_n = Q, B_n = B, R_n = R$, etc. Consider the polynomials

$$p_B(t) = \det(I + tB) = 1 + \sum_{k=1}^{n} S_k(B) t^k. \tag{18}$$

The coefficients $S_k(B)$ are the elementary symmetric functions of the eigenvalues of B; we have

$$S_k(B) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \det B_{i_1 i_2 \ldots i_k},$$

where the summation is over all *principal* $k \times k$ submatrices $B_{i_1 i_2 \ldots i_k}$ of B (that is the submatrices formed by the k rows and k columns with indices $i, j \in \{i_1 < i_2 < \cdots < i_k\}$). We have the following well-known identity, by the Binet-Cauchy Theorem (see [MO, p. 503]).

2.1. Proposition. *If B is an $(n \times n)$-matrix of the form $B = RR^*$ for some rectangular matrix R, then*

$$S_k(B) = \sum_{R_{k\times k}} |\det R_{k\times k}|^2$$

for all $1 \le k \le n$, where the summation is over all *$k \times k$ submatrices $R_{k\times k}$ of R (for which we have avoided exhibiting row and column indices to simplify notation).*

The following lemma and the definitions preceding it are taken from [FG]. For the sake of completeness we present a detailed proof of the lemma, based on the sketch provided in [FG, p. 853]. This approach replaces a similar but weaker lemma originally used by the author.

2.2.1. Definition. A $(0,1)$ *matrix* is a rectangular matrix with each entry equal to 0 or 1. We also refer to this as the (0,1) *property* of a matrix.

2.2.2. Definition. A (0,1) matrix M is said to have the *interval property* if, after some permutation of the columns of M, in each row all 1 entries occur consecutively in some interval (or the row is identically zero).

2.2.3. Definition. A rectangular matrix M is said to be *totally unimodular* if each square submatrix of M has determinant equal to 0, 1, or -1. (In particular, the entries of M are necessarily 0, 1, or -1.)

The following matrix M gives an example of the interval property:

$$M = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The matrices Q_n generated by the polynomial D for any $n, N \ge 1$ clearly have the interval property. For $n = 4, N = 3$ this matrix is

$$Q_4 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

2.3. Lemma [FG]. *If M is a $(0,1)$ matrix with the interval property, then M is totally unimodular. In particular, each of the matrices Q_n is totally unimodular.*

Proof [FG]. Let M be a (0,1) matrix with the interval property. It is easy to see that deleting any row or column preserves the interval property. Hence, any square submatrix of M still has the interval property. Thus, it suffices to show that if a square (0,1) matrix S has the interval property, then $\det S \in \{0, 1, -1\}$. We prove this by induction on the number of 1 entries in S. Let K be the number of 1 entries in S. If $K = 0$, then $S = 0$, and the result holds. Assume now that $K \geq 1$ and that the result holds whenever the number of ones is less than K. We permute the columns of S so that on each row the ones occur in an interval. We may assume that no row is identically zero, and, consequently, for each row index i there is a unique interval $[m_i, n_i]$ of column indices j where the ones occur:

$$\{j : S_{ij} = 1\} = [m_i, n_i] := \{j : m_i \leq j \leq n_i\}.$$

Suppose that the m_i are all distinct. Then we may permute the rows so that $m_1 < m_2 < \cdots < m_d$, where d is the number of rows (dimension) of the square matrix S. Thus, S becomes upper triangular, and we are done.

Finally, we suppose that the m_i are not all distinct. Take any two rows $i \neq i'$ with $m_i = m_{i'}$. We may assume that $n_i \leq n_{i'}$. Subtracting row i from row i' gives a new square matrix S' that still has both the (0,1) property and the interval property, but with the number of ones strictly less than K. Thus, $\det S = \pm \det S' \in \{0, 1, -1\}$ by the induction hypothesis. •

Proof of Theorem 1.2. As above, we write

$$A_n = A = QQ^*, \quad B_n = B = RR^*$$

for the matrices generated by D and f, respectively. For our special f, the matrices Q and R have exactly the same dimensions (they are both $n \times (n+N-1)$). Thus, in Proposition 2.1, for each k and each fixed submatrix $Q_{k\times k}$ of Q we may consider the corresponding submatrix $R_{k\times k}$ of R given by the same choices of rows and columns. We claim that

$$|\det Q_{k\times k}| \leq |\det R_{k\times k}|. \tag{19}$$

To prove the claim, observe that

$$\det Q_{k\times k} \in \{0, 1, -1\} \tag{20}$$

by Lemma 2.3. But mod 2 the coefficients of f are exactly the same as those of D for our special class of f. Consequently, $R = Q$ mod 2 and $\det Q_{k\times k} = \det R_{k\times k}$ mod 2. So if $(m)_2 \in \{0,1\}$ denotes the representative of any integer m mod 2, then

$$|\det Q_{k\times k}| = (\det Q_{k\times k})_2 = (\det R_{k\times k})_2 \le |\det R_{k\times k}|.$$

Therefore, $S_k(A) \le S_k(B)$ for all k and $\det(I+tA) \le \det(I+tB)$ for all $t \ge 0$, by Proposition 2.1 and (18). After putting $t = 1/\lambda$, this becomes

$$\det(\lambda I + A) \le \det(\lambda I + B), \quad \lambda \ge 0,$$

so the first part of the theorem is proved. Next, for a diagonalizable matrix such as A or B we have

$$\det(I+tA) = \prod_{i=1}^{n}(1+t\lambda_i(A)).$$

Taking the logarithms of both sides of our inequality, we obtain

$$\sum_{i=1}^{n}\log(1+t\lambda_i(A)) \le \sum_{i=1}^{n}\log(1+t\lambda_i(B)), \quad t \ge 0. \tag{21}$$

Now we can multiply both sides of this inequality by any positive function $g(t)$ and integrate with respect to t to obtain further inequalities involving sums of some function of the eigenvalues. Let us try to find a weight function $g(t)$ such that $\lambda^p = \int_0^\infty \log(1+t\lambda)g(t)dt$ for some range of p. After making the substitution $s = t\lambda$, we see that a reasonable guess for g is $g(t) = ct^r$ for some r. Then checking these formal integrals for convergence at $t = 0$ and ∞, we arrive at the result that for any $0 < p < 1$ and any $\lambda \ge 0$

$$\lambda^p = \int_0^\infty \log(1+t\lambda)t^{-1-p}dt/C_p, \quad \text{where } C_p = \int_0^\infty \log(1+t)t^{-1-p}dt.$$

(We can check that this formal result is actually valid by repeating the above substitution.) Therefore,

$$\sum_{i=1}^{n}\lambda_i(A)^p \le \sum_{i=1}^{n}\lambda_i(B)^p, \quad 0 < p < 1.$$

For $p = 0$, inequality (11) is precisely the determinant inequality (10) with $\lambda = 0$. Finally, we consider the range $1 \le p \le 2$. First, we note that $\mathrm{tr}(A) = nN = \mathrm{tr}(B)$, whence

$$\sum_{i=1}^{n} \lambda_i(A) = \sum_{i=1}^{n} \lambda_i(B).$$

Therefore, by (21),

$$\sum_{i=1}^{n} \{t\lambda_i(A) - \log(1 + t\lambda_i(A))\} \ge \sum_{i=1}^{n} \{t\lambda_i(B) - \log(1 + t\lambda_i(B))\}$$

for all $t \ge 0$. Observe that $s - \log(1 + s) \ge 0, s \ge 0$. Applying weight functions as above, we see that

$$\lambda^p = \int_0^\infty \{t\lambda - \log(1 + t\lambda)\} t^{-1-p} dt / C_p', \quad \text{where } C_p' := \int_0^\infty \{t - \log(1 + t)\} t^{-1-p} dt,$$

for any $1 < p < 2$ and any $\lambda \ge 0$. Therefore,

$$\sum_{i=1}^{n} \lambda_i(A)^p \ge \sum_{i=1}^{n} \lambda_i(B)^p, \quad 1 < p < 2.$$

The case of $p = 2$ follows by continuity. •

We note that the proof of (19) works whenever Q is a totally unimodular matrix and R is the matrix defined by putting arbitrary $\pm$ signs on *each* entry of Q. Thus, the matrices $A = QQ^*$ and $B = RR^*$ satisfy (10)–(12). Also, the result (19) implies that for any fixed choice of indices $1 \le i_1 < i_2 < \cdots < i_k \le n$ we have

$$\det A_{i_1 i_2 \ldots i_k} \le \det B_{i_1 i_2 \ldots i_k}. \tag{22}$$

(This follows by using Proposition 2.1 to expand the determinant of such a fixed submatrix of A or B.) Now we could ask whether condition (22) for an arbitrary pair of positive definite Hermitian $(n \times n)$-matrices A and B with equal traces *a priori* implies the full range of p-norm inequalities (8) instead of merely the range (12) obtained in the proof. The following example shows that the answer is no, and that the range of p-norm inequalities we have obtained is the best possible if only the information in (22) is used. In fact, our example is defined starting with a totally unimodular matrix Q, so that it shares all of the structure used in the proof. We define

$$Q := \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad A := QQ^* = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix},$$

$$R := \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B := RR^* = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

The eigenvalues of A are 0, 3, 3, and those of B are 1, 1, 4. Q is totally unimodular, and R was obtained by changing the sign of an entry of Q. Thus, our proof yields (10)—(12). But (8) fails for all $p > 2$, as can be verified by using the eigenvalues. In view of this example, we could modify our question by also insisting that all entries A_{ij} of A should be *nonnegative.* (This would at least give (8) for all *integers* $p \geq 2$, since (22) for $k = 1, 2$ would imply that the entries B_{ij} satisfy $A_{ii} = B_{ii}$ and $|B_{ij}| \leq |A_{ij}| = A_{ij}$.) However, even this modified question has a negative answer. A 4×4 counterexample is provided by taking the first row of Q_4 to be $[1,1,0,0,1,0,0,0]$, the first row of R_4 to be $[1,1,0,0,-1,0,0,0]$, and then taking forward shifts for the subsequent three rows. This Q_4 turns out to be totally unimodular, yet the p-norm inequalities (8) fail somewhere in the interval $2 < p < 3$ (for the corresponding matrices $A_4 := Q_4Q_4{}^*$ and $B_4 := R_4R_4{}^*$). We omit the verifications.

Returning to the general case (1) of $f(z)$, we remark that the strong condition (22) is trivially false. For example, for $D(z) = 1 + z, f(z) = 1 + z^2$, and $n = 3$, we have

$$A = Q_3Q_3^* = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad B = R_3R_3^* = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Thus, we have a counterexample to (22) with $k = 2$, for the subdeterminants $\det A_{13} = 4 > 3 = \det B_{13}$. However, we have found no counterexamples to the following conjecture, for the general case (1) of $f(z)$.

2.4. Conjecture. $S_k(A_n) \leq S_k(B_n)$ *for all* $n \geq 1,\ 1 \leq k \leq n.$

By the proof of Theorem 1.2, this conjecture implies inequalities (10)–(15), and in particular the original 1-norm case (3) of the sharp Littlewood conjecture.

§3. Notes and comments

3.1. Failure of majorization. We note that (4) and (5) are clearly true for $p = 0, 2, \infty$. (For $p = 0$ observe that $\log|f(z)|$ is subharmonic while $\log|D(z)|$ is harmonic in $|z| < 1$.) Could (4) and (5) simply be consequences of the majorization property [MO] for the pair $w(\theta) = |D(e^{i\theta})|^2$, $v(\theta) = |f(e^{i\theta})|^2$? We say that $w \geq 0$ majorizes $v \geq 0$ if $\int_0^{2\pi} w(\theta)d\theta = \int_0^{2\pi} v(\theta)d\theta$ and one of the following three equivalent conditions is fulfilled:

$$\int_0^{2\pi t} w^*(\theta)d\theta \geq \int_0^{2\pi t} v^*(\theta)d\theta, \quad 0 \leq t \leq 1, \tag{23}$$

$$\int_0^{2\pi} (w - \lambda)_+ d\theta \geq \int_0^{2\pi} (v - \lambda)_+ d\theta, \quad \lambda \geq 0, \tag{24}$$

$$\int_0^{2\pi} \phi(w)d\theta \geq \int_0^{2\pi} \phi(v)d\theta \quad \text{for all convex } \phi : [0,\infty) \to \mathbb{R}, \tag{25}$$

where w^*, v^* are the decreasing rearrangements. The author has proved that for $N = 3$ the answer is yes; see [Kel]. However, in general the answer is no, because by a remark of Pichorides [P] one can construct an $f(z)$ with a double root on $|z| = 1$. In fact, already for $N = 4$ we have double root examples such as $f_1(z) = (1+z)(1+z^3) = 1 + z + z^3 + z^4$ and $f_2(z) = (1+z)(1-z^2) = 1 + z - z^2 - z^3$. But, these are not counterexamples to (4) or (5) by the following additional result of the author (see [Kel]):

$$\int_0^{2\pi} \phi(|1+z|^2)\psi(|1+z^2|^2)d\theta \geq \int_0^{2\pi} \phi(|1+az|^2)\psi(|1+bz^m|^2)d\theta, \tag{26}$$

for any convex ϕ and nondecreasing $\psi \geq 0$, and $m \geq 2, |a| = |b| = 1$, where $z = e^{i\theta}$. This result will reappear in connection with Conjecture 3.3.2. For the moment we note that the choices $\phi(x) = \pm x^p, \psi(x) = x^p, p > 0$ yield (4) and (5) for $f(z) = (1+az)(1+bz^m)$.

3.2. Szegő's infimum with a constraint parameter. Now we sketch the heuristic arguments which led to Toeplitz matrices and their subdeterminants.

3.2.1. Proposition. *For* (1) *and* (2) *let*

$$|f(e^{i\theta})|^2 = \sum_{n\in\mathbb{Z}} b_n e^{in\theta}, \quad |D(e^{i\theta})|^2 = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}.$$

Then $\{a_n\}$ *is symmetric decreasing. If* $\{b_n^*\}$ *denotes the symmetric decreasing rearrangement of* $\{|b_n|\}$, *then*

$$\sum_{n=0}^{k} a_n \geq \sum_{n=0}^{k} b_n^*, \quad k \geq 0. \tag{27}$$

This result is a special case of the rearrangement theory (for arbitrary coefficients) used for the proof of (5) with $p = 4, 6, 8, \dots$ (see [HL28] and [G]). Probably, that method also gives (8) for $p = 2, 3, 4, \dots$, with the help of the identity $\|A\|_p^p = \mathrm{tr}(A^p)$. For $p = \infty$ an alternative is to use $\|A\|_\infty = \sup_{\|x\|_2=1}\langle x, Ax\rangle$. (In fact, the rearrangement problems originated from this context; see [HLP26]. Also, "arrays" corresponding to row submatrices of R_n (see (16)) were already used in [HL28, p. 109], in the form of a diagram illustrating a proof. But our initial motivation for the matrix approach was different, as will be seen below.) Strangely, (27) is the discrete version of the majorization property (23), but for the Fourier coefficients. Unfortunately, property (27) and the rearrangement theory are too weak to formally imply the desired inequalities (4) and (5) for other p such as $p = 1$, by a result of Lehmer [L] (see also [Keo, K91]).

Instead, as our first item of heuristic motivation, we note the following consequence of (27):

$$\begin{aligned} H(k, w) &:= \inf_{1\leq m_1<\dots<m_k} \int_0^{2\pi} \sum_{j=1}^{k} |1 - z^{m_j}|^2 w\, d\theta \\ &\leq \inf_{1\leq m_1<\dots<m_k} \int_0^{2\pi} \sum_{j=1}^{k} |1 - z^{m_j}|^2 v\, d\theta =: H(k, v), \end{aligned} \tag{28}$$

where $w(\theta) = |D(e^{i\theta})|^2$, $v(\theta) = |f(e^{i\theta})|^2$, $z = e^{i\theta}$, $k \geq 1$. This is a naive attempt to relate properties of the coefficients $\hat{f}(m)$ to properties of $|f|$. At this point it is useful to observe that our main problem (the conjecture (4) and (5)) fits into the framework of the uncertainty principle [HJ]. It does so because intuitively $\hat{D}(m)$ is the function that is "closest" to being constant and smooth among all admissible $\hat{f}(m)$, and thus $|D|$ should have a correspondingly opposite extremal property. The properties (4) and (5) are indeed consistent with $|D|$ being the "farthest" from a constant function. For $k = 1$ we can view (28) as a very literal form of the original uncertainty principle; momentum corresponds to $\frac{\partial}{\partial x}$, which could here correspond to a difference operator such as $f(z) \to (1-z)f(z)$. Thus $(1-z)f(z)$ is a specific way of testing the constancy or smoothness of $\hat{f}(m)$. On the other hand, the smallness of $\|(1-z)D(z)\|_2$ tells us that most of $|D|$ must have been concentrated near $z = 1$. Now, we actually have no preferred origin on $|z| = 1$ since our problem concerns p-norms and thus the overall distribution of $|f|$. Thus the presence of arbitrary terms $(1 - z^{m_j})f(z)$ and all $k \geq 1$ is natural.

As our second item we recall that Szegő's theorem on the geometric mean (see [GS, §3.1(a), p. 44]) applied to a continuous weight $w \geq 0$ states:

$$\exp\left(\int_0^{2\pi} \log w(\theta)d\theta/2\pi\right) = \inf_{\{p_j\}} \int_0^{2\pi} \left|1 - \sum_{j\geq 1} p_j z^j\right|^2 w(\theta)d\theta/2\pi. \tag{29}$$

Thirdly, the majorization property (23) can be restated as $J(t,w) \leq J(t,v)$ by using the functional

$$J(t,w) := \int_{2\pi t}^{2\pi} w^*(\theta)d\theta/2\pi = \inf_{|E|\leq 2\pi t} \int_0^{2\pi} (1 - \chi_E(\theta))w(\theta)d\theta/2\pi$$
$$= \inf_{\|g\|_1\leq t} \int_0^{2\pi} |1 - g|wd\theta/2\pi, \quad t \geq 0. \tag{30}$$

By our remarks so far, we know that (28) is too weak, (30) is too strong, whereas (29) correctly gives the endpoint $p = 0$ of the desired range of inequalities (4). This suggests creating a new "hybrid" functional such as

$$I(t,w) := \inf_{\|P\|_2\leq t} \int_0^{2\pi} |1 - P|^2 wd\theta/2\pi, \quad t \geq 0, \tag{31}$$

where $P = P(z)$ ranges over the (analytic) polynomials with $P(0) = 0$ as in (29), but has the constraint indicated.

3.2.2. Conjecture. *For* (1), (2) *and for* $w(\theta) = |D(e^{i\theta})|^2$, $v(\theta) = |f(e^{i\theta})|^2$, *we have*

$$I(t,w) \le I(t,v) \quad \textit{for all } t \ge 0. \tag{32}$$

This conjecture was made by analogy with (28)–(30) and in the hope that (32) would be strong enough to imply some range of p-norm inequalities. After some effort, this hope was fulfilled by the following proposition. It is actually a precise analog of the equivalence (23) $\iff$ (24). (We give only an outline of the proof since the result now serves only as motivation.)

3.2.3. Proposition. *Let* $w, v \ge 0$ *be continuous functions with* $\int_0^{2\pi} w d\theta/2\pi = \int_0^{2\pi} v d\theta/2\pi = 1$. *Then* (32) *is equivalent to*

$$\int_0^{2\pi} \log(\lambda + w) d\theta \le \int_0^{2\pi} \log(\lambda + v) d\theta, \quad \lambda \ge 0, \tag{33}$$

and this implies $\|w\|_p \le \|v\|_p$ *for* $0 \le p \le 1$ *and* $\|w\|_p \ge \|w\|_p$ *for* $1 \le p \le 2$.

Outline of the proof. Assuming (32), we take a Legendre transform L of both sides as follows:

$$\begin{aligned}
L(\lambda, w) &:= \inf_{t\ge0}(I(t,w) + \lambda t^2) \\
&= \inf_{t\ge0}\left(\inf_{\|P\|_2\le t} \int_0^{2\pi} |1-P|^2 w d\theta/2\pi + \lambda t^2\right) \\
&= \inf_{t\ge0}\left(\inf_{\|P\|_2 = t} \int_0^{2\pi} |1-P|^2 w d\theta/2\pi + \lambda t^2\right) \\
&= \inf_{t\ge0}\left(\inf_{\|P\|_2 = t} \int_0^{2\pi} |1-P|^2 w d\theta/2\pi + \lambda \int_0^{2\pi} |1-P|^2 d\theta/2\pi\right) - \lambda \\
&= \inf_{\{P;\text{analytic};P(0)=0\}} \left(\int_0^{2\pi} |1-P|^2 (w+\lambda) d\theta/2\pi\right) - \lambda \\
&= \exp\left(\int_0^{2\pi} \log(w+\lambda) d\theta/2\pi\right) - \lambda,
\end{aligned}$$

where we have applied Szegő's theorem (29) to the weight $(w+\lambda)$ to get the last line. Clearly $L(\lambda, w) \leq L(\lambda, v)$ for all $\lambda \geq 0$, which gives (33). Since the Legendre transformation is an involution, we can prove the converse by taking Legendre transforms on both sides of the latter inequality. The derivation of the p-norm inequalities from (33) was given in the proof of Theorem 1.2. •

This Legendre transform technique is well known in real interpolation theory [BL]. (It is also one way to prove the result (23) $\Longleftrightarrow$ (24).) In view of Proposition 3.2.3, we can now remark that Conjecture 3.2.2 is true for the special case (9), by (13). Moreover, Conjecture 2.4 implies Conjecture 3.2.2 via (13). Thus Conjecture 2.4 provides a discrete set of conditions which is more fundamental than (32). Intuitively, (32) is a quantitative way of saying that the forward shifts $z^j D(z)$ of $D(z)$ are "more linearly dependent" than the forward shifts $z^j f(z)$ of $f(z)$, and Conjecture 2.4 is a more explicit form of the same notion: each subdeterminant $\det R_{k\times k}$ in Proposition 2.1 can be thought of as a measure of the "amount of linear independence" (or k-volume) in that submatrix of the forward shift matrix R.

However, there is probably something even more fundamental which we are still missing, because Conjecture 2.4 does not seem to be strong enough to imply (8) for the range $2 < p \leq \infty$. Also, the functional (31) is clearly an *ad hoc* construction, for at least two reasons. First, the choice of the 2-norm in the constraint was made only for simplicity. Second, it is likely that some discrete set of conditions is more fundamental, rather than the continuum of conditions given by the parameter t. Nevertheless, the results and conjectures in this paper are mostly consequences of investigating this *ad hoc* functional. It clearly involves the Toeplitz form generated by w, and the treatment in [GS] naturally leads to considering the finite sections of this quadratic form.

3.3. Other functionals. For the functional (31), removing the condition that P is an analytic polynomial gives the functional

$$K(t, w) := \inf_{\|g\|_2 \leq t} \int_0^{2\pi} (1-g)^2 w \, d\theta / 2\pi, \tag{34}$$

where g varies over, say, real L^2 functions orthogonal to 1 (this orthogonality turns out to be irrelevant for our discussion). For this functional, the conjecture $K(t, |D|^2) \leq K(t, |f|^2)$ for $t \geq 0$, analogous to Conjecture 3.2.2, is false. Any of the double root examples, such as $f = f_1$ and $f = f_2$ in Subsection 3.1, are counterexamples. This is seen by considering $\|f\|_p := (\int_0^{2\pi} |f(e^{i\theta})|^p d\theta/2\pi)^{1/p}$ for small *negative* values of p, and by using inequalities (36) of the following statement.

3.3.1. Proposition. *Let* $w, v \geq 0$ *be continuous functions with* $\int_0^{2\pi} w d\theta/2\pi = \int_0^{2\pi} v d\theta/2\pi = 1$. *Then the property* $K(t,w) \leq K(t,v)$, $t \geq 0$, *is equivalent to*

$$\int_0^{2\pi} \frac{1}{\lambda + w} d\theta \geq \int_0^{2\pi} \frac{1}{\lambda + v} d\theta, \quad \lambda \geq 0. \tag{35}$$

Moreover, (35) *implies*

$$\|w\|_p \leq \|v\|_p, \; -1 \leq p \leq 1; \quad \|w\|_p \geq \|w\|_p, \; 1 \leq p \leq 2. \tag{36}$$

Outline of the proof. For (35) take Legendre transforms as in the proof of Proposition 3.2.3 and compute the resulting infimum by Lagrange multipliers (or by the result of Kolmogorov [Kol]). For (36), use (35) and the method of proof of (11) and (12) in Theorem 1.2. It turns out that precisely the same proposition is true if we remove the condition $g \perp 1$ in the definition (34) of K, and the proof is similar. ●

The preceding counterexamples show that the endpoint $p = 0$ in the sharp Littlewood conjectures is itself rather sharp. Now we address the undesirable endpoint $p = 4$ in (15), which corresponds to $p = 2$ in Proposition 3.2.3. It is clear that the presence of this endpoint is due to our *ad hoc* choice of $\|P\|_2$ in constructing the functional (31); one can predict directly from (31) that such an endpoint will occur simply by considering very small t and using duality. In fact, we could make this endpoint move to ∞ by using the constraint $\|P\|_1 \leq t$ instead of $\|P\|_2 \leq t$ in the definition (31). We have not tried this because the 1-norm and the analyticity of P are difficult to work with simultaneously. If we use $\|P\|_1 \leq t$ but remove analyticity, then we no longer have the lower endpoint at $p = 0$, as was already seen for the functional K and (36) of Proposition 3.3.1. One possibly interesting result of such tinkering is the functional

$$\begin{aligned} G(t,w) &:= \inf_{\int_0^{2\pi} \log_+ \frac{1}{|1-g|} d\theta/2\pi \leq t} \int_0^{2\pi} (1-g)^2 w d\theta/2\pi, \\ &= \inf_{\rho \geq 0, \|\rho\|_1 \leq 2t} \int_0^{2\pi} e^{-\rho} w d\theta/2\pi, \end{aligned} \tag{37}$$

where no analyticity conditions are imposed. It turns out that the following three conditions are equivalent if $w, v \geq 0$ are nice functions with $\int_0^{2\pi} w d\theta/2\pi = \int_0^{2\pi} v d\theta/2\pi$:

$$G(t, w) \leq G(t, v), \quad t \geq 0; \tag{38}$$

$$\int_0^{2\pi} \beta_\lambda(w) d\theta \geq \int_0^{2\pi} \beta_\lambda(v) d\theta, \quad \lambda \geq 0, \tag{39}$$

where

$$\beta_\lambda(x) := (x - \lambda)_+ - \lambda \log_+(x/\lambda), \quad x \geq 0; \tag{40}$$

$$\int_0^{2\pi} \beta(w) d\theta \geq \int_0^{2\pi} \beta(v) d\theta \tag{41}$$

for any β of the form

$$\beta(x) = \int_0^x \phi(t) dt/t \quad \text{for some convex } \phi : [0, \infty) \to \mathbb{R}. \tag{42}$$

By using (39) or (41) it is easy to show that these conditions imply that $\|w\|_p \leq \|v\|_p$ $(0 \leq p \leq 1)$ and $\|w\|_p \geq \|v\|_p$ $(1 \leq p \leq \infty)$, and that the endpoint $p = 0$ is sharp. It also can be checked that (41) $\Rightarrow$ (33), i.e., these conditions are stronger than those obtained with the help of the functional (31). Next, define

$$\|A\|_\beta = \sum_{k=1}^n \beta(\lambda_k(A)) \tag{43}$$

where $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ are the eigenvalues of the positive definite Hermitian $(n \times n)$-matrix A, and $\beta : [0, \infty) \to \mathbb{R}$.

3.3.2. Conjecture. *For all* $n \geq 1$ *we have* $||A_n||_\beta \geq ||B_n||_\beta$ *for all* β *of the form* (42), *where* A_n, B_n *are defined as in Conjecture* 1.1. *In particular, by the Szegő limit theorem we would have* (41) *for* $w = |D|^2, v = |f|^2$.

The following are some reasons for making this conjecture.

1. The result (26) implies (39), for the examples $f(z) = (1 + az)(1 + bz^m)$ considered in (26). Moreover, in a sense, (39) is the strongest result of the form $\int_0^{2\pi} F(w)d\theta \geq \int_0^{2\pi} F(v)d\theta$ that can be deduced formally from (26); this is a remarkable coincidence that was not anticipated when we defined the functional (37). Briefly, the point is that in (26) we would like the integrand to change from the form $\phi(X)\psi(Y)$ into the form $F(XY)$. Now at first it seems that the only way to achieve this is to choose $\phi(x) = \pm x^p, \psi(x) = x^p, p > 0$. But in fact if we introduce a parameter $s > 0$, we can observe that

$$\int_0^\infty \phi(sX)\psi(Y/s)ds/s = F(XY)$$

for some F. It turns out that trying $\phi(x) = (x - \alpha)_+$ and $\psi(x) = \chi_{[\mu,\infty)}(x)$ gives precisely the family $F = \beta_\lambda$ in (40).

2. I have proved Conjecture 3.3.2 for $n = 3$, for the special case where $f(z)$ has the form (1) with all $c_k = \pm 1$, for any N. Here is an outline of the proof. First, it actually suffices to prove that (i) $\det(A_3) \leq \det(B_3)$ and (ii) $||A_3||_\infty \geq ||B_3||_\infty$. This suffices because we prove a lemma stating that on the measure space $\{1, 2, 3\}$ of 3 points with uniform measure, for any nonnegative w_i, v_i with $w_1 + w_2 + w_3 = v_1 + v_2 + v_3$, the two conditions (a) $w_1w_2w_3 \leq v_1v_2v_3$, and (b) $\max\{w_i\} \geq \max\{v_i\}$, are *equivalent* to (39) (where the integral means the sum over 3 points). Next, (ii) follows from the rearrangement theory and the identity $||A||_\infty = \sup_{||x||_2=1}\langle x, Ax\rangle$. (As mentioned in Subsection 3.2, this works for any n.) Finally (i) is not difficult to verify for $n = 3$ by using Proposition 2.1 for the case where $k = 3$. The trick is to use a row reduction on the matrix Q_3, by subtracting row 2 from row 1; then we can verify that any other matrix R_3 gives a larger value of $\det(R_3R_3^*)$.

3. I have tested Conjecture 3.3.2 on a computer (using the family (40) for a large number of λ) for various examples of $f(z)$ and found no counterexamples. Some of the computations included matrices of dimensions up to about 30×30, but I cannot claim that my search has been systematic. It is interesting that on the other hand, the majorization property for the eigenvalues already fails for 3×3 matrices, for example with $f(z) = 1 + z - z^3 - z^4$; i.e., Conjecture 3.3.2 fails for 3×3 matrices if we allow β to range over all convex functions instead of the family (42).

3.4. Two more remarks on Toeplitz matrices.

3.4.1. Proposition. *Fix $N \geq 1, m \geq 1$, and let $f(z) = D(z^m) = 1 + z^m + z^{2m} + \cdots + z^{(N-1)m}$. Then for all $n \geq 1$ and all* convex *β we have $\|A_n\|_\beta \geq \|B_n\|_\beta$, where A_n, B_n are defined as in Conjecture* 1.1.

Proof. The rows of R_n (see (16)) can be grouped into m subsets that are mutually orthogonal. This enables us to put the matrix B_n into a block diagonal form after permuting rows and columns. This block diagonal form is the same as the matrix A_n but with zeros outside of certain diagonal blocks. For example, for $N = 3, m = 2, n = 5$ we get

$$A_n = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 \\ 2 & 3 & 2 & 1 & 0 \\ 1 & 2 & 3 & 2 & 1 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}, \quad B_n \sim \begin{bmatrix} 3 & 2 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 & 3 \end{bmatrix}.$$

Now, a theorem of Fan (see [MO, p. 255]) immediately gives the desired result: $\|A_n\|_\beta \geq \|B_n\|_\beta$ for all convex β. •

As $n \to \infty$, this result does not give us any new cases of the sharp Littlewood conjectures, since $|D(z)|$ and $|D(z^m)|$ obviously have the same distribution functions on $|z| = 1$. Nevertheless, it confirms Conjectures 3.3.2 and 2.4 for $f(z) = D(z^m)$ (because majorization implies Conjecture 2.4 by a result of Schur; see [MO]). Proposition 3.4.1 suggested the following observation and conjecture on subsections of arbitrary Toeplitz Hermitian matrices. Although the result does not have a direct relationship to the sharp Littlewood conjectures, it does show that truncations of Toeplitz matrices are well behaved with respect to some p-norms.

3.4.2. Proposition. *Let A_n be any positive definite Hermitian Toeplitz $(n \times n)$-matrix, and let A_{n-1} be the principal $(n-1) \times (n-1)$ section of A_n. Let $\|A_n\|_{p,n}$ and $\|A_{n-1}\|_{p,n-1}$ be defined using* averages *over the eigenvalues*:

$$\|A_n\|_{p,n} = \left(\frac{1}{n}\sum_{k=1}^{n} \lambda_k(A_n)^p\right)^{1/p}; \quad \|A_{n-1}\|_{p,n-1} = \left(\frac{1}{n-1}\sum_{k=1}^{n-1} \lambda_k(A_{n-1})^p\right)^{1/p}.$$

Then for all $\lambda \geq 0$ we have

$$\det(\lambda I_n + A_n)^{1/n} \leq \det(\lambda I_{n-1} + A_{n-1})^{1/(n-1)}, \tag{44}$$

where I_k denotes the identity $(k \times k)$-matrix. It follows that

$$||A_n||_{p,n} \leq ||A_{n-1}||_{p,n-1}, \quad 0 \leq p \leq 1, \tag{45}$$

$$||A_n||_{p,n} \geq ||A_{n-1}||_{p,n-1}, \quad 1 \leq p \leq 2. \tag{46}$$

Proof. The case where $\lambda = 0$ in (44) was proved in [GS]. This implies the general case since $(\lambda I_n + A_n)$ is still positive definite Hermitian Toeplitz. The normalized p-norm inequalities follow from (44) as in the proof of Theorem 1.2. •

We note that for arbitrary convex β, the inequality $||A_n||_{\beta,n} \geq ||A_{n-1}||_{\beta,n-1}$ would fail for the normalized versions of the $||\cdot||_\beta$ functionals (43). This can be seen for $n = 3$ with the matrix A_3 whose first row is $[2, 1, 0]$. Thus "majorization" in the sense of [MO] fails again (for the eigenvalue distributions with respect to normalized counting measure). On the other hand, the interlacing property of eigenvalues tells us that $||A_n||_\infty \geq ||A_{n-1}||_\infty$. Therefore it seems natural to conjecture that the result (46) does extend to the intermediate values $2 < p < \infty$. What about $p < 0$? Perhaps these questions have already been dealt with in the vast literature on Toeplitz matrices.

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Department of Mathematics and Statistics
805 Sherbrooke Street West
McGill University
Montréal, Québec
H3A 2K6
Canada
E-mail: klemes@math.mcgill.ca

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