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On Jackson's inequality for a generalized modulus of continuity in L_2

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Abstract. The value of the sharp constant $\varkappa$ in the Jackson type inequality in the space $L_2(\mathbb{T}^d)$

$$E_{n-1}(f) \leqslant \varkappa \overline{\omega}_\psi(f,T) \tag{1}$$

is studied for the generalized modulus of continuity

$$\overline{\omega}_\psi(f,T) = \max_{t\in T}\biggl(\sum_s \psi(st)|\widehat{f}_s|^2\biggr)^{1/2}.$$

The value $\overset{*}{\varkappa}$ of the minimum sharp constant in inequality (1) is found.

A class of generalized moduli of continuity is introduced which contains the moduli $\widetilde{\omega}_{a,r}(f,\delta) := \sup_{0\leqslant t\leqslant\delta} \|\Delta_{a^{r-1}t}\cdots\Delta_{at}\Delta_t f\|_2$, with even a. The relation $\varkappa = \overset{*}{\varkappa}$ is proved in this class for all $\delta \geqslant \pi/n$.

Bibliography: 25 titles.

§ 1. Introduction

In approximation theory by *Jackson's inequality* one usually means the following relation between the value $d(f,L,X)$ of the best approximation of a function f in a normed function space X by elements of a subspace L and the structure characterization of the function f in terms of some seminorm (or quasi-seminorm) $|\cdot|_X$:

$$d(f,L,X) \leqslant K(L,X)|f|_X \qquad \text{for all } f\in X.$$

The greatest lower bound of the $K(L,X)$ is called the *sharp constant in Jackson's inequality.*

In the present paper we restrict our study of bounds for the sharp constant in Jackson's inequality to the case when $X = L_2(\mathbb{T}^d)$ and L is the space of functions $g \in L_2(\mathbb{T}^d)$ with spectrum lying in some fixed subset Λ of $\mathbb{Z}^d$. We characterize the

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smoothness of functions by the so-called generalized θ- and ψ-moduli of continuity. In § 2 we find the minimum sharp constant for Jackson's inequalities of this type.

In § 4, in the one-dimensional case, we find the sharp constant in Jackson's inequality

$$E_{n-1}(f) \leqslant K\omega_\psi(f,\delta) \text{ for each } f \in L_2(\mathbb{T})$$

which holds for all $\delta \geqslant \pi/n$ on a certain subset of the class of generalized moduli of continuity ω_ψ containing, in particular, the moduli of the form $\widetilde{\omega}_{a,r}(f,\delta) := \sup_{0\leqslant t\leqslant\delta} \|\Delta_{a^{r-1}t}\cdots\Delta_{at}\Delta_t f\|_2$ with even a.

1.1. Let $f\colon \mathbb{R}^d \to \mathbb{C}$ be a 2π-periodic function of d variables belonging to $L_2(\mathbb{T}^d)$, where the d-dimensional torus $\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$ is regarded as the d-dimensional cube $[0,2\pi]^d$ with identified opposite faces and let

$$\|f\|_2^2 = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} |f(x)|^2\,dx.$$

For $\mathbf{s} \in \mathbb{Z}^d$, $\mathbf{x} \in \mathbb{T}^d$ we set[1]

$$e_{\mathbf{s}}(\mathbf{x}) := e^{i\mathbf{s}\mathbf{x}}, \qquad \mathbf{s}\mathbf{x} = (\mathbf{s},\mathbf{x}) = \sum_{k=1}^{d} s_k x_k.$$

In a similar way $e_k(y) := e^{iky}$ for $k \in \mathbb{Z}$, $y \in \mathbb{T}$.

Let $\{\theta_k\}_{k\in\mathbb{Z}}$ be a fixed complex number sequence such that the series $\sum_{k\in\mathbb{Z}}\theta_k$ converges absolutely. Fixing a vector $\mathbf{t} \in \mathbb{T}^d$ we define a difference operator $\Delta_{\mathbf{t}}^\theta$ on the set of trigonometric polynomials $\mathcal{T} = \operatorname{lin}\{e_{\mathbf{s}}(\,\cdot\,) : \mathbf{s} \in \mathbb{Z}^d\}$ by the formula

$$\Delta_{\mathbf{t}}^\theta f(\mathbf{x}) \equiv \Delta_{\mathbf{t}}^\theta(f,\mathbf{x}) := \sum_{k\in\mathbb{Z}} \theta_k f(\mathbf{x}+k\mathbf{t}) \quad \text{for all } f \in \mathcal{T}.$$

From this formula we conclude, in particular, that

$$\Delta_{\mathbf{t}}^\theta e_{\mathbf{s}}(\,\cdot\,) = \sum_{k\in\mathbb{Z}} \theta_k e_{\mathbf{s}}(k\mathbf{t}) e_{\mathbf{s}}(\,\cdot\,) = \sum_{k\in\mathbb{Z}} \theta_k e_k(\mathbf{s}\mathbf{t}) e_{\mathbf{s}}(\,\cdot\,) \quad \text{for all } \mathbf{s} \in \mathbb{Z}^d$$

and

$$\biggl|\sum_{k\in\mathbb{Z}} \theta_k e_k(\mathbf{s}\mathbf{t})\biggr| \leqslant \sum_{k\in\mathbb{Z}} |\theta_k| \qquad \text{for each } \mathbf{s} \in \mathbb{Z}^d.$$

This means that the operator $\Delta_{\mathbf{t}}^\theta$ can be extended from the linear manifold $\mathcal{T}$ to a bounded linear operator on the space $L_2(\mathbb{T}^d)$ with norm

$$\|\Delta_{\mathbf{t}}^\theta\|_{L_2\to L_2} = \sup_{\mathbf{s}\in\mathbb{Z}^d}\biggl|\sum_{k\in\mathbb{Z}} \theta_k e_k(\mathbf{s}\mathbf{t})\biggr|$$

(see [1], § 16.1). It is a natural condition that a generalized difference operator must vanish at constants. One can easily verify that this is equivalent to the equality $\sum_{k\in\mathbb{Z}}\theta_k = 0$.

[1] We use 'plain' (not bold) script for scalars.

Let $\theta^{\vee}(\,\cdot\,)$ be a 2π-periodic function with Fourier coefficients θ_k. It follows from the condition $\sum_{k\in\mathbb{Z}}|\theta_k|<\infty$ that the series $\sum_{k\in\mathbb{Z}}\theta_k e_k(y)$ converges uniformly. Hence $\theta^{\vee}(\,\cdot\,)\in C(\mathbb{T})$ and

$$\|\Delta^{\theta}_{\mathbf{t}}\|_{L_2\to L_2}=\sup_{\mathbf{s}\in\mathbb{Z}^d}\biggl|\sum_{k\in\mathbb{Z}}\theta_k e_k(\mathbf{st})\biggr|=\sup_{\mathbf{s}\in\mathbb{Z}^d}|\theta^{\vee}(\mathbf{st})|\leqslant\|\theta^{\vee}(\,\cdot\,)\|_{C(\mathbb{T})}\qquad\text{for all }\mathbf{t}. \tag{1.1}$$

We set by definition $\psi_\theta(\,\cdot\,)=|\theta^{\vee}(\,\cdot\,)|^2$. It follows from the representation $f(x+k)=\sum_{\mathbf{s}\in\mathbb{Z}^d}\widehat{f}_{\mathbf{s}}e(x+k)$ and the definition of the operator Δ^{θ} that

$$\begin{aligned}\Delta^{\theta}f(x)&=\sum_{k\in\mathbb{Z}}\theta_k f(x+k)=\sum_{k\in\mathbb{Z}}\theta_k\biggl(\sum_{\mathbf{s}\in\mathbb{Z}^d}\widehat{f}_{\mathbf{s}}e(x+k)\biggr)\\&=\sum_{\mathbf{s}\in\mathbb{Z}^d}\biggl(\sum_{k\in\mathbb{Z}}\theta_k e_k(x)\biggr)\widehat{f}_{\mathbf{s}}e(x)=\sum_{\mathbf{s}\in\mathbb{Z}^d}\theta^{\vee}(x)\widehat{f}_{\mathbf{s}}e(x),\end{aligned}$$

where all equalities hold for almost all x. Bearing in mind Parseval's equality we now obtain

$$\|\Delta^{\theta}_{\mathbf{t}}f(\,\cdot\,)\|_2^2=\sum_{\mathbf{s}\in\mathbb{Z}^d}|\theta^{\vee}(\mathbf{st})|^2|\widehat{f}_{\mathbf{s}}|^2=\sum_{\mathbf{s}\in\mathbb{Z}^d}\psi_\theta(\mathbf{st})|\widehat{f}_{\mathbf{s}}|^2\quad\text{for all }f\in L_2(\mathbb{T}^d). \tag{1.2}$$

The function $\theta^{\vee}(\,\cdot\,)$ can be interpreted as the restriction to the unit circle of the so-called *characteristic function of the difference operator* Δ^{θ}_{t}, which we define by the formula

$$\chi_\theta(z)=\sum_{k\in\mathbb{Z}}\theta_k z^k,\qquad z\in\mathbb{C}.$$

Thus, $\theta^{\vee}(t)=\chi_\theta(e_1(t))$.

We also point out that if at least one component of the vector $\mathbf{t}/\pi$ is irrational, then

$$\|\Delta^{\theta}_{\mathbf{t}}\|_{L_2\to L_2}=\|\theta^{\vee}(\,\cdot\,)\|_{C(\mathbb{T})}. \tag{1.3}$$

Indeed, we can assume without loss of generality that the first component of $\mathbf{t}/\pi$ is irrational. Then the sequence $\{st_1\}_{s\in\mathbb{Z}}$ taken modulo 2π is dense on $\mathbb{T}$, therefore

$$\|\theta^{\vee}(\,\cdot\,)\|_{C(\mathbb{T})}\geqslant\sup_{\mathbf{s}\in\mathbb{Z}^d}|\theta^{\vee}(\mathbf{st})|\geqslant\sup_{s\in\mathbb{Z}}|\theta^{\vee}(st_1)|=\|\theta^{\vee}(\,\cdot\,)\|_{C(\mathbb{T})}.$$

It now follows from (1.1) that

$$\|\Delta^{\theta}_{\mathbf{t}}\|_{L_2\to L_2}=\sup_{\mathbf{s}\in\mathbb{Z}^d}|\theta^{\vee}(\mathbf{st})|=\|\theta^{\vee}(\,\cdot\,)\|_{C(\mathbb{T})}. \tag{1.4}$$

We also observe that if all the quantities θ_k are real, then the equalities

$$\begin{aligned}\biggl|\sum_{k\in\mathbb{Z}}\theta_k e_k(\tau)\biggr|^2&=\biggl(\sum_{k\in\mathbb{Z}}\theta_k e_k(\tau)\biggr)\biggl(\overline{\sum_{k\in\mathbb{Z}}\theta_k e_k(\tau)}\biggr)=\sum_{k,s\in\mathbb{Z}}\theta_k\theta_s e_k(\tau)e_s(-\tau)\\&=\sum_{s,k\in\mathbb{Z}}\theta_s\theta_k e_s(-\tau)e_k(-(-\tau))=\biggl|\sum_{k\in\mathbb{Z}}\theta_k e_k(-\tau)\biggr|^2\end{aligned}$$

imply that $\psi(\,\cdot\,)=\psi_\theta(\,\cdot\,)=|\theta^{\vee}(\,\cdot\,)|^2$ is an even function.

For a fixed sequence θ and a closed subset T of $[0,2\pi]^d$ we now define the *generalized θ-modulus of continuity* as follows:

$$\omega_\theta(f,T) := \max_{\mathbf{t}\in T} \|\Delta^\theta_{\mathbf{t}} f(\,\cdot\,)\|_2 \qquad \forall\, f\in L_2(\mathbb{T}^d).$$

In a similar way, for a fixed non-negative continuous 2π-periodic function $\psi(\,\cdot\,)$ and a closed subset T of $[0,2\pi]^d$ we define the *generalized ψ-modulus of continuity* by the formula

$$\overline{\omega}_\psi(f,T) := \max_{\mathbf{t}\in T} \sqrt{\sum_{\mathbf{s}\in\mathbb{Z}^d} \psi(\mathbf{s}\mathbf{t})|\widehat{f}_{\mathbf{s}}|^2}\,.$$

By (1.2) we obtain $\omega_\theta(f,T)=\overline{\omega}_\psi(f,T)$ for ψ and θ satisfying the equality $\psi(\,\cdot\,)=|\theta^\vee(\,\cdot\,)|^2$.

Remark 1. By the Cauchy criterion the condition $\sum_k |\theta_k|<\infty$ implies that there exists a finite Borel measure $\sigma^\theta_t(\,\cdot\,)$ on $\mathbb{T}^d$ that is a $*$-weak limit of partial sums of the series $\sum_k \theta_k\delta_{k\mathbf{t}}(\,\cdot\,)$. Here $\delta_{k\mathbf{t}}(\,\cdot\,)$ is the Dirac measure concentrated at the point $k\mathbf{t} \pmod{(2\pi)^d}$ and

$$\int_{\mathbb{T}^d} f(x)\,d\delta_{k\mathbf{t}} = f(k\mathbf{t} \pmod{(2\pi)^d}) \quad \text{for all } f\in C(\mathbb{T}^d).$$

It immediately follows that

$$\Delta^\theta_{\mathbf{t}} f(\mathbf{x}) = \sigma^\theta_t * f(\mathbf{x}) \qquad \text{for almost all } \mathbf{x}$$

for each $f\in L_2(\mathbb{T}^d)$. Hence

$$\omega_\theta(f,T) \equiv \sup_{\mathbf{t}\in T} \|\Delta^\theta_{\mathbf{t}}\|_2 = \sup_{\mathbf{t}\in T} \|\sigma^\theta_t * f\|_2, \qquad f\in L_2(\mathbb{T}^d).$$

Now, setting $\sigma_{(t)}(\,\cdot\,) := \sigma^\theta_t(\,\cdot\,)$ we deduce that for $d=1$ and $p=2$ the quantity $\omega_\theta(f,[0,\delta])$ is the value of the (σ,p)-modulus of continuity

$$\omega_{\sigma,p}(f,\delta) := \sup_{0\leqslant t\leqslant \delta} \|\sigma_{(t)} * f\|_p$$

introduced by Shapiro in [2] (see also [3], [4]).

1.2. Let Λ be a proper subset of $\mathbb{Z}^d$ containing the origin $\mathbf{0}$. We denote by $E_\Lambda(f)=E_\Lambda(f)_2$ the best approximation to $f\in L_2(\mathbb{T}^d)$ by functions with spectrum concentrated in Λ:

$$E_\Lambda(f)_2 = \sqrt{\sum_{\mathbf{k}\notin\Lambda} |\widehat{f}_{\mathbf{k}}|^2}\,.$$

Let T be a closed subset of $\mathbb{T}^d$. We denote by $\varkappa^{(d)}_\theta(\Lambda,T)$ the *sharp constant* in Jackson's inequality in $L_2(\mathbb{T}^d)$ for a generalized θ-modulus of continuity:

$$E_\Lambda(f) \leqslant C\omega_\theta(f,T) \quad \text{for all } f\in L_2(\mathbb{T}^d).$$

That is,[2]

$$\varkappa_\theta^{(d)}(\Lambda, T) = \sup\biggl\{\frac{E_\Lambda(f)}{\omega_\theta(f,T)} : f \in L_2(\mathbb{T}^d)\biggr\}.$$

Remark. We observe that even for $d = 1$ and the classical difference there exists a positive number sequence $\{\delta_k\}_{k=1}^\infty$ approaching zero such that for no k is the interval $T_k = [0, \delta_k]$ an extremal set in the following minimization problem for sharp constants:

$$\sup\biggl\{\frac{E_{n-1}(f)}{\omega(f,T)} : f \in L_2(\mathbb{T}^d)\biggr\} \to \inf,$$

$\operatorname{meas} T = \delta_k$, T is a closed subset of $\mathbb{T}$ (see [5]).

In a similar way, if a generalized modulus of continuity is defined in terms of a function ψ, then we set

$$\overline{\varkappa}_\psi^{(d)}(\Lambda, T) := \sup\biggl\{\frac{E_\Lambda(f)}{\overline{\omega}_\psi(f,T)} : f \in L_2(\mathbb{T}^d)\biggr\}.$$

The function $\overline{\varkappa}_\psi^{(d)}(\Lambda, \cdot\,)$ is monotone in the following sense:

$$A \subset B \quad \Longrightarrow \quad \overline{\varkappa}_\psi^{(d)}(\Lambda, A) \geqslant \overline{\varkappa}_\psi^{(d)}(\Lambda, B). \tag{1.5}$$

Note that if a non-negative 2π-periodic function $\psi(t)$ is continuous and vanishes at the point $t = 0$ and the set Λ is finite, then $\overline{\varkappa}_\psi^{(d)}(\Lambda, T) = +\infty$ for each countable closed set T (see Lemma 3 for the proof).

For $d = 1$ we set $\omega_\theta(f, \delta) := \omega_\theta(f, [0, \delta])$ and $\overline{\omega}_\psi(f, \delta) := \overline{\omega}_\psi(f, [0, \delta])$. Correspondingly, for $n \in \mathbb{N}$ we set $\varkappa_\theta(n, \delta) := \varkappa_\theta^{(1)}\bigl(\{-n+1, \dots, n-1\}, [0, \delta]\bigr)$ and $\overline{\varkappa}_\psi(n, \delta) := \overline{\varkappa}_\psi^{(1)}\bigl(\{-n+1, \dots, n-1\}, [0, \delta]\bigr)$

In the case of arbitrary $d \in \mathbb{N}$ and the sequence θ corresponding to the classical difference of order r we write $\varkappa_r^{(d)}(\Lambda, T)$ in place of $\varkappa_\theta^{(d)}(\Lambda, T)$.

Definition 1.1. The quantity

$$\overset{*}{\varkappa}{}_\psi^{(d)}(\Lambda) := \min\bigl\{\overline{\varkappa}_\psi^{(d)}(\Lambda, T) : T \text{ is a closed subset of } \mathbb{T}^d\bigr\}$$

is called the *minimum sharp constant in Jackson's inequality for a generalized ψ-modulus of continuity in $L_2(\mathbb{T}^d)$*.

It immediately follows from property (1.5) that $\overset{*}{\varkappa}{}_\psi^{(d)}(\Lambda) = \overline{\varkappa}_\psi^{(d)}(\Lambda, [0, 2\pi]^d)$. In the case of Jackson's inequality for the classical modulus of order r we denote this constant by $\overset{*}{\varkappa}{}_r^{(d)}(\Lambda)$.

1.3. Throughout this section $d = 1$. Let $i(n)$ be the number of ones in the binary notation of $n \in \mathbb{N}$ and let

$$\gamma_n := \begin{cases} 0 & \text{if } i(n) \equiv 0 \pmod 2; \\ 1 & \text{if } i(n) \equiv 1 \pmod 2. \end{cases}$$

[2]Here and in what follows we assume that $0/0 = 0$; $c/0 = +\infty$ for $c > 0$.

The sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ is known as the *Thue–Morse sequence*[3] [7], [8]. We point out several properties of this sequence. First of all it is easy to verify by induction that

$$\prod_{k=0}^{r-1}(1-z^{2^k}) = 1 + \sum_{k=1}^{2^r-1}(-1)^{\gamma_k} z^k \tag{1.6}$$

for all $r \in \mathbb{N}$. On the other hand, we can obtain the quantities γ_k by the following procedure.

We construct recursively words u_i, v_i on the alphabet of two letters, "0" and "1", $u_0 := (0)$, $v_0 := (1)$, $u_{i+1} := (u_i, v_i)$, $v_{i+1} := (v_i, u_i)$. Hence $u_r = (0, \gamma_1, \dots, \gamma_{2^r-1})$. The finite difference operator

$$\widetilde{\Delta}_t^r f(x) = \widetilde{\Delta}_t^r(f,x) := f(x) + \sum_{k=1}^{2^r-1}(-1)^{\gamma_k} f(x+kt)$$

if called the *Thue–Morse difference of order* r. It is obvious that $\widetilde{\Delta}_t^r f = \Delta_t^{\widetilde{\theta}} f$ for all $f \in L_2(\mathbb{T})$, where

$$\widetilde{\theta}_k = \widetilde{\theta}_k(r) = \begin{cases} 0 & \text{if } k \notin \{0,\dots,2^r-1\}; \\ (-1)^{\gamma_k} & \text{if } k \in \{0,\dots,2^r-1\}. \end{cases}$$

(Here $\gamma_0 := 0$.) Taking account of formula (1.6) we see that the characteristic function of $\widetilde{\Delta}_t^r$ is the algebraic polynomial $\prod_{k=0}^{r-1}(1-z^{2^k})$, and therefore

$$\widetilde{\theta}^{\vee}(x) = \chi_{\widetilde{\theta}}(e_1(x)) = 1 + \sum_{k=1}^{2^r-1}(-1)^{\gamma_k} e_k(x) = \prod_{k=0}^{r-1}(1-e_{2^k}(x)).$$

Hence

$$\psi_r(x) := |\widetilde{\theta}^{\vee}(x)|^2 = 2^{2r}\prod_{k=0}^{r-1}\sin^2(2^{k-1}x) = 2^r\prod_{k=0}^{r-1}(1-\cos(2^k x)). \tag{1.7}$$

We point out one important property of the operator $\widetilde{\Delta}_t^r$: its L_2-norm has the estimate $(2/\sqrt{3}\,)\cdot 3^{r/2}$ (see Lemma 7). For a comparison we now calculate the L_2-norm of the 'classical' Newton–Gregory difference of order r for t incommensurate with π:

$$\Delta_t^r f(\,\cdot\,) = \bigl(\Delta_t\bigr)^r f(\,\cdot\,) = \sum_{k=0}^{r}(-1)^k C_r^k f(\,\cdot\, + kt).$$

It is obvious that the characteristic function of the classical difference is the polynomial $\chi(z) := (1-z)^r$. We have $\theta^{\vee}(t) = \chi(e_1(t)) = (1-e_1(t))^r$. Hence $|\theta^{\vee}(t)| = 2^r|\sin^r(t/2)|$, and by (1.4) we obtain

$$\|\Delta_t^r\|_{L_2\to L_2} = \|\theta^{\vee}(\,\cdot\,)\|_{C(\mathbb{T})} = \|2^r\sin^r(\,\cdot\,/2)\|_{C(\mathbb{T})} = 2^r.$$

[3]Sometimes it is called the *Prouhet–Thue–Morse sequence* or even the *Prouhet sequence* since some its properties were studied in [6].

We now derive a formula connecting the Thue–Morse difference of order r with the classical difference Δ_t^r. To this end we consider the translation operator $E_t[f(\,\cdot\,)] := f(\,\cdot\, + t)$. Clearly, the operator E_t is linear and $E_{kt} = (E_t)^k$. It follows by (1.6) that

$$\widetilde{\Delta}_t^r = \prod_{k=0}^{r-1} \Delta_{2^k t} = \prod_{k=0}^{r-1} (I - (E_t)^{2^k}),$$

where I is the identity operator. Moreover, $I - (E_t)^{2^k} = \bigl(\sum_{s=0}^{2^k-1} (E_t)^s\bigr)(I - E_t)$ and therefore

$$\widetilde{\Delta}_t^r f = \Biggl(\prod_{k=0}^{r-1} \sum_{s=0}^{2^k-1} (E_t)^s\Biggr)(I - E_t)^r f = \Biggl(\prod_{k=0}^{r-1} \sum_{s=0}^{2^k-1} (E_t)^s\Biggr)\Delta_t^r f. \tag{1.8}$$

We conclude from the last formula that the Thue–Morse difference of order r annihilates the space of algebraic polynomials of degree $r-1$ or less.

It is convenient to use the notation $\widetilde{\omega}_r(f,\delta)$ for the θ-modulus of continuity corresponding to the Thue–Morse difference of order r and the set $[0,\delta]$, $\delta > 0$.

We now present several important properties of the Thue–Morse differences, which are similar to the corresponding properties of the 'classical' differences (see Lemmas 6 and 8):

(a) $\widetilde{\Delta}_h^r f(x) = (-1)^r \int_x^{x+h} dt_1 \int_{t_1}^{t_1+2h} dt_2 \int_{t_2}^{t_2+2^2h} dt_3 \cdots \int_{t_{r-1}}^{t_{r-1}+2^{r-1}h} f^{(r)}(t_r)\,dt_r$, $f \in W_1^r(\mathbb{T})$, $h > 0$;[4]

(b) $\int_{\mathbb{T}} f(x)\widetilde{\Delta}_{-h}^r\varphi(x)\,dx = \int_{\mathbb{T}} \widetilde{\Delta}_h^r f(x)\varphi(x)\,dx$, $f,\varphi \in L_2(\mathbb{T})$;

(c) let $f \in L_p([t_1,t_2])$ and $[\alpha,\beta] \subset (t_1,t_2)$; then $\|\widetilde{\Delta}_h^r(f,\,\cdot\,)\|_{L_p([\alpha,\beta])} = o(h^r)$ as $h \to +0$, $\iff$ $f(x) = P(x)$ for almost all $x \in [\alpha,\beta]$, where $P \in \mathcal{P}_{r-1}$ is an algebraic polynomial of degree at most $r-1$;[5]

(d) $\widetilde{\omega}_r(f,t) \leqslant 2^{r(r-1)/2} t^r \|f^{(r)}\|_2$, $f \in W_2^r(\mathbb{T})$, $t > 0$;

(e) $\widetilde{\omega}_r(f,n\delta) \leqslant n^r \widetilde{\omega}_r(f,\delta)$, $f \in L_2(\mathbb{T})$, $n \in \mathbb{N}$, $\delta > 0$;

(f) $\widetilde{\omega}_r(f,\delta) \overset{\delta\to 0+}{=} o(\delta^r) \implies \widetilde{\omega}_r(f,\,\cdot\,) \equiv 0$, $f \in L_2(\mathbb{T})$;

(g) if $f \in W_2^r(\mathbb{T})$, $\int_0^{2\pi} f(x)\,dx = 0$ and $t/(2\pi)$ is an irrational quantity with bounded convergents,[6] then there exists a function $g \in L_2(\mathbb{T})$ such that

$$f(x) = \widetilde{\Delta}_t^r g(x) \quad \text{for almost all } x.$$

[4] Here and in what follows $W_p^r(\mathbb{T})$ is the Sobolev space of 2π-periodic functions with absolutely continuous $(r-1)$th derivative such that $f^{(r)} \in L_p(\mathbb{T})$.

[5] We define the norm in $L_p(I)$, where I is a finite interval, as $\|f\|_{L_p(I)} = \left(\int_I |f(x)|^p\,dx\right)^{1/p}$.

[6] Recall that the boundedness of convergents to a quantity $\alpha > 0$ means the uniform boundedness of the coefficients a_k of the continued fraction expansion

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}.$$

We finally note that the Thue–Morse difference is a particular case (for $a = 2$) of the finite difference operator $\widetilde{\Delta}_t^{a,r}$ defined by the formula

$$\widetilde{\Delta}_t^{a,r} f(x) = \widetilde{\Delta}_t^{a,r}(f,x) := \biggl(\prod_{k=0}^{r-1} \Delta_{a^k t}\biggr) f(x).$$

(We call it the *generalized Thue–Morse difference*.) In another special case $a = 1$ the difference $\widetilde{\Delta}_t^{a,r} f(x)$ is identical to the classical difference $\Delta_t^r f(x)$.

For simplicity assume that a is a positive integer. Then the characteristic function of the above operator is the algebraic polynomial $\chi_{a,r}(z) = \prod_{k=0}^{r-1}(1 - z^{a^k})$ and, accordingly,

$$\begin{aligned}\psi_{a,r}(t) &:= |\chi_{a,r}(e_1(t))|^2 = \biggl|\prod_{k=0}^{r-1}(1 - e_{a^k}(t))\biggr|^2 \\ &= 2^{2r}\prod_{k=0}^{r-1}\sin^2\biggl(\frac{a^k}{2}\,t\biggr) = 2^r\prod_{k=0}^{r-1}(1 - \cos(a^k t)). \qquad (1.9)\end{aligned}$$

We denote by $\widetilde{\omega}_{a,r}(\,\cdot\,,\cdot\,)$ the modulus of continuity corresponding to the generalized Thue–Morse difference. Let us also introduce our notation for the sharp constant in Jackson's inequality for the generalized Thue–Morse difference:

$$\widetilde{\varkappa}_{a,r}(n,\delta) := \sup\biggl\{\frac{E_n(f)_2}{\widetilde{\omega}_{a,r}(f,\delta)} : f \in L_2(\mathbb{T})\biggr\}.$$

We now present some known results on sharp constants in Jackson's inequality for the classical modulus of continuity in $L_2(\mathbb{T})$ that are close to our research.

Theorem A [9]. *Let $n \in \mathbb{N}$ and suppose that $\delta \geqslant \pi/n$. Then*

$$\varkappa_1(n,\delta) = \frac{1}{\sqrt{2}}\,.$$

Remark 2. It was proved in [10] that the estimate of the sharp constant in Theorem A is best possible in the following sense: $\varkappa_1(n,\delta) > 1/\sqrt{2}$ for $\delta \in (0,\pi/n)$ and $n \in \mathbb{N}$.

Theorem B [11]. *Let $n, r \in \mathbb{N}$, $r \geqslant 2$. Then*

$$\varkappa_r\biggl(n, \frac{2\pi}{n}\biggr) \leqslant \frac{1}{\sqrt{C_{2r}^r}}\,.$$

Moreover, if $n > r$, then

$$\varkappa_r\biggl(n, \frac{2\pi}{n}\biggr) = \frac{1}{\sqrt{C_{2r}^r}}\,.$$

Theorem C [12]. *Let $r \in \mathbb{N}$, $r \geqslant 2$. Then*

$$\varkappa_r(1, 2\pi) = \frac{1}{\sqrt{C_{2r}^r}}\,.$$

We note that

$$C_{2r}^r = \sum_{k=0}^{r} (C_r^k)^2 = \frac{1}{2\pi}\int_0^{2\pi} 4^r \sin^{2r}\left(\frac{x}{2}\right) dx = \frac{1}{2\pi}\int_0^{2\pi} \psi_{1,r}(x)\, dx. \tag{1.10}$$

Theorem D [13].[7] *Let $V \subset \mathbb{R}^d$ be a bounded convex centrally symmetric closed body and let $\overline{V} = V \cap \mathbb{Z}^d$. Then*

$$\overset{*}{\varkappa}{}_1^{(d)}(\overline{V}) = \frac{1}{\sqrt{2}}\,.$$

We now indicate several results on lower bounds for sharp constants in Jackson's inequality with generalized modulus of continuity in $L_2(\mathbb{T})$.

Definition 1.2. We denote by $C^+(\mathbb{T})$ the class of continuous non-negative 2π-periodic functions not vanishing identically, and by $C_0^+(\mathbb{T})$ its subclass of functions vanishing at $x = 0$:

$$C_0^+(\mathbb{T}) := \bigl\{f(\,\cdot\,) \in C(\mathbb{T}) : f(\,\cdot\,) \geqslant 0,\ f(0) = 0,\ f(\,\cdot\,) \not\equiv 0\bigr\}.$$

We now consider the function class

$$\Psi^V(\mathbb{T}) := \biggl\{\psi(\,\cdot\,) \in C_0^+(\mathbb{T}) : \psi(-x) \equiv \psi(x),\\ \frac{1}{t}\int_0^t \psi(x)\, dx \leqslant \frac{1}{\pi}\int_0^\pi \psi(x)\, dx \quad \forall\, t \in (0, \pi)\biggr\}.$$

The class $\Psi^V(\mathbb{T})$ was introduced by Vasil'ev in [14]. It was proved there that

$$\overline{\varkappa}_\psi(n, \delta) \leqslant \left(\frac{1}{2\pi}\int_0^{2\pi} \psi(x)\, dx\right)^{-1/2} \quad \text{for } \delta \geqslant \frac{7}{5}\,\frac{\pi}{n}\,, \quad n \in \mathbb{N}.$$

An even function $\psi \in C_0^+(\mathbb{T})$ belongs to the class $\Psi^V(\mathbb{T})$ if, for example, ψ is non-decreasing on $[0, \pi]$. In particular, the function $\psi^r(t) = 4^r \sin^{2r}(t/2)$ corresponding to the classical difference of order r belongs to $\Psi^V(\mathbb{T})$. However, it is not difficult to show that in the case of the function $\psi_r(t) = 4^r \prod_{k=0}^{r-1} \sin^2(2^{k-1} t)$ corresponding to the Thue–Morse difference the condition

$$\frac{1}{t}\int_0^t \psi_r(x)\, dx \leqslant \frac{1}{\pi}\int_0^\pi \psi_r(x)\, dx \qquad \text{for each } t \in (0, \pi)$$

fails for $r \geqslant 2$.

[7] Formally, in the cited paper the author investigates the problem of the approximation of functions in $L_2(\mathbb{R}^d)$ by entire functions of exponential type. However, we can easily transform the corresponding statement into this theorem using the standard periodization techniques.

Remark. We now show that for $r \in \mathbb{N}$, $r \geqslant 2$ the function

$$\psi_r(t) = 4^r \prod_{k=0}^{r-1} \sin^2(2^{k-1}t)$$

does not belong to $\Psi^V(\mathbb{T})$. We claim that for such r we have the relation

$$\frac{1}{\pi-\varepsilon}\int_0^{\pi-\varepsilon} \psi_r(x)\,dx = J_\psi + \frac{\varepsilon}{\pi}J_\psi + o(\varepsilon) \quad \text{as } \varepsilon \to 0, \tag{1.11}$$

where $J_\psi = \dfrac{1}{\pi}\displaystyle\int_0^\pi \psi_r(x)\,dx$. Indeed,

$$\begin{aligned}
\frac{1}{\pi-\varepsilon}\int_0^{\pi-\varepsilon} \psi_r(x)\,dx &= \frac{1}{\pi-\varepsilon}\biggl(\int_0^\pi - \int_{\pi-\varepsilon}^\pi\biggr) \\
&= \frac{1}{\pi}\Bigl(1-\frac{\varepsilon}{\pi}\Bigr)^{-1}\biggl(\int_0^\pi - \int_{\pi-\varepsilon}^\pi\biggr) = \frac{1}{\pi}\Bigl(1+\frac{\varepsilon}{\pi}+o(\varepsilon)\Bigr)\biggl(\int_0^\pi - \int_{\pi-\varepsilon}^\pi\biggr) \\
&= J_\psi + \frac{\varepsilon}{\pi}J_\psi + o(\varepsilon) - \frac{1}{\pi}\Bigl(1+\frac{\varepsilon}{\pi}+o(\varepsilon)\Bigr)\int_{\pi-\varepsilon}^\pi \psi_r(x)\,dx.
\end{aligned}$$

To prove (1.11) it is sufficient to show that $\displaystyle\int_{\pi-\varepsilon}^\pi \psi_r(x)\,dx = O(\varepsilon^{2r-1})$ as $\varepsilon \to 0$. In the last integral we make the change of the variable $z = \pi - x$ and denote z by x again:

$$\begin{aligned}
\int_{\pi-\varepsilon}^\pi \psi_r(x)\,dx &= 2^r \int_0^\varepsilon (1+\cos x)\prod_{k=1}^{r-1}(1-\cos(2^k x))\,dx \\
&= 2^r \int_0^\varepsilon (2+o(x))\prod_{k=1}^{r-1}(2^{2k-1}x^2 + o(x^2))\,dx \\
&= 2^{r+1}\cdot 2^{1+3+5+\dots+(2r-1)}\int_0^\varepsilon (x^{2(r-1)} + o(x^{2(r-1)}))\,dx \\
&= C_r\cdot\varepsilon^{2r-1} + o(\varepsilon^{2r-1}), \qquad \varepsilon \to 0.
\end{aligned}$$

We have thus proved equality (1.11). It follows from (1.11), in particular, that for $r \in \mathbb{N}$, $r \geqslant 2$, there exists ε_0 such that for each $\varepsilon \in (0,\varepsilon_0)$,

$$\frac{1}{\pi-\varepsilon}\int_0^{\pi-\varepsilon} \psi_r(x)\,dx > J_\psi.$$

Hence the function $\psi_r(x)$ does not belong to the class $\Psi^V(\mathbb{T})$.

Theorem E [14]. *Let $\psi \in \Psi^V(\mathbb{T})$, $n \in \mathbb{N}$, and assume that $\delta > 0$. Then*

$$\overline{\varkappa}_\psi(n,\delta) \geqslant \biggl(\frac{1}{2\pi}\int_0^{2\pi}\psi(x)\,dx + \omega\Bigl(\psi,\frac{\delta}{2}\Bigr)_\infty\biggr)^{-1/2}.$$

Here $\omega(\psi,h)_\infty = \max_{0\leqslant t\leqslant h}\|\Delta_t\psi(\,\cdot\,)\|_{C(\mathbb{T})}$ is the usual 'uniform' modulus of continuity.

Theorem F [5].[8] *Let* $r, n \in \mathbb{N}$, *let* $\theta_k \in \mathbb{R}$, $\sum_{k=0}^{r} \theta_k = 0$, $\theta_k = 0$ *for* $k \notin \{0, \dots, r\}$, *and assume that* $0 < \delta < 2\pi/r$. *Then*

$$\varkappa_\theta(n, \delta) \geqslant \frac{1}{\sqrt{\sum_{k=0}^{r} \theta_k^2}} \,.$$

We now present a result on the sharp constant in Jackson's inequality for the Thue–Morse modulus in $L_2(\mathbb{T})$.

Theorem G [15].[9] *Let* $n \in \mathbb{N}$, $r = 2, 3, \dots, 7$, *and assume that*

$$n > \frac{9}{1 + 2\sqrt{7}}\, 2^{r-1}, \qquad \frac{9}{1 + 2\sqrt{7}}\, \frac{\pi}{n} \leqslant \delta < \frac{\pi}{2^{r-1}} \,.$$

Then

$$\widetilde{\varkappa}_{2,r}(n, \delta) = \frac{1}{\sqrt{2^r}} \,.$$

Note that $2^r = 1^2 + \sum_{k=1}^{2^r-1} |(-1)^{\gamma_k}|^2$, where $\{\gamma_k\}$ is the Thue–Morse sequence.

We now present the main statements of this paper.

Definition 1.3. We say that a subset Λ of $\mathbb{Z}^d$ belongs to the *class* Λ_*^d if it contains the origin $\mathbf{0}$ and its intersection with at least one of the coordinate half-axes is a finite set. Thus,

$$\Lambda_*^d = \bigl\{\Lambda \subset \mathbb{Z}^d : \mathbf{0} \in \Lambda, (\operatorname{card}(\Lambda \cap \epsilon_j^+) < \infty \text{ or } \operatorname{card}(\Lambda \cap \epsilon_j^-) < \infty) \text{ for such } j\bigr\},$$

where $\epsilon_j^{+(-)} = \{\lambda \in \mathbb{Z}^d : \lambda_j \geqslant (\leqslant)\, 0,\ \lambda_k = 0,\ k \neq j\}$.

Theorem 1. *Let* $\psi \in C_0^+(\mathbb{T})$, $\Lambda \in \Lambda_*^d$. *Then*

$$\overset{*}{\varkappa}{}_\psi^{(d)}(\Lambda) = \biggl(\frac{1}{2\pi} \int_0^{2\pi} \psi(x)\, dx\biggr)^{-1/2} .$$

Setting $\psi(x) = 4\sin^2(x/2)$ we obtain Theorem D as a particular case of the above result.

Corollary 1.1. *Let* $\psi \in C_0^+(\mathbb{T})$, $\psi(-x) \equiv \psi(x)$, $n \in \mathbb{N}$, *and assume that* $\delta \geqslant \pi$. *Then*

$$\overline{\varkappa}_\psi(n, \delta) = \biggl(\frac{1}{2\pi} \int_0^{2\pi} \psi(x)\, dx\biggr)^{-1/2} .$$

This statement easily yields Theorem E.

[8] For simplicity we formulate Theorem F in the special case of finite differences with constant coefficients. In the cited paper a similar result was proved for all difference operators the coefficients of which are continuous functions of the step t satisfying the condition $\sum_{k=0}^{r} \theta_k(0) = 0$.

[9] In fact, in [15] this equality was also announced in the case when in the definition of a sharp constant one considers approximations by an arbitrary subspace of dimension $2n - 1$.

Corollary 1.2. *Let n, $r \in \mathbb{N}$, $r \geqslant 2$. Then*

$$\varkappa_r(n,\pi) = \frac{1}{\sqrt{C_{2r}^r}}\,.$$

In particular, this result improves Theorem C and allows us to drop the constraint $n > r$ in the second part of Theorem B.

Corollary 1.3. *Let T be a closed subset of $[0,2\pi]^d$, let $\Lambda \in \Lambda_*^d$, $\{\theta_s\}_{s\in\mathbb{Z}} \subset \mathbb{C}$, assume that $0 < \sum_{s\in\mathbb{Z}} |\theta_s| < \infty$, $\sum_{s\in\mathbb{Z}} \theta_s = 0$, and let $n \in \mathbb{N}$. Then*

$$\varkappa_\theta^{(d)}(\Lambda, T) \geqslant \frac{1}{\sqrt{\sum_{s\in\mathbb{Z}} |\theta_s|^2}}\,;$$

moreover,

$$\varkappa_\theta^{(d)}(\Lambda, [0,2\pi]^d) = \frac{1}{\sqrt{\sum_{s\in\mathbb{Z}} |\theta_s|^2}}\,.$$

In the case $d = 1$ this corollary yields Theorem F.

Definition 1.4. For $\psi \in C_0^+(\mathbb{T})$ we set

$$J_\psi(x) := \frac{1}{2\pi}\int_0^{2\pi x} \psi(\tau)\,d\tau, \qquad \overline{J}_\psi(x) := \min_{0\leqslant\delta\leqslant 1} \frac{1}{2\pi}\int_{2\pi\delta}^{2\pi(\delta+x)} \psi(\tau)\,d\tau.$$

It is clear that

$$\overline{J}_\psi(1) = J_\psi(1) = J_\psi := \frac{1}{2\pi}\int_0^{2\pi} \psi(\tau)\,d\tau.$$

Theorem 2. *Let $\psi \in C_0^+(\mathbb{T})$, $\psi(-x) \equiv \psi(x)$, $n \in \mathbb{N}$, $\gamma \geqslant 1$. Then*

$$\overline{\varkappa}_\psi\biggl(n, \gamma\frac{\pi}{n}\biggr) \leqslant \biggl(\min_{[\gamma]/2\leqslant x\leqslant[\gamma]/2+1} \frac{J_\psi(x)}{x}\biggr)^{-1/2}.$$

Corollary 1.4. *Let $\psi \in C_0^+(\mathbb{T})$, $\psi(-x) \equiv \psi(x)$, $n \in \mathbb{N}$, $\gamma \geqslant 1$. Then*

$$\overline{\varkappa}_\psi\biggl(n, \gamma\frac{\pi}{n}\biggr) \leqslant \sqrt{1 + \frac{1}{[\gamma]}}\biggl(\frac{1}{2\pi}\int_0^{2\pi} \psi(x)\,dx\biggr)^{-1/2}.$$

We set $\square_n := \{\mathbf{k} \in \mathbb{Z}^d : |k_j| < n,\ j = 1,\dots,d\}$.

Theorem 3. *Let $\psi \in C_0^+(\mathbb{T})$, $n \in \mathbb{N}$, $\gamma \geqslant 1$. Then*

$$\overline{\varkappa}_\psi^{(d)}\biggl(\square_n, \biggl[0, \gamma\frac{2\pi}{n}\biggr]^d\biggr) \leqslant \biggl(\min_{[\gamma]\leqslant x\leqslant[\gamma]+1} \frac{\overline{J}_\psi(x)}{x}\biggr)^{-1/2}.$$

Corollary 1.5. *Let $\psi \in C_0^+(\mathbb{T})$, $n \in \mathbb{N}$, $\gamma \geqslant 1$. Then*

$$\overline{\varkappa}_\psi^{(d)}\biggl(\square_n, \biggl[0, \gamma\frac{2\pi}{n}\biggr]^d\biggr) \leqslant \sqrt{1+\frac{1}{[\gamma]}}\biggl(\frac{1}{2\pi}\int_0^{2\pi}\psi(x)\,dx\biggr)^{-1/2}.$$

Corollary 1.6. *Let $a \in \mathbb{N}$ be an odd integer, $a \geqslant 3$, let $n \in \mathbb{N}$, and assume that $\gamma \geqslant 1$. Then*

$$\frac{1}{\sqrt{2^r}} \leqslant \widetilde{\varkappa}_{a,r}\biggl(n, \gamma\frac{\pi}{n}\biggr) \leqslant \sqrt{1+\frac{1}{[\gamma]}}\,\frac{1}{\sqrt{2^r}}\,.$$

Definition 1.5. We now define a function class Υ. We shall write $\xi(\,\cdot\,) \in \Upsilon$ if the following conditions hold:

(1) $\xi(\,\cdot\,) \in C(\mathbb{R})$;
(2) $\xi(x) \geqslant 0$ for all x;
(3) $\xi(x) \equiv \xi(\pi + x)$, that is, $\xi(\,\cdot\,)$ is a π-periodic function;
(4) $\xi(x) \equiv \xi(-x)$, that is, $\xi(\,\cdot\,)$ is an even function.

Definition 1.6. Let $\xi(\,\cdot\,) \in \Upsilon$. Then we set

$$h_\xi(x) := -\int_0^\pi \cos(xt)\xi(xt)\sin t\,dt. \tag{1.12}$$

Definition 1.7. Let $\mathbf{b} = \{b_k\}$ be a fixed finite set of positive numbers. Then we denote by $\Psi(\mathbf{b})$ the set of even functions $\psi(\,\cdot\,) \in C_0^+(\mathbb{T})$ such that the representation

$$\int_0^\pi \psi(xt)\frac{\sin t}{2}\,dt = \frac{1}{2\pi}\int_0^{2\pi}\psi(t)\,dt + \sum_k h_{\xi_k}(b_k x) \quad \text{for each } x \geqslant \widehat{b} := \max_k \frac{1}{b_k} \tag{1.13}$$

holds for some $\xi_k(\,\cdot\,) \in \Upsilon$.

Remark 3. The functions $\psi_{a,r}(\,\cdot\,)$ introduced above belong to the class $\Psi(\mathbf{b})$ for even a and $\mathbf{b} = (a^{r-1}, a^{r-2}, \dots, a^0)$ (see Lemma 15). On the other hand, for $r \geqslant 2$ the function $\psi_{1,r}(t) := 2^r(1-\cos t)^r$ 'corresponding' to the classical modulus of continuity of order r does not belong to the class $\Psi(\mathbf{b})$ for any $\mathbf{b}$.[10]

Theorem 4. *Let $\mathbf{b} = \{b_k\}$ be a finite set of positive numbers and let $\psi \in \Psi(\mathbf{b})$, $n \in \mathbb{N}$. Then*

$$\overline{\varkappa}_\psi(n,\delta) = \biggl(\frac{1}{2\pi}\int_0^{2\pi}\psi(x)\,dx\biggr)^{-1/2}$$

for all $\delta \geqslant \min\{\widehat{b}\pi/n, \pi\}$.

[10]Indeed, it is proved in Lemma 13 that $h_\xi(y) \geqslant 0$ for all $\xi \in \Upsilon$ and $y \geqslant 1$. It follows that $\sum_k h_{\xi_k}(b_k x) \geqslant 0$ for all $x \geqslant \widehat{b}$. On the other hand,

$$\int_0^\pi \psi_{1,r}(t)\frac{\sin t}{2}\,dt \equiv 2^r\int_0^\pi (1-\cos t)^r\frac{\sin t}{2}\,dt = \frac{4^r}{r+1} < C_{2r}^r = \frac{1}{2\pi}\int_0^{2\pi}\psi_{1,r}(t)\,dt.$$

Hence the representation (1.13) is impossible.

Corollary 1.7. *Let $a \in \mathbb{N}$ be an even integer and let $n, r \in \mathbb{N}$, $\delta \geqslant \pi/n$. Then*

$$\widetilde{\varkappa}_{a,r}(n,\delta) = \frac{1}{\sqrt{2^r}}\,.$$

Remark 4. For $r = 1$ Corollary 1.7 yields Theorem A and for $a = 2$ it yields Theorem G.

§ 2. Proofs of Theorems 1–3 and corollaries to them.

Definition 2.1. For a fixed closed subset T of $[0,2\pi]^d$ we denote by $\mathcal{M}(T)$ the space of finite Borel measures on T, by $\mathcal{M}^+(T) = \{\mu \in \mathcal{M}(T) : \mu \geqslant 0\}$ its subset of non-negative measures, and we set $B\mathcal{M}^+(T) = \{\mu \in \mathcal{M}^+(T) : \mu(T) \leqslant 1\}$.

The following two lemmas give us a twofold description of a sharp constant in Jackson's inequality for the generalized modulus of continuity. The idea of the reduction of the problem of finding a sharp constant in Jackson's inequality in L_2 to the calculation of the distance from a point to a convex set in the space of continuous functions is apparently due to Arestov (see, for instance, [16]).

Lemma 1. *Let T be a closed subset of $[0,2\pi]^d$, let $\psi(\,\cdot\,) \in C^+(\mathbb{T})$ and $\Lambda \subsetneqq \mathbb{Z}^d$. Then*

$$\inf\biggl\{\frac{\max_{\mathbf{t}\in T}\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}}\psi(\mathbf{k}\mathbf{t})}{\sum_{\mathbf{k}\notin\Lambda} c_{\mathbf{k}}} : c_{\mathbf{k}} \geqslant 0,\ 0 < \sum_{\mathbf{k}\notin\Lambda} c_{\mathbf{k}} < \infty\biggr\} = \max_{\mu\in B\mathcal{M}^+(T)}\ \inf_{\mathbf{k}\notin\Lambda}\int_T \psi(\mathbf{k}\mathbf{t})\,d\mu(\mathbf{t}).$$

Proof. Consider the following set of continuous functions on T:

$$A_\psi := \biggl\{g \in C(T) : g(\mathbf{t}) = \sum_{\mathbf{k}\notin\Lambda} c_{\mathbf{k}}\psi(\mathbf{k}\mathbf{t}),\ c_{\mathbf{k}} \geqslant 0,\ \sum_{\mathbf{k}\notin\Lambda} c_{\mathbf{k}} = 1\biggr\}.$$

It is easy to see that A_ψ is a convex set. The definition of A_ψ yields the inequality

$$\inf\biggl\{\frac{\max_{\mathbf{t}\in T}\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}}\psi(\mathbf{k}\mathbf{t})}{\sum_{\mathbf{k}\notin\Lambda} c_{\mathbf{k}}} : c_{\mathbf{k}} \geqslant 0,\ 0 < \sum_{\mathbf{k}\notin\Lambda} c_{\mathbf{k}} < \infty\biggr\} = \inf\{\|g(\,\cdot\,)\|_{C(T)} : g(\,\cdot\,) \in A_\psi\}.$$

By the duality formula for the distance from a point to a convex set in a normed space [17],

$$\inf\{\|g\|_{C(T)} : g \in A_\psi\} = \max\Bigl\{\inf_{g\in A_\psi}\langle g^*, g\rangle : g^* \in C^*(T),\ \|g^*\|_* \leqslant 1\Bigr\}.$$ [11]

[11] Here $C^*(T)$ is the dual space of $C(T)$ and $\|\cdot\|_*$ is the standard norm in $C^*(T)$.

By the Riesz–Radon theorem on the general form of a continuous linear functional on $C(T)$ we obtain $\langle g^*, g\rangle = \int_T g(\mathbf{t})\,d\mu(\mathbf{t})$, where $\mu \in \mathcal{M}(T)$. Consequently,

$$\inf\{\|g\|_{C(T)} : g \in A_\psi\} = \max\biggl\{\inf_{g\in A_\psi} \int_T g(\mathbf{t})\,d\mu(\mathbf{t}) : \mu(\,\cdot\,) \in \mathcal{M}(T),\ \|\mu\|_* \leqslant 1\biggr\}.$$

Moreover,

$$\begin{aligned}\inf_{g\in A_\psi} \int_T g(\mathbf{t})\,d\mu(\mathbf{t}) &= \inf\biggl\{\sum_{\mathbf{k}\notin\Lambda} c_{\mathbf{k}} \int_T \psi(\mathbf{kt})\,d\mu(\mathbf{t}) : c_{\mathbf{k}} \geqslant 0,\ \sum_{\mathbf{k}\notin\Lambda} c_{\mathbf{k}} = 1\biggr\}\\ &= \inf_{\mathbf{k}\notin\Lambda} \int_T \psi(\mathbf{kt})\,d\mu(\mathbf{t}).\end{aligned}$$

To complete the proof we observe that

$$\max_{\mu\in\mathcal{M}(T),\ \|\mu\|_*\leqslant 1}\ \inf_{\mathbf{k}\notin\Lambda} \int_T \psi(\mathbf{kt})\,d\mu(\mathbf{t}) = \max_{\mu\in B\mathcal{M}^+(T)}\ \inf_{\mathbf{k}\notin\Lambda} \int_T \psi(\mathbf{kt})\,d\mu(\mathbf{t}),$$

since ψ is a non-negative function.

Lemma 2. *Let T be a closed subset of $[0,2\pi]^d$, let $\psi(\,\cdot\,) \in C^+(\mathbb{T})$ and $\Lambda \subsetneqq \mathbb{Z}^d$. Then*

$$\overline{\varkappa}_\psi^{(d)}(\Lambda, T) = \biggl(\max_{\mu\in B\mathcal{M}^+(T)}\ \inf_{\mathbf{k}\notin\Lambda} \int_T \psi(\mathbf{kt})\,d\mu(\mathbf{t})\biggr)^{-1/2}.$$

Proof. Since $\overline{\omega}_\psi^2(f,T) = \max_{\mathbf{t}\in T} \sum_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{f}_{\mathbf{k}}|^2 \psi(\mathbf{kt})$, it follows that

$$\begin{aligned}(\overline{\varkappa}_\psi^{(d)}(\Lambda, T))^2 &= \sup\biggl\{\frac{\sum_{\mathbf{k}\notin\Lambda} c_{\mathbf{k}}}{\max_{\mathbf{t}\in T}\sum_{\mathbf{k}\in\mathbb{Z}^d} \psi(\mathbf{kt})c_{\mathbf{k}}} : c_{\mathbf{k}} \geqslant 0,\ 0 < \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}} < \infty\biggr\}\\ &= \sup\biggl\{\frac{\sum_{\mathbf{k}\notin\Lambda} c_{\mathbf{k}}}{\max_{\mathbf{t}\in T}\sum_{\mathbf{k}\in\mathbb{Z}^d} \psi(\mathbf{kt})c_{\mathbf{k}}} : c_{\mathbf{k}} \geqslant 0,\ 0 < \sum_{\mathbf{k}\notin\Lambda} c_{\mathbf{k}} < \infty\biggr\}\\ &= \biggl(\inf\biggl\{\frac{\max_{\mathbf{t}\in T}\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}}\psi(\mathbf{kt})}{\sum_{\mathbf{k}\notin\Lambda} c_{\mathbf{k}}} : c_{\mathbf{k}} \geqslant 0,\ 0 < \sum_{\mathbf{k}\notin\Lambda} c_{\mathbf{k}} < \infty\biggr\}\biggr)^{-1}.\end{aligned}$$

Using Lemma 1 we now obtain the required equality.

Lemma 3. *Let $\psi \in C_0^+(\mathbb{T})$, let $\Lambda \subset \mathbb{Z}^d$ be a finite set containing 0, and let T be a countable subset of $[0,2\pi]^d$. Then*

$$\overline{\varkappa}_\psi^{(d)}(\Lambda, T) = +\infty.$$

Proof. In view of Lemma 2, it is sufficient to prove the equality

$$\max_{\mu\in B\mathcal{M}^+(T)}\ \inf_{\mathbf{k}\notin\Lambda} \int_T \psi(\mathbf{kt})\,d\mu(\mathbf{t}) = 0.$$

We enumerate the points in T in an arbitrary order: $T = \{2\pi\mathbf{t}_s\}_{s=1}^{\infty}$. Then

$$\int_T \psi(\mathbf{kt})\, d\mu(\mathbf{t}) = \sum_{s=1}^{\infty} \psi(2\pi\mathbf{kt}_s)\alpha_s,$$

where $\alpha_s = \mu(2\pi\mathbf{t}_s) \geqslant 0$.

Since the series $\sum_{s=1}^{\infty} \psi(2\pi\mathbf{kt}_s)\alpha_s$ converges uniformly with respect to $\mathbf{k} \in \mathbb{Z}^d$, it follows that for arbitrary $\varepsilon > 0$ one can find $M = M(\varepsilon) \in \mathbb{N}$ such that $\sum_{s=M+1}^{\infty} \psi(2\pi\mathbf{kt}_s)\alpha_s < \varepsilon/2$ for all $\mathbf{k} \in \mathbb{Z}^d$.

By assumption $\psi(\,\cdot\,)$ is continuous, 2π-periodic, and $\psi(0) = 0$, therefore one can find $\delta > 0$ such that $\psi(x) < \varepsilon/2$ once $\|x/(2\pi)\| < \delta$. (Here $\|y\|$ is the distance from y to the nearest integer.)

By Dirichlet's theorem on simultaneous approximation ([18], Chapter 2, § 1) applied to the set $\{\mathbf{t}_1, \dots, \mathbf{t}_M\}$ there exists an infinite sequence $\{\mathbf{k}_l\}_{l=1}^{\infty} \subset \mathbb{Z}^d$ such that

$$\left\| \frac{1}{2\pi}(2\pi\mathbf{t}_s\mathbf{k}_l) \right\| = \|\mathbf{t}_s\mathbf{k}_l\| < \delta \quad \text{for each } l \in \mathbb{N}, \quad 1 \leqslant s \leqslant M.$$

(We assume without loss of generality that $|\mathbf{k}_l| > \max\{|\mathbf{k}| : \mathbf{k} \in \Lambda\}$ for all l.) Consequently, $0 \leqslant \psi(2\pi\mathbf{t}_s\mathbf{k}_l) < \varepsilon/2$ for all $l \in \mathbb{N}$, $1 \leqslant s \leqslant M$, and therefore

$$\sum_{s=1}^{M} \psi(2\pi\mathbf{t}_s\mathbf{k}_l)\alpha_s \leqslant \max_{1\leqslant s\leqslant M} \psi(2\pi\mathbf{t}_s\mathbf{k}_l) \sum_{s=1}^{M} \alpha_s < \frac{\varepsilon}{2} \int_T d\mu(\mathbf{t}) \leqslant \frac{\varepsilon}{2} \quad \text{for all } l \in \mathbb{N}.$$

Hence

$$0 \leqslant \inf_{\mathbf{k}\notin\Lambda} \int_T \psi(\mathbf{kt})\, d\mu(\mathbf{t}) \leqslant \liminf_{l\to\infty} \int_T \psi(\mathbf{k}_l\mathbf{t})\, d\mu(\mathbf{t}) < \varepsilon.$$

Taking into account the fact that ε is an arbitrary positive number we obtain the required equality.

Remark 5. For $d = 1$ and the classical difference this result was proved in [19].

For compactness we set $B\mathcal{M}^+ := B\mathcal{M}^+([0, 2\pi]^d)$.

Lemma 4 (the main lemma). *Let* $\psi \in C_0^+(\mathbb{T})$ *and* $\Lambda \in \Lambda_*^d$. *Then*

$$\max_{\mu\in B\mathcal{M}^+} \inf_{\mathbf{k}\notin\Lambda} \int_{[0,2\pi]^d} \psi(\mathbf{kt})\, d\mu(\mathbf{t}) = \frac{1}{2\pi} \int_0^{2\pi} \psi(x)\, dx.$$

Proof. We assume without loss of generality that the set $\Lambda \cap \{\mathbf{k} : \mathbf{k} = (k, 0, \dots, 0),\ k \in \mathbb{Z}_+\}$ is finite.

Let $K = \max\{k_1 : \mathbf{k} = (k_1, 0, \dots, 0) \in \Lambda\}$. Then

$$\begin{aligned}\inf_{\mathbf{k}\notin\Lambda} \int_{[0,2\pi]^d} \psi(\mathbf{kt})\, d\mu(\mathbf{t}) &\leqslant \inf_{\mathbf{k}\notin\Lambda,\ \mathbf{k}=(k,0,\dots,0)} \int_{[0,2\pi]^d} \psi(\mathbf{kt})\, d\mu(\mathbf{t}) \\ &\leqslant \inf_{k>K,\, k\in\mathbb{Z}} \int_{[0,2\pi]^d} \psi(kt_1)\, d\mu(\mathbf{t}).\end{aligned}$$

We split the integral $\int_{[0,2\pi]^d} \psi(kt_1)\,d\mu(\mathbf{t})$ into two:

$$\int_{[0,2\pi]^d} \psi(kt_1)\,d\mu(\mathbf{t}) = \int_{\Omega_1} \psi(kt_1)\,d\mu(\mathbf{t}) + \int_{\Omega_2} \psi(kt_1)\,d\mu(\mathbf{t}),$$

where $\Omega_1 = \{\mathbf{t} \in [0,2\pi]^d : t_1/\pi \in \mathbb{Q}\}$, $\Omega_2 = [0,2\pi]^d \setminus \Omega_1$, and $\mathbb{Q}$ is the set of rational numbers.

We enumerate the points in $[0,2\pi] \cap 2\pi\mathbb{Q}$ in an arbitrary manner and shall write $[0,2\pi] \cap 2\pi\mathbb{Q} = \{2\pi p_s/q_s\}_{s=1}^{\infty}$. (Here $p_s \in \mathbb{Z}$, $q_s \in \mathbb{N}$.) By σ-additivity we obtain

$$\int_{\Omega_1} \psi(kt_1)\,d\mu(\mathbf{t}) = \sum_{s=1}^{\infty} \psi\biggl(k2\pi \frac{p_s}{q_s}\biggr)\alpha_s,$$

where $0 \leqslant \alpha_s = \mu(\{\mathbf{t} \in [0,2\pi]^d : t_1 = 2\pi p_s/q_s\})$, $\sum_{s=1}^{\infty} \alpha_s \leqslant 1$.

We fix arbitrary $\varepsilon > 0$ and select $M = M(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{s=M+1}^{\infty} \alpha_s < \frac{\varepsilon}{\|\psi(\,\cdot\,)\|_{C(\mathbb{T})}}\,.$$

Then

$$\sum_{s=M+1}^{\infty} \psi\biggl(2\pi k \frac{p_s}{q_s}\biggr)\alpha_s < \varepsilon \qquad \text{for all } k \in \mathbb{Z}.$$

Let $Q = \operatorname{LCM}\{q_1, \dots, q_M\}$.[12] Then $\psi(l2\pi Q p_s/q_s) = \psi(0) = 0$ for all $l \in \mathbb{N}$, $1 \leqslant s \leqslant M$. Hence $\sum_{s=1}^{M} \psi(l2\pi Q p_s/q_s)\alpha_s = 0$ for all $l \in \mathbb{N}$ and

$$\int_{\Omega_1} \psi(lQt_1)\,d\mu(\mathbf{t}) = \sum_{s=M+1}^{\infty} \psi\biggl(l2\pi Q \frac{p_s}{q_s}\biggr)\alpha_s \leqslant \varepsilon \quad \text{for all } l \in \mathbb{N}.$$

This yields

$$\begin{aligned}\inf_{l\in\mathbb{N}} \int_{[0,2\pi]^d} \psi(lQt_1)\,d\mu(\mathbf{t}) &= \inf_{l\in\mathbb{N}} \biggl(\int_{\Omega_1} \psi(lQt_1)\,d\mu(\mathbf{t}) + \int_{\Omega_2} \psi(lQt_1)\,d\mu(\mathbf{t}) \biggr) \\ &\leqslant \varepsilon + \inf_{l\in\mathbb{N}} \int_{\Omega_2} \psi(lQt_1)\,d\mu(\mathbf{t}) = \varepsilon + \inf_{N\in\mathbb{N}} \min_{1\leqslant l\leqslant N} \int_{\Omega_2} \psi(lQt_1)\,d\mu(\mathbf{t}).\end{aligned}$$

We set

$$\overline{\psi}_N(x) := \frac{1}{N} \sum_{l=1}^{N} \psi(lx) \qquad \text{for all } x \in \mathbb{R}.$$

It can be easily verified that

$$\min_{1\leqslant l\leqslant N} \int_{\Omega_2} \psi(lQt_1)\,d\mu(\mathbf{t}) \leqslant \frac{1}{N} \sum_{l=1}^{N} \int_{\Omega_2} \psi(lQt_1)\,d\mu(\mathbf{t}) = \int_{\Omega_2} \overline{\psi}_N(Qt_1)\,d\mu(\mathbf{t}).$$

[12] Without loss of generality we assume that $Q > K$.

Since for each $t_1 \in [0, 2\pi] \setminus 2\pi\mathbb{Q}$ the sequence $\{lQt_1 \pmod{2\pi}\}_{l=1}^{\infty}$ is uniformly distributed on $\mathbb{T}$, it follows by Weyl's criterion of uniform distribution ([20], Chapter 4, § 4) that

$$\lim_{N\to\infty} \overline{\psi}_N(Qt_1) = \frac{1}{2\pi}\int_0^{2\pi} \psi(x)\,dx \quad \text{for all } t_1 \notin 2\pi\mathbb{Q}.$$

We also have $|\overline{\psi}_N(Qt_1)| \leqslant \max_{1\leqslant l\leqslant N} |\psi(lQt_1)| \leqslant \|\psi(\,\cdot\,)\|_{C(\mathbb{T})}$; hence $|\overline{\psi}_N(Qt_1)|$ is uniformly bounded with respect to t_1 and N.

Hence by Lebesgue's dominated convergence theorem ([21], Chapter 5, § 5)

$$\lim_{N\to\infty} \int_{\Omega_2} \overline{\psi}_N(Qt_1)\,d\mu(\mathbf{t}) = \int_{\Omega_2} \biggl(\frac{1}{2\pi}\int_0^{2\pi} \psi(x)\,dx\biggr)\,d\mu(\mathbf{t}) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \psi(x)\,dx.$$

Since $Q \geqslant K + 1$, it follows that

$$\begin{aligned}\inf_{k\geqslant K+1,\, k\in\mathbb{N}} \int_{[0,2\pi]^d} \psi(kt)\,d\mu(\mathbf{t}) &\leqslant \inf_{l\in\mathbb{N}} \int_{[0,2\pi]^d} \psi(lQt_1)\,d\mu(\mathbf{t}) \\ &\leqslant \varepsilon + \lim_{N\to\infty} \int_{\Omega_2} \overline{\psi}_N(Qt_1)\,d\mu(\mathbf{t}) \leqslant \varepsilon + \frac{1}{2\pi}\int_0^{2\pi} \psi(x)\,dx.\end{aligned}$$

Taking account of the fact that ε is an arbitrary positive number we now conclude that

$$\inf_{k\geqslant K+1,\, k\in\mathbb{N}} \int_{[0,2\pi]^d} \psi(kt_1)\,d\mu(\mathbf{t}) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \psi(x)\,dx.$$

Summarizing our arguments we obtain

$$\inf_{\mathbf{k}\notin\Lambda} \int_{[0,2\pi]^d} \psi(\mathbf{kt})\,d\mu(\mathbf{t}) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \psi(x)\,dx \qquad \forall\, \mu \in B\mathcal{M}^+.$$

On the other hand,

$$\max_{\mu\in B\mathcal{M}^+} \inf_{\mathbf{k}\notin\Lambda} \int_{[0,2\pi]^d} \psi(\mathbf{kt})\,d\mu(\mathbf{t}) \geqslant \inf_{\mathbf{k}\notin\Lambda} \int_{[0,2\pi]^d} \psi(\mathbf{kt})\,\frac{d\mathbf{t}}{(2\pi)^d}\,.$$

At the same time it follows by the 2π-periodicity of $\psi(\,\cdot\,)$ that

$$\begin{aligned}\int_{[0,2\pi]^d} \psi(\mathbf{kt})\,\frac{d\mathbf{t}}{(2\pi)^d} &= \frac{1}{(2\pi)^d}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_0^{2\pi} \psi(k_1t_1)\,dt_1\,dt_2\cdots dt_d \\ &= \frac{1}{2\pi}\int_0^{2\pi} \psi(k_1t_1)\,dt_1 = \frac{1}{2\pi k_1}\int_0^{2\pi k_1} \psi(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} \psi(x)\,dx\end{aligned}$$

for all $\mathbf{k} \in \mathbb{Z}^d$, $\mathbf{k} \neq 0$ (we assume for definiteness that $k_1 \neq 0$). The proof is complete.

Proof of Theorem 1. By Lemma 2,

$$\overline{\varkappa}_{\psi}^{(d)}(\Lambda, [0,2\pi]^d) = \biggl(\max_{\mu \in B\mathcal{M}^+} \inf_{\mathbf{k} \notin \Lambda} \int_{[0,2\pi]^d} \psi(\mathbf{kt})\, d\mu(\mathbf{t}) \biggr)^{-1/2}.$$

Consequently, using Lemma 4 we obtain the required equality.

Proof of Corollary 1.1. By definition,

$$\begin{aligned} \overline{\omega}_{\psi}(f,2\pi)^2 &= \max_{0 \leqslant t \leqslant 2\pi} \sum_{s \in \mathbb{Z}} \psi(st) |\widehat{f}_s|^2 \\ &= \max\biggl\{ \max_{0 \leqslant t \leqslant \pi} \sum_{s \in \mathbb{Z}} \psi(st) |\widehat{f}_s|^2,\ \max_{\pi \leqslant t \leqslant 2\pi} \sum_{s \in \mathbb{Z}} \psi(st) |\widehat{f}_s|^2 \biggr\}. \end{aligned}$$

Since ψ is an even 2π-periodic function, it follows that $\psi(s(2\pi - \tau)) \equiv \psi(s\tau)$. Consequently,

$$\begin{aligned} \max_{\pi \leqslant t \leqslant 2\pi} \sum_{s \in \mathbb{Z}} \psi(st) |\widehat{f}_s|^2 &= \max_{0 \leqslant \tau \leqslant \pi} \sum_{s \in \mathbb{Z}} \psi(s(2\pi - \tau)) |\widehat{f}_s|^2 \\ &= \max_{0 \leqslant \tau \leqslant \pi} \sum_{s \in \mathbb{Z}} \psi(s\tau) |\widehat{f}_s|^2 = \overline{\omega}_{\psi}(f,\pi)^2. \end{aligned}$$

Bearing in mind the definition of $\overline{\varkappa}_{\psi}$ we conclude that $\overline{\varkappa}_{\psi}(n,\pi) = \overline{\varkappa}_{\psi}(n,2\pi)$. It remains to use Theorem 1 and to take account of the fact that $\overline{\varkappa}_{\psi}(n,\,\cdot\,)$ is a non-decreasing function for each fixed n.

Proof of Corollary 1.2. Bearing in mind relations (1.10) we immediately deduce the required result from Theorem B and Corollary 1.1.

Proof of Corollary 1.3. As usual, we set

$$\psi_{\theta}(t) = \biggl| \sum_{k \in \mathbb{Z}} \theta_k e_k(t) \biggr|^2.$$

Then

$$\frac{1}{2\pi} \int_0^{2\pi} \psi_{\theta}(x)\, dx = \sum_{k \in \mathbb{Z}} |\theta_k|^2.$$

It follows by Theorem 1 and property (1.5) that

$$\overline{\varkappa}_{\psi_\theta}^{(d)}(\Lambda, T) \geqslant \overline{\varkappa}_{\psi_\theta}^{(d)}(\Lambda, [0,2\pi]^d) = \biggl(\frac{1}{2\pi} \int_0^{2\pi} \psi_{\theta}(x)\, dx \biggr)^{-1/2} = \biggl(\sum_{k \in \mathbb{Z}} |\theta_k|^2 \biggr)^{-1/2}.$$

Lemma 5. *Let $\psi(\,\cdot\,) \in C^+(\mathbb{T})$, $\psi(-x) \equiv \psi(x)$, and suppose that $\gamma > 0$. Then*

$$\overline{\varkappa}_\psi\biggl(n, \gamma\frac{\pi}{n}\biggr) \leqslant \biggl(\max_{\mu \in B\mathcal{M}^+([0,\pi])} \inf_{x \geqslant \gamma} \int_0^\pi \psi(xt)\,d\mu(t)\biggr)^{-1/2}.$$

Proof. By Lemma 2,

$$\overline{\varkappa}_\psi\biggl(n, \gamma\frac{\pi}{n}\biggr) = \biggl(\max_{\mu \in B\mathcal{M}^+([0,\gamma\pi/n])} \inf_{|k| \geqslant n} \int_0^{\gamma\pi/n} \psi(kt)\,d\mu(t)\biggr)^{-1/2}.$$

At the same time, since $\psi(\,\cdot\,)$ is even, it follows that

$$\begin{aligned}\inf_{|k| \geqslant n} \int_0^{\gamma\pi/n} \psi(kt)\,d\mu(t) &= \inf_{k \geqslant n} \int_0^{\gamma\pi/n} \psi(kt)\,d\mu(t) \\ &= \inf_{k \geqslant n} \int_0^\pi \psi\biggl(\gamma\frac{k}{n}\tau\biggr)\,d\widetilde{\mu}(\tau) \geqslant \inf_{x \geqslant \gamma} \int_0^\pi \psi(x\tau)\,d\widetilde{\mu}(\tau).\end{aligned}$$

(Here $\widetilde{\mu} \in B\mathcal{M}^+([0,\pi])$ is the measure defined by the equality $\widetilde{\mu}(A) = \mu(\gamma/nA)$ for each Borel subset A of $[0,\pi]$.)

Proof of Theorem 2. We have

$$\begin{aligned}\max_{\mu \in B\mathcal{M}^+([0,\pi])} \inf_{x \geqslant \gamma} \int_0^\pi \psi(xt)\,d\mu(t) &\geqslant \inf_{x \geqslant \gamma} \frac{1}{\pi} \int_0^\pi \psi(xt)\,dt \\ &= \inf_{x \geqslant \gamma} \frac{1}{\pi x} \int_0^{\pi x} \psi(t)\,dt \geqslant \inf_{x \geqslant [\gamma]} \frac{1}{\pi x} \int_0^{\pi x} \psi(t)\,dt.\end{aligned}$$

Representing x in the form $x = [\gamma] + 2s + y$, $s \in \mathbb{Z}_+$, $0 \leqslant y < 2$, we obtain

$$\frac{1}{\pi} \int_0^{\pi x} \psi(t)\,dt = \frac{1}{\pi} \int_0^{\pi(2s+[\gamma])} \psi(t)\,dt + \frac{1}{\pi} \int_{\pi(2s+[\gamma])}^{\pi(2s+[\gamma]+y)} \psi(t)\,dt.$$

Since ψ is even and 2π-periodic,

$$\begin{gathered}\frac{1}{\pi} \int_0^{\pi(2s+[\gamma])} \psi(t)\,dt = (2s + [\gamma])J_\psi, \\ \frac{1}{\pi} \int_{\pi(2s+[\gamma])}^{\pi(2s+[\gamma]+y)} \psi(t)\,dt = \frac{1}{\pi} \int_{\pi[\gamma]}^{\pi([\gamma]+y)} \psi(t)\,dt.\end{gathered}$$

We denote the last integral by $j_\psi(y)$. Hence we obtain

$$\inf_{x \geqslant [\gamma]} \frac{1}{\pi x} \int_0^{\pi x} \psi(t)\,dt = \inf_{0 \leqslant y < 2} \inf_{s \in \mathbb{Z}_+} \frac{([\gamma] + 2s)J_\psi + j_\psi(y)}{[\gamma] + 2s + y}.$$

Since the function

$$H(s,y) := \frac{([\gamma] + 2s)J_\psi + j_\psi(y)}{[\gamma] + 2s + y}$$

is monotone in s for each $y \in [0,2)$, $\lim_{s\to\infty} H(s,y) = J_\psi$, and $H(0,0) = J_\psi$, it follows that

$$\begin{aligned}
\inf_{0\leqslant y<2}\inf_{s\in\mathbb{Z}_+} H(s,y) &= \min_{0\leqslant y\leqslant 2}\min\{H(0,y),J_\psi\} = \min_{0\leqslant y\leqslant 2} H(0,y)\\
&= \min_{0\leqslant y\leqslant 2}\frac{[\gamma]J_\psi + j_\psi(y)}{[\gamma]+y}\\
&= \min_{0\leqslant y\leqslant 2}\frac{1}{[\gamma]+y}\biggl(\frac{1}{\pi}\int_0^{\pi[\gamma]}\psi(t)\,dt + \frac{1}{\pi}\int_{\pi[\gamma]}^{\pi([\gamma]+y)}\psi(t)\,dt\biggr)\\
&= \min_{[\gamma]\leqslant z\leqslant[\gamma]+2}\frac{1}{\pi z}\int_0^{\pi z}\psi(t)\,dt = \min_{[\gamma]/2\leqslant x\leqslant[\gamma]/2+1}\frac{1}{2\pi x}\int_0^{2\pi x}\psi(t)\,dt\\
&= \min_{[\gamma]/2\leqslant x\leqslant[\gamma]/2+1}\frac{J_\psi(x)}{x}.
\end{aligned}$$

For the completion of the proof is remains to use Lemma 5.

Proof of Corollary 1.4. Because ψ is an even non-negative 2π-periodic function, it follows that

$$\frac{1}{\pi z}\int_0^{\pi z}\psi(t)\,dt \geqslant \frac{[z]}{z}\,\frac{1}{2\pi}\int_0^{2\pi}\psi(x)\,dx \quad \text{for all } z\geqslant 1,$$

and therefore

$$\begin{aligned}
\min_{[\gamma]/2\leqslant x\leqslant[\gamma]/2+1}\frac{J_\psi(x)}{x} &= \min_{[\gamma]\leqslant z\leqslant[\gamma]+2}\frac{1}{\pi z}\int_0^{\pi z}\psi(t)\,dt\\
&\geqslant \min_{[\gamma]\leqslant z\leqslant[\gamma]+2}\frac{[z]}{z}\,\frac{1}{2\pi}\int_0^{2\pi}\psi(x)\,dx = \frac{[\gamma]}{[\gamma]+1}\,\frac{1}{2\pi}\int_0^{2\pi}\psi(x)\,dx. \qquad (2.1)
\end{aligned}$$

Hence the right hand side of the inequality proved in Theorem 2 has the estimate

$$\biggl(\frac{[\gamma]}{[\gamma]+1}\,\frac{1}{2\pi}\int_0^{2\pi}\psi(x)\,dx\biggr)^{-1/2} = \sqrt{1+\frac{1}{[\gamma]}}\biggl(\frac{1}{2\pi}\int_0^{2\pi}\psi(x)\,dx\biggr)^{-1/2}.$$

Proof of Theorem 3. By Lemma 2,

$$\begin{aligned}
&\overline{\varkappa}_\psi^{(d)}\biggl(\square_n,\biggl[0,\gamma\frac{2\pi}{n}\biggr]^d\biggr)\\
&\qquad= \biggl(\max_{\mu\in B\mathcal{M}^+([0,\gamma2\pi/n]^d)}\ \inf_{|\mathbf{k}|\geqslant n}\int_{[0,\gamma2\pi/n]^d}\psi(\mathbf{kt})\,d\mu(\mathbf{t})\biggr)^{-1/2}.
\end{aligned}$$

(Here $|\mathbf{k}| := |\mathbf{k}|_\infty = \max_{1\leqslant j\leqslant d}|k_j|$.)

It is clear that

$$\max_{\mu\in B\mathcal{M}^+([0,\gamma 2\pi/n]^d)} \inf_{|\mathbf{k}|\geqslant n} \int_{[0,\gamma 2\pi/n]^d} \psi(\mathbf{kt})\,d\mu(\mathbf{t})$$
$$\geqslant \inf_{|\mathbf{k}|\geqslant n}\left(\frac{n}{\gamma 2\pi}\right)^d \int_{[0,\gamma 2\pi/n]^d} \psi(\mathbf{kt})\,d\mathbf{t} = \min_{1\leqslant j\leqslant d} \inf_{|k_j|\geqslant n,\,\mathbf{k}\in\mathbb{Z}^d}\left(\frac{n}{\gamma 2\pi}\right)^d$$
$$\times\int_0^{\gamma 2\pi/n}\cdots\int_0^{\gamma 2\pi/n}\psi(k_1t_1+\dots+k_dt_d)\,dt_1\cdots dt_d$$
$$=\inf_{|k_1|\geqslant n}\inf_{k_2,\dots,k_d\in\mathbb{Z}}\left(\frac{n}{\gamma 2\pi}\right)^{d-1}\int_{[0,\gamma 2\pi/n]^{d-1}}\frac{n}{\gamma 2\pi}\int_0^{\gamma 2\pi/n}\psi(k_1t_1+\overline{k}\overline{t})\,dt_1\,d\overline{t},$$

where $\overline{k}=(k_2,\dots,k_d)$, $\overline{t}=(t_2,\dots,t_d)$.

For $\delta\in\mathbb{R}$ we set $\psi_\delta(x):=\psi(x+\delta)$. Obviously,

$$\int_0^{2\pi}\psi_\delta(t)\,dt=\int_0^{2\pi}\psi(t)\,dt.$$

Assume that $k\geqslant n$. Setting $x:=\gamma k/n$ we represent x in the form $x=[\gamma]+s+y$, $s\in\mathbb{Z}_+$, $0\leqslant y<1$, and arguing as in the proof of Theorem 2 obtain the following equalities:

$$\frac{n}{\gamma 2\pi}\int_0^{\gamma 2\pi/n}\psi_\delta(k\tau)\,d\tau=\frac{n}{\gamma 2\pi k}\int_0^{\gamma 2\pi k/n}\psi_\delta(t)\,dt=\frac{1}{2\pi x}\int_0^{2\pi x}\psi_\delta(t)\,dt$$
$$=\frac{1}{[\gamma]+s+y}\left(([\gamma]+s)\,\frac{1}{2\pi}\int_0^{2\pi}\psi_\delta(t)\,dt+\frac{1}{2\pi}\int_0^{2\pi y}\psi_\delta(t)\,dt\right)$$
$$=\frac{1}{[\gamma]+s+y}\left(([\gamma]+s)\,\frac{1}{2\pi}\int_0^{2\pi}\psi(t)\,dt+\frac{1}{2\pi}\int_0^{2\pi y}\psi_\delta(t)\,dt\right).$$

Consequently, setting $\overline{\delta}:=\overline{k}\overline{t}$ we obtain

$$\inf_{k\geqslant n}\inf_{k_2,\dots,k_d\in\mathbb{Z}}\left(\frac{n}{\gamma 2\pi}\right)^{d-1}\int_{[0,\gamma 2\pi/n]^{d-1}}\frac{n}{\gamma 2\pi}\int_0^{\gamma 2\pi/n}\psi(k_1t_1+\overline{k}\overline{t})\,dt_1\,d\overline{t}$$
$$\geqslant\inf_{x\geqslant[\gamma]}\inf_{k_2,\dots,k_d\in\mathbb{Z}}\left(\frac{n}{\gamma 2\pi}\right)^{d-1}\int_{[0,\gamma 2\pi/n]^{d-1}}\left(\frac{1}{2\pi x}\int_0^{2\pi x}\psi(t+\overline{k}\,\overline{t})\,dt\right)d\overline{t}$$
$$\geqslant\inf_{0\leqslant y\leqslant 1}\inf_{s\in\mathbb{Z}_+}\inf_{\overline{k}\in\mathbb{Z}^{d-1}}\left(\frac{n}{\gamma 2\pi}\right)^{d-1}\int_{[0,\gamma 2\pi/n]^{d-1}}\left(\frac{[\gamma]+s}{[\gamma]+s+y}J_\psi\right.$$
$$\left.+\frac{1}{[\gamma]+s+y}\,\frac{1}{2\pi}\int_0^{2\pi y}\psi_{\bar\delta}(t)\,dt\right)d\overline{t}=\inf_{0\leqslant y\leqslant 1}\inf_{s\in\mathbb{N}}\left(\frac{[\gamma]+s}{[\gamma]+s+y}J_\psi\right.$$
$$\left.+\frac{1}{[\gamma]+s+y}\inf_{\overline{k}\in\mathbb{Z}^{d-1}}\left(\frac{n}{\gamma 2\pi}\right)^{d-1}\int_{[0,\gamma 2\pi/n]^{d-1}}\frac{1}{2\pi}\int_0^{2\pi y}\psi_{\bar\delta}(t)\,dt\,d\overline{t}\right).$$

At the same time,

$$\begin{aligned}
&\inf_{\overline{k}\in\mathbb{Z}^{d-1}}\left(\frac{n}{\gamma 2\pi}\right)^{d-1}\int_{[0,\gamma 2\pi/n]^{d-1}}\frac{1}{2\pi}\int_0^{2\pi y}\psi_{\overline{\delta}}(t)\,dt\,d\overline{t}\\
&\quad\geqslant\left(\frac{n}{\gamma 2\pi}\right)^{d-1}\int_{[0,\gamma 2\pi/n]^{d-1}}\inf_{\overline{k}\in\mathbb{Z}^{d-1}}\frac{1}{2\pi}\int_0^{2\pi y}\psi_{\overline{\delta}}(t)\,dt\,d\overline{t}\\
&\quad\geqslant\left(\frac{n}{\gamma 2\pi}\right)^{d-1}\int_{[0,\gamma 2\pi/n]^{d-1}}\inf_{0\leqslant\delta\leqslant 2\pi}\frac{1}{2\pi}\int_0^{2\pi y}\psi_{\delta}(t)\,dt\,d\overline{t}\\
&\quad=\inf_{0\leqslant\delta\leqslant 2\pi}\frac{1}{2\pi}\int_0^{2\pi y}\psi_\delta(t)\,dt=\inf_{0\leqslant\delta\leqslant 2\pi}\frac{1}{2\pi}\int_\delta^{2\pi y+\delta}\psi(t)\,dt=\overline{J}_\psi(y).
\end{aligned}$$

Hence

$$\begin{aligned}
&\inf_{k\geqslant n}\inf_{k_2,\dots,k_d\in\mathbb{Z}}\left(\frac{n}{\gamma 2\pi}\right)^{d-1}\int_{[0,\gamma 2\pi/n]^{d-1}}\frac{n}{\gamma 2\pi}\int_0^{\gamma 2\pi/n}\psi(k_1t_1+\overline{k}\,\overline{t})\,dt_1\,d\overline{t}\\
&\qquad\geqslant\inf_{0\leqslant y\leqslant 1}\inf_{s\in\mathbb{Z}_+}\frac{([\gamma]+s)J_\psi+\overline{J}_\psi(y)}{[\gamma]+s+y}\,.
\end{aligned}$$

Since the function

$$H(s,y):=\frac{([\gamma]+s)J_\psi+\overline{J}_\psi(y)}{[\gamma]+s+y}$$

is monotone in s for each $y\in[0,1]$ and $\lim_{s\to\infty}H(s,y)=J_\psi$, it follows that

$$\begin{aligned}
\min_{0\leqslant y\leqslant 1}\inf_{s\in\mathbb{N}}H(s,y)&=\min_{0\leqslant x\leqslant 1}\min\{H(0,x),J_\psi\}\\
&=\min_{0\leqslant x\leqslant 1}H(0,x)=\min_{[\gamma]\leqslant x\leqslant[\gamma]+1}\frac{\overline{J}_\psi(x)}{x}\,.
\end{aligned}$$

In the penultimate equality we use the simple relation $H(0,0)=J_\psi$, and in the last equality the property $[\gamma]J_\psi+\overline{J}_\psi(y)=\overline{J}_\psi([\gamma]+y)$.

Finally, using the equality

$$\int_0^{\gamma 2\pi/n}\psi_\delta(kt)\,dt=\int_0^{\gamma 2\pi/n}\psi_{\delta-|k|\gamma 2\pi/n}(|k|s)\,ds\quad\text{for all } k<0$$

we reduce the case $k_1\leqslant -n$ to the (already considered) case $k_1\geqslant n$.

Proof of Corollary 1.5. Similarly to the proof of Corollary 1.4 we observe that from the non-negativity and the 2π-periodicity of ψ one obtains easily the inequality $\overline{J}_\psi(x)\geqslant([x]/(2\pi))\displaystyle\int_0^{2\pi}\psi(x)\,dx$ (for $x\geqslant 1$); we also observe that

$$\min_{[\gamma]\leqslant x\leqslant[\gamma]+1}\frac{[x]}{x}=\frac{[\gamma]}{[\gamma]+1}\,.$$

Hence

$$\left(\frac{[\gamma]}{[\gamma]+1}\,\frac{1}{2\pi}\int_0^{2\pi}\psi(x)\,dx\right)^{-1/2} \geqslant \left(\min_{[\gamma]\leqslant x\leqslant[\gamma]+1}\frac{\overline{J}_\psi(x)}{x}\right)^{-1/2}$$
$$\geqslant \overline{\varkappa}_\psi^{(d)}\left(\square_n,\left[0,\gamma\frac{2\pi}{n}\right]^d\right),$$

where the last inequality is the result of Theorem 3.

Proof of Corollary 1.6. It follows from equality (1.9) that[13]

$$\frac{1}{2\pi}\int_0^{2\pi}\psi_{a,r}(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi}2^r\prod_{k=0}^{r-1}(1-\cos(a^kx))\,dx = 2^r$$

for positive integers a, $a \geqslant 2$. Hence by Corollary 1.1 we obtain $\widetilde{\varkappa}_{a,r}(n,\delta) = 1/\sqrt{2^r}$ for each $\delta \geqslant \pi$, therefore

$$\widetilde{\varkappa}_{a,r}(n,\delta) \geqslant \frac{1}{\sqrt{2^r}} \text{ for each } \delta > 0.$$

On the other hand,

$$\widetilde{\varkappa}_{a,r}\left(n,\gamma\frac{\pi}{n}\right) \leqslant \sqrt{1+\frac{1}{[\gamma]}}\,\frac{1}{\sqrt{2^r}} \text{ for each } \gamma \geqslant 1$$

by Corollary 1.4. A comparison of this estimate with the lower bound yields the required result for $a \in \mathbb{N}$, $a \geqslant 2$, and, in particular, for odd $a \geqslant 3$.

§ 3. Properties of the Thue–Morse moduli

We now study in detail properties of the moduli constructed from the Thue–Morse difference $\widetilde{\Delta}_t^r(f,x)$ or the generalized difference $\widetilde{\Delta}_t^{a,r}(f,x)$.

To this end we consider the characteristic polynomial of the difference operator $\widetilde{\Delta}_t^r$:

$$\chi_{2^r-1}(z) = (1-z)(1-z^2)(1-z^4)\cdots(1-z^{2^{r-1}}) = \sum_{k=0}^{2^r-1}c_{k,r}z^k. \tag{3.1}$$

One can easily prove by induction on r that the coefficients $c_{k,r}$ of the polynomial $\chi_{a^r-1}(z)$ are equal to ±1. We observed in § 1 that $c_{k,r} = (-1)^{\gamma_k}$, where $\gamma_0 = 1$, and $\{\gamma_k\}_{k=1}^{+\infty}$ is the Thue–Morse sequence. For $r = 1, 2, 3$ the polynomials $\chi_{2^r-1}(z)$ are as follows:

$$\chi_1(z) = 1-z,$$
$$\chi_3(z) = 1-z-z^2+z^3,$$
$$\chi_7(z) = 1-z-z^2+z^3-z^4+z^5+z^6-z^7.$$

[13] For $a \geqslant 2$, $a \in \mathbb{N}$, the equality $\frac{1}{2\pi}\int_0^{2\pi}2^r\prod_{k=0}^{r-1}(1-\cos(a^kx))\,dx = 2^r$ follows from the representation $\prod(1-\cos(a^kx)) = 1+\sum\alpha_k\cos kx$, which clearly holds for such a with some $\alpha_k \in \mathbb{R}$.

One can easily show that the difference $\widetilde{\Delta}_t^{a,r}(f,x)$ satisfies the following relations:

$$\begin{aligned}
\widetilde{\Delta}_h^{a,1}(f,x) &= \Delta_h^1(f,x) = \Delta_h(f,x) = f(x) - f(x+h),\\
\widetilde{\Delta}_h^{a,2}(f,x) &= \widetilde{\Delta}_h^{a,1}(f,x) - \widetilde{\Delta}_h^{a,1}(f,x+ah)\\
&= f(x) - f(x+h) - f(x+ah) + f(x+(a+1)h),\\
\widetilde{\Delta}_h^{a,3}(f,x) &= \widetilde{\Delta}_h^{a,2}(f,x) - \widetilde{\Delta}_h^{a,2}(f,x+a^2h),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\widetilde{\Delta}_h^{a,r}(f,x) &= \widetilde{\Delta}_h^{a,r-1}(f,x) - \widetilde{\Delta}_h^{a,r-1}(f,x+a^{r-1}h).
\end{aligned}$$

These equalities yield, in particular, the relation

$$\widetilde{\Delta}_h^{a,r}(f,x) = \Delta_h \Delta_{ah} \cdots \Delta_{a^{r-1}h}(f,x),$$

where $\Delta_h(f,x) = f(x) - f(x+h)$ is the 'ordinary' difference. In § 1.1 we defined the modulus of continuity in L_2 for $\widetilde{\Delta}_h^{a,r}(f,x)$ as $\widetilde{\omega}_{a,r}(f,\delta) = \sup_{0\leqslant h\leqslant\delta} \|\widetilde{\Delta}_h^{a,r}(f,\,\cdot\,)\|_{L_2}$. This modulus can be regarded as a special case (for $p=2$) of the modulus

$$\widetilde{\omega}_{a,r}(f,\delta)_p := \sup_{0\leqslant h\leqslant\delta} \|\widetilde{\Delta}_h^{a,r}(f,\,\cdot\,)\|_{L_p}.$$

Here and throughout we assume that $p \in [1,+\infty]$.

Lemma 6. *The generalized Thue–Morse difference $\widetilde{\Delta}_h^{a,r}(f,x)$, $a\in\mathbb{N}$, has the following properties.*

(1) *Let f be a function in $W_1^r(\mathbb{T})$. Then the following integral representation holds*:

$$\widetilde{\Delta}_h^{a,r}(f,x) = (-1)^r \int_x^{x+h} dt_1 \int_{t_1}^{t_1+ah} dt_2 \int_{t_2}^{t_2+a^2h} dt_3 \cdots \int_{t_{r-1}}^{t_{r-1}+a^{r-1}h} f^{(r)}(t_r)\,dt_r.$$

(2) *Let $f\in L_p(\mathbb{T})$, $\varphi\in L_{p'}(\mathbb{T})$, $1/p+1/p'=1$. Then*

$$\int_{\mathbb{T}} f(x)\widetilde{\Delta}_{-h}^{r}(\varphi,x)\,dx = \int_{\mathbb{T}} \widetilde{\Delta}_h^{r}(f,x)\varphi(x)\,dx.$$

(3) *Let $f\in L_p([t_1,t_2])$ and $[\alpha,\beta]\subset(t_1,t_2)$. Then there exists $h(\alpha,\beta,t_1,t_2)>0$ such that for $|h|\leqslant h(\alpha,\beta,t_1,t_2)$ the operator $\widetilde{\Delta}_h^{a,r}$ annihilates the space $\mathcal{P}_{r-1}$ of algebraic polynomials of degree $r-1$; more precisely, $\|\widetilde{\Delta}_h^{a,r}(f,\,\cdot\,)\|_{L_p([\alpha,\beta])} = o(h^r)$ as $h\to+0 \iff f(x)=P(x)$ for almost all $x\in[\alpha,\beta]$, where $P\in\mathcal{P}_{r-1}$.*

(4) *Let $f\in W_p^r(\mathbb{T})$. Then $\widetilde{\omega}_{a,r}(f,t)_p \leqslant a^{r(r-1)/2}t^r\|f^{(r)}\|_p$ for all $t\geqslant 0$.*

(5) *For each $f\in L_p(\mathbb{T})$ the inequality $\widetilde{\omega}_{a,r}(f,n\delta)_p \leqslant n^r\widetilde{\omega}_{a,r}(f,\delta)_p$ holds for all $n\in\mathbb{N}$, $\delta\geqslant 0$.*

(6) *Let $f(x)$ be a function in the space $L_p(\mathbb{T})$ and let $\widetilde{\omega}_{a,r}(f,\delta)_p = o(\delta^r)$ as $\delta\to+0$. Then $\widetilde{\omega}_{a,r}(f,\delta)_p \equiv 0$, that is, the function $g(\delta) := \widetilde{\omega}_{a,r}(f,\delta)_p$ cannot decrease more rapidly than δ^r as $\delta\to+0$, provided that $f(x)$ is not constant a.e.*

(7) *Let $f(x)$ be a non-constant function in $W_p^r(\mathbb{T})$. Then*

$$\widetilde{\omega}_{a,r}(f,\delta)_p \asymp \delta^r \quad \textit{as } \delta \to +0.$$

Proof. Using the identity $\widetilde{\Delta}_h^{a,r}(f,x) = \widetilde{\Delta}_h^{a,r-1}(f,x) - \widetilde{\Delta}_h^{a,r-1}(f,x+a^{r-1}h)$ one can easily prove the equality in part (1) by induction on r. We now prove the second property. Obviously,

$$\int_{\mathbb{T}} f(x)\Delta_{-h}(\varphi,x)\,dx = \int_{\mathbb{T}} \Delta_h(f,x)\varphi(x)\,dx.$$

Hence

$$\begin{aligned}
\int_{\mathbb{T}} f(x)\widetilde{\Delta}_{-h}^{a,r}(\varphi,x)\,dx &= \int_{\mathbb{T}} f(x)\Delta_{-h}\Delta_{-ah}\cdots\Delta_{-a^{r-1}h}(\varphi,x)\,dx \\
&= \int_{\mathbb{T}} \Delta_h(f,x)\Delta_{-ah}\cdots\Delta_{-a^{r-1}h}(\varphi,x)\,dx \\
&= \cdots = \int_{\mathbb{T}} \Delta_{a^{r-1}h}\cdots\Delta_{ah}\Delta_h(f,x)\varphi(x)\,dx. \qquad (3.2)
\end{aligned}$$

Taking into account the commutativity property of difference operators, which means that $\Delta_{h_1}\Delta_{h_2} = \Delta_{h_2}\Delta_{h_1}$ (this is easy to see from the definition) we conclude that (3.2) yields property (2).

We now prove (3). Let $g(x)$ be a function in the space $C^r([t_1,t_2])$. Then by the integral mean value theorem [22] it follows that

$$\lim_{h\to 0} a^{-r(r-1)/2}h^{-r}\widetilde{\Delta}_h^{a,r}(g,x) = g^{(r)}(x) \qquad \forall\, x\in[\alpha,\beta]. \qquad (3.3)$$

By Hölder's inequality we obtain (for h sufficiently small in absolute value)

$$\begin{aligned}
\left|\int_\alpha^\beta f(x)a^{-r(r-1)/2}h^{-r}\widetilde{\Delta}_{-h}^{a,r}(\varphi,x)\,dx\right| &= \left|\int_\alpha^\beta a^{-r(r-1)/2}h^{-r}\widetilde{\Delta}_h^{a,r}(f,x)\varphi(x)\,dx\right| \\
&\leqslant a^{-r(r-1)/2}h^{-r}\int_\alpha^\beta |\widetilde{\Delta}_h^{a,r}(f,x)|\cdot|\varphi(x)|\,dx \\
&\leqslant a^{-r(r-1)/2}h^{-r}\|\widetilde{\Delta}_h^{a,r}(f,\,\cdot\,)\|_{L_p([\alpha,\beta])}\|\varphi\|_{L_{p'}([\alpha,\beta])}
\end{aligned}$$

for an arbitrary infinitely smooth function $\varphi\in C^\infty([t_1,t_2])$.

Letting $h\to +0$ and using (3.3) we obtain that $\displaystyle\int_\alpha^\beta f(x)\varphi^{(r)}(x)\,dx = 0$. However, in accordance with [23], Chapter 5, if a function $f(x)$ from the space $L_{\mathrm{loc}}([t_1,t_2])$ satisfies the equality

$$\int_\alpha^\beta f(x)\varphi^{(r)}(x)\,dx = 0 \quad \text{for all } \varphi\in C^\infty([t_1,t_2]),$$

then there exists an algebraic polynomial P of degree at most $r-1$ such that $f(x) = P(x)$ for almost all $x\in[\alpha,\beta]$. The proof of property (3) is thus complete.

We now prove property (4). In the equality of part (1) we make successively the following changes of the variable: $t_k^* = t_k - x,\ \ k = 1, \dots, r$:

$$\begin{aligned}\widetilde{\Delta}_h^{a,r}(f,x) &= (-1)^r \int_x^{x+h} dt_1 \int_{t_1}^{t_1+ah} dt_2 \int_{t_2}^{t_2+a^2h} dt_3 \cdots \int_{t_{r-1}}^{t_{r-1}+a^{r-1}h} f^{(r)}(t_r)\,dt_r \\ &= (-1)^r \int_0^h dt_1^* \int_{t_1^*+x}^{t_1^*+x+ah} dt_2 \int_{t_2}^{t_2+a^2h} dt_3 \cdots \int_{t_{r-1}}^{t_{r-1}+a^{r-1}h} f^{(r)}(t_r)\,dt_r \\ &= \cdots = (-1)^r \int_0^h dt_1^* \int_{t_1^*}^{t_1^*+ah} dt_2^* \int_{t_2^*}^{t_2^*+a^2h} dt_3^* \cdots \int_{t_{r-1}^*}^{t_{r-1}^*+a^{r-1}h} f^{(r)}(t_r^*+x)\,dt_r^*.\end{aligned}$$

Applying now the generalized Minkowski inequality (see [24], Chapter 1) to the last equality we obtain

$$\begin{aligned}\|\widetilde{\Delta}_h^{a,r}(f,\,\cdot\,)\|_p &\leqslant \int_0^h dt_1^* \int_{t_1^*}^{t_1^*+ah} dt_2^* \int_{t_2^*}^{t_2^*+a^2h} dt_3^* \\ &\quad\times \cdots \times \int_{t_{r-1}^*}^{t_{r-1}^*+a^{r-1}h} \|f^{(r)}(t_r^*+\,\cdot\,)\|_p\,dt_r^* = a^{r(r-1)/2}h^r\|f^{(r)}\|_p\end{aligned}$$

for all $h \geqslant 0$.

Property (5) is a consequence of the identity

$$\widetilde{\Delta}_{nh}^{a,r}(f,x) = \sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-1}\cdots\sum_{k_r=0}^{n-1}\widetilde{\Delta}_h^{a,r}(f, x+k_1h+ak_2h+\cdots+a^{r-1}k_rh).$$

This identity is obvious for $r = 1$. For $r \in \mathbb{N}$, $r \geqslant 2$ we can prove it by induction on r with the help of the equality $\widetilde{\Delta}_h^{a,r}(f,x) = \widetilde{\Delta}_h^{a,r-1}(f,x) - \widetilde{\Delta}_h^{a,r-1}(f,x+a^{r-1}h)$.

We now prove property (6). Using property (5) we conclude that

$$\widetilde{\omega}_{a,r}(f,\delta)_p = \widetilde{\omega}_{a,r}\biggl(f,\frac{n\delta}{n}\biggr)_p \leqslant n^r\widetilde{\omega}_{a,r}\biggl(f,\frac{\delta}{n}\biggr)_p = \delta^r\frac{\widetilde{\omega}_{a,r}(f,\delta/n)_p}{(\delta/n)^r}.$$

Passing to the limit as $n \to +\infty$ and assuming the f satisfies the condition $\widetilde{\omega}_{a,r}(f,h)_p = o(h^r)$ as $\delta \to +0$ we obtain the identity $\widetilde{\omega}_{a,r}(f,\delta) \equiv 0$.

We prove property (7). By property (5), $\widetilde{\omega}_{a,r}(f,\lambda\delta)_p \leqslant (\lambda+1)^r\widetilde{\omega}_{a,r}(f,\delta)_p$ for $\lambda > 0$, $\delta \geqslant 0$. Hence for each $t \in (0,1]$ we obtain

$$\widetilde{\omega}_{a,r}(f,t)_p \geqslant \frac{1}{(1+1/t)^r}\,\omega_{a,r}(f,1)_p \geqslant 2^{-r}\widetilde{\omega}_{a,r}(f,1)_pt^r = Ct^r.$$

We obtain that $C > 0$ by property (6). Now, using property (4) we obtain the required equivalence.

We shall regard the difference $\widetilde{\Delta}_t^r$ as a continuous linear operator in $L_2(\mathbb{T})$. We point out one important property of this operator: for almost all t its norm is less in order (as $r \to \infty$) than both the norm of the 'classical' difference operator of order r and the norm of the generalized Thue–Morse difference $\widetilde{\Delta}_t^{a,r}$ for all $a \in \mathbb{N}$, $a > 2$. However we observe that for the computation of the value of $\widetilde{\Delta}_t^{a,r}(f,x)$ at a point x we use the values of f at 2^r points, while for the computation of $\Delta_t^r(f,x)$ we use its values at $r+1$ points.

We set $K_{a,r}(t) := \|\widetilde{\Delta}_t^{a,r}\|^2_{L_2\to L_2}$ and prove the following result.

Lemma 7. *Assume that $t > 0$ and let $a, r \in \mathbb{N}$. Then*

$$K_{a,r}(t) \leqslant 4^r \qquad \text{if } a \text{ is odd}, \tag{3.4}$$

$$K_{a,r}(t) \leqslant \left(2\sin\frac{a\pi}{2(a+1)}\right)^{2r} \sin^{-2}\frac{a\pi}{2(a+1)} \qquad \text{if } a \text{ is even}. \tag{3.5}$$

If $t/\pi \in \mathbb{R}\setminus\mathbb{Q}$, that is, t/π is irrational, then

$$K_{a,r}(t) = 4^r \qquad \text{if } a \text{ is odd}, \tag{3.6}$$

$$\left(2\sin\frac{a\pi}{2(a+1)}\right)^{2r} \leqslant K_{a,r}(t) \leqslant \left(2\sin\frac{a\pi}{2(a+1)}\right)^{2r} \sin^{-2}\frac{a\pi}{2(a+1)} \qquad \text{if } a \text{ is even}. \tag{3.7}$$

Remark 6. If t/π is irrational, then $K_{a,r}(t)$ does not depend on t (see (1.3)), therefore in this case we write $K_{a,r}$ in place of $K_{a,r}(t)$.

Remark 7. In the case $a = 2$, $r \in \mathbb{N}$, $t/\pi \in \mathbb{R}\setminus\mathbb{Q}$ we have a two-sided estimate.[14]

Remark 8. The function $g(a) = 2\sin(a\pi/(2(a+1)))$ increases on $[1,\infty)$, therefore for each $k \in \mathbb{N}$ there exists $r_0 \in \mathbb{N}$ such that

$$3^r \leqslant K_{2,r} < K_{4,r} < K_{6,r} < \cdots < K_{2k,r} < K_{1,r} = K_{3,r} = K_{5,r} = K_{2k-1,r} = 4^r$$

for all $r \geqslant r_0$ and $t/\pi \in \mathbb{R}\setminus\mathbb{Q}$.

Remark 9. Carrying out more detailed estimates of $\|\psi_{a,r}(\,\cdot\,)\|_{C(\mathbb{T})}$ than in the proof of Lemma 7 (the estimates (3.7) are not sufficient in that case) we can show that

$$3^r < K_{2,r} < K_{4,r} < \cdots < K_{2k,r} < \cdots < K_{1,r} = K_{3,r} = \cdots = K_{2k-1,r} = \cdots = 4^r$$

for $r \in \mathbb{N}$, $r \geqslant 2$, and $t/\pi \in \mathbb{R}\setminus\mathbb{Q}$

Proof. By (1.1) and (1.9) we obtain $\|\widetilde{\Delta}_t^{a,r}\|^2_{L_2\to L_2} \leqslant \|\psi_{a,r}(\,\cdot\,)\|_{C(\mathbb{T})}$. Recall that $\psi_{a,r}(t) = 4^r\prod_{\nu=0}^{r-1}\sin^2(a^\nu t/2)$. Hence $\|\psi_{a,r}(\,\cdot\,)\|_{C(\mathbb{T})} \leqslant 4^r$ and the estimate (3.4) holds.

We now prove the estimate (3.5). First we point out that the equalities

$$\psi_{a,r}(x) = 4\sin^2\frac{x}{2}\,\psi_{a,r-1}(ax) = \psi_{a,1}(x)\psi_{a,r-1}(ax) = \psi_{a,2}(x)\psi_{a,r-2}(a^2x) \tag{3.8}$$

hold for all $r \geqslant 2$ and $a \in \mathbb{N}$.

For $r = 1$ we have $\|\psi_{a,1}\|_{C(\mathbb{T})} = 4$. By induction on $r = 2, 3, \dots$ we prove the inequality

$$\|\psi_{a,r}\|_C \leqslant 4^r\left(\sin\frac{a\pi}{2(a+1)}\right)^{2r-2}. \tag{3.9}$$

[14] The upper bound $\max_x 2^{2r}\prod_{k=0}^{r-1}\sin^2(2^{k-1}x) \leqslant \frac{4}{3}3^r$ was also established in [25].

(We bear in mind that $\|\widetilde{\Delta}_t^{a,r}\|^2_{L_2\to L_2}\leqslant\|\psi_{a,r}\|_C$.) For $r=2$ we prove the estimate (3.9) separately on the intervals $[0,a\pi/(a+1)]$ and $[a\pi/(a+1),\pi]$. Then we have

$$\max_{x\in[0,a\pi/(a+1)]}\psi_{a,2}(x)=\max_{x\in[0,a\pi/(a+1)]}4\sin^2\frac{x}{2}\,\psi_{a,1}(ax)\leqslant 16\sin^2\frac{a\pi}{2(a+1)}$$
$$=\biggl(2\sin\frac{a\pi}{2(a+1)}\biggr)^4\sin^{-2}\frac{a\pi}{2(a+1)}\,.$$

On $[a\pi/(a+1),\pi]$ one must prove a stronger inequality

$$\max_{x\in[a\pi/(a+1),\pi]}\psi_{a,2}(x)=\max_{x\in[a\pi/(a+1),\pi]}16\sin^2\frac{x}{2}\sin^2\frac{ax}{2}\leqslant\biggl(2\sin\frac{a\pi}{2(a+1)}\biggr)^4. \tag{3.10}$$

To this end we shall show that the function $\psi_{a,2}(x)$ is non-increasing on the interval $[a\pi/(a+1),\pi]$. In fact,

$$\biggl(\frac{\psi_{a,2}(x)}{4}\biggr)'=\bigl((1-\cos x)(1-\cos(ax))\bigr)'=\sin x(1-\cos(ax))+a\sin(ax)(1-\cos x)$$
$$=\sin x\sin ax\biggl(\frac{1-\cos(ax)}{\sin(ax)}+a\,\frac{1-\cos x}{\sin x}\biggr)=\sin x\sin ax\biggl(\tan\frac{ax}{2}+a\tan\frac{x}{2}\biggr).$$

For $x\in[a\pi/(a+1),\pi]$ we have the equalities $\sin x\geqslant 0$ and $\sin ax\leqslant 0$. Consequently,

$$\operatorname{sign}\psi'_{a,2}(x)=(-1)\operatorname{sign}\biggl(\tan\frac{ax}{2}+a\tan\frac{x}{2}\biggr).$$

We shall show that $|\tan(ax/2)|\leqslant a\tan(x/2)$ for $x\in[a\pi/(a+1),\pi]$. After the substitution $x=\pi-y\pi/(a+1)$, $y\in[0,1]$, we see that the last inequality is equivalent to the relation $|\tan(ay\pi/(2(a+1)))|\leqslant a\cot(y\pi/(2(a+1)))$, which we shall prove.

Since

$$\tan\frac{ay\pi}{2(a+1)}\leqslant\tan\frac{ay\pi}{2(a+1)}=\tan\biggl(\frac{\pi}{2}-\frac{\pi}{2(a+1)}\biggr)=\cot\frac{\pi}{2(a+1)}\,,$$
$$\cot\frac{y\pi}{2(a+1)}\geqslant\cot\frac{\pi}{2(a+1)}$$

for $y\in[0,1]$ and $a\in\mathbb{N}$, $a\geqslant 2$, it follows that

$$\tan\frac{ay\pi}{2(a+1)}\leqslant\cot\frac{y\pi}{2(a+1)}\leqslant a\cot\frac{y\pi}{2(a+1)}\,.$$

Hence $\operatorname{sign}\psi'_{a,2}(x)=-1$ for $x\in[a\pi/(a+1),\pi)$, $a\in\mathbb{N}$, $a\geqslant 2$. Consequently, the function $\psi_{a,2}(x)$ is non-increasing on the interval $[a\pi/(a+1),\pi]$.

Assume that we have already proved inequality (3.9) for $r=2,3,\dots,k$, $k\geqslant 2$. We shall prove it also for $r=k+1$.

As before, we find separate estimates of the function $\psi_{a,k+1}$ on $[0, a\pi/(a+1)]$ and $[a\pi/(a+1), \pi]$. We have

$$\max_{x\in[0,a\pi/(a+1)]} \psi_{a,k+1}(x) = \max_{x\in[0,a\pi/(a+1)]} 4\sin^2\frac{x}{2}\,\psi_{a,k}(ax)$$
$$\leqslant 4\sin^2\frac{a\pi}{2(a+1)}\,\|\psi_{a,k}(a\,\cdot\,)\|_{C(\mathbb{T})} \leqslant \biggl(2\sin\frac{a\pi}{2(a+1)}\biggr)^{2r}\sin^{-2}\frac{a\pi}{2(a+1)}\,.$$

For the interval $[a\pi/(a+1), \pi]$ we use (3.8), (3.10), and the monotonicity of the function $\psi_{a,2}(x)$:

$$\max_{x\in[a\pi/(a+1),\pi]} \psi_{a,k+1}(x) = \max_{x\in[a\pi/(a+1),\pi]} \psi_{a,2}(x)\psi_{a,k-1}(ax)$$
$$\leqslant \biggl(2\sin\frac{a\pi}{2(a+1)}\biggr)^{4}\|\psi_{a,k-1}(a\,\cdot\,)\|_{C(\mathbb{T})} \leqslant \biggl(2\sin\frac{a\pi}{2(a+1)}\biggr)^{2r}\sin^{-2}\frac{a\pi}{2(a+1)}\,.$$

We have thus proved (3.9) and (3.5). On the other hand if $t/\pi \in \mathbb{R}\setminus\mathbb{Q}$, then

$$\|\widetilde{\Delta}_t^{a,r}\|^2_{L_2\to L_2} = \|\psi_{a,r}(\,\cdot\,)\|_{C(\mathbb{T})} = \max_t 2^{2r}\prod_{k=0}^{r-1}\sin^2\biggl(a^k\frac{t}{2}\biggr)$$

by (1.3) and (1.9). For odd a and non-negative integer k the equality $\sin^2(a^k\pi/2)=1$ yields $\|\widetilde{\Delta}_t^{a,r}\|^2_{L_2\to L_2} = 4^r$. Hence (3.6) holds.

We now prove the first inequality in (3.7). To this end for even a we prove the equalities

$$\sin^2\frac{a\pi}{2(a+1)} = \sin^2\frac{a^2\pi}{2(a+1)} = \dots = \sin^2\frac{a^k\pi}{2(a+1)} \tag{3.11}$$

by induction on $k = 1, 2, \dots$. Assume that equality (3.11) holds for some $k \in \mathbb{N}$. We claim that (3.11) holds also for $k+1$. It follows from the relation $a^{k+1} = (a+1)a^k - a^k$ and the assumption that a is even that

$$\sin^2\frac{a^{k+1}\pi}{2(a+1)} = \sin^2\biggl(\frac{a^k\pi}{2} - \frac{a^k\pi}{2(a+1)}\biggr) = \sin^2\frac{a^k\pi}{2(a+1)}\,.$$

The proof of (3.11) is complete. Consequently,

$$\|\psi_r(\,\cdot\,)\|_{C(\mathbb{T})} \geqslant \psi_{a,r}\biggl(\frac{a\pi}{a+1}\biggr) = 4^r\prod_{\nu=0}^{r-1}\sin^2\frac{a^{\nu+1}\pi}{2(a+1)}$$
$$= 4^r\prod_{\nu=0}^{r-1}\sin^2\frac{a\pi}{2(a+1)} = \biggl(2\sin\frac{a\pi}{2(a+1)}\biggr)^{2r}$$

which proves inequalities (3.7).

Lemma 8. *If* $f \in W_2^r(\mathbb{T})$, $\int_0^{2\pi} f(x)\,dx = 0$, $a \in \mathbb{N}$, *the quantity* $t/(2\pi)$ *is irrational and has bounded convergents, then there exists a function* $g \in L_2(\mathbb{T})$ *such that*

$$f(x) = \widetilde{\Delta}_t^{a,r} g(x) \qquad \textit{for almost all } x. \tag{3.12}$$

Proof. We define a sequence $\{\lambda_s\}_{s\in\mathbb{Z}}$ by the formula

$$\lambda_s = \begin{cases} \left((is)^r \prod_{k=0}^{r-1} (1 - e_{a^k}(st))\right)^{-1} & \text{if } s \neq 0; \\ 0 & \text{if } s = 0, \end{cases}$$

and define a linear operator T_λ on the set of trigonometric polynomials by the formula

$$T_\lambda[e_s(\,\cdot\,)] = \lambda_s e_s(\,\cdot\,).$$

Bearing in mind that

$$\frac{d^r}{dx^r} e_s(x) = (is)^r e_s(x)$$

we obtain

$$T_\lambda[\widetilde{\Delta}_t^{a,r} P^{(r)}(\,\cdot\,)] = P(\,\cdot\,)$$

for each trigonometric polynomial $P(\,\cdot\,)$ with mean value zero.

Note that

$$\begin{aligned} |\lambda_s|^{-1} &= |s|^r 2^r \prod_{k=0}^{r-1} \left|\sin\left(a^{k-1} s \frac{t}{2}\right)\right| = |s|^r 2^r \prod_{k=0}^{r-1} \left|\sin\left(a^{k-1} s\pi \frac{t}{2\pi}\right)\right| \\ &\geqslant |s|^r 2^r \prod_{k=0}^{r-1} 2 \left\|a^{k-1} s \frac{t}{2\pi}\right\|, \end{aligned}$$

where $\|y\|$ is the distance from y to the nearest integer.

It is well known ([18], Chapter 1, § 5) that the boundedness of convergents of a quantity α means the existence of $C(\alpha) > 0$ such that $|s|\,\|s\alpha\| \geqslant C(\alpha)$ for all $s \in \mathbb{Z}$, $s \neq 0$. Consequently,

$$\begin{aligned} |s|^r 2^{2r} \prod_{k=0}^{r-1} \left\|a^{k-1} s \frac{t}{2\pi}\right\| &= 2^{2r} \cdot a^0 \cdots a^{1-r} \prod_{k=0}^{r-1} a^k |s| \left\|a^k s \frac{t}{2\pi}\right\| \\ &\geqslant 2^{2r} a^{-r(r-1)/2} C^r\left(\frac{t}{2\pi}\right), \end{aligned}$$

and therefore the absolute values of the elements of the sequence $\{\lambda_s\}_{s\in\mathbb{Z}}$ are uniformly bounded. It follows (see [1], § 16.1) that T_λ can be extended to a continuous linear translation-invariant operator in $L_2(\mathbb{T})$ (which we also denote by T_λ).

Thus,

$$\widetilde{\Delta}_t^{a,r} T_\lambda[f^{(r)}](x) = T_\lambda[\widetilde{\Delta}_t^{a,r} f^{(r)}](x) = f(x) \text{ for almost all } x.$$

Hence for the function $f(\,\cdot\,)$ we have the representation (3.12) in which $g(\,\cdot\,)$ can be set equal to $T_\lambda f^{(r)}$.

§ 4. Proof of Theorem 4

Recall that the notation $\xi(\,\cdot\,) \in \Upsilon$ means that the function $\xi(\,\cdot\,)$ satisfies the following assumptions:

(1) $\xi(\,\cdot\,) \in C(\mathbb{R})$;
(2) $\xi(x) \geqslant 0$;
(3) $\xi(x) = \xi(\pi + x)$, that is, $\xi(\,\cdot\,)$ is π-periodic;
(4) $\xi(x) = \xi(-x)$, that is, $\xi(\,\cdot\,)$ is even.

The basic idea of the proof of Theorem 4 is the demonstration of the non-negativity of $h_\xi(x)$ for all $x \geqslant 1$ and $\xi(\,\cdot\,) \in \Upsilon$ (see Lemma 13).

We shall prove some auxiliary lemmas.

Lemma 9. *Let* $\xi(x) \in \Upsilon$, $k = 2l + 1$, $l \in \mathbb{Z}_+$. *Then*

$$-\int_0^{k\pi/x} \xi(xt)\cos(xt)\sin t\,dt = 2\cos\biggl(\frac{k\pi}{2x}\biggr)\int_0^{\pi/(2x)} \xi(xt) \times \cos(xt)\biggl(\sin\biggl(\frac{k\pi}{2x} - t\biggr) - \frac{\sin(l\pi/x)}{\cos(\pi/(2x))}\cos t\biggr)\,dt$$

for all $x \geqslant 1$.

Proof. We split the integral into two:

$$-\int_0^{k\pi/x} \xi(xt)\cos(xt)\sin t\,dt = I_1 + I_2,$$

where

$$I_1 = -\int_0^{k\pi/(2x)} \xi(xt)\cos(xt)\sin t\,dt, \qquad I_2 = -\int_{k\pi/(2x)}^{k\pi/x} \xi(xt)\cos(xt)\sin t\,dt.$$

In the integral I_2 we make the change of the variable $t = k\pi/x - z$, where z is a new variable. Replacing z by t again and bearing in mind that k is odd we obtain

$$I_2 = \int_0^{k\pi/(2x)} \xi(xt)\cos(xt)\sin\biggl(\frac{k\pi}{x} - t\biggr)\,dt.$$

We continue the transformations as follows:

$$\begin{aligned} I_1 + I_2 &= \int_0^{k\pi/(2x)} \xi(xt)\cos(xt)\biggl(\sin\biggl(\frac{k\pi}{x} - t\biggr) - \sin t\biggr)\,dt \\ &= 2\cos\biggl(\frac{k\pi}{2x}\biggr)\int_0^{k\pi/(2x)} \xi(xt)\cos(xt)\sin\biggl(\frac{k\pi}{2x} - t\biggr)\,dt \\ &= 2\cos\biggl(\frac{k\pi}{2x}\biggr)\biggl(\int_0^{\pi/(2x)} + \int_{\pi/(2x)}^{3\pi/(2x)} + \dots + \int_{(l-1/2)\pi/x}^{(l+1/2)\pi/x}\biggr) \\ &= 2\cos\biggl(\frac{k\pi}{2x}\biggr)(J_0 + J_1 + \dots + J_l). \end{aligned}$$

In the integrals J_s, $s=1,\dots,l$, we make the change of the variable $t=s\pi/x-z$, where z is a new variable. Replacing z by t again we obtain

$$J_s=(-1)^s\int_{-\pi/(2x)}^{\pi/(2x)}\xi(xt)\cos(xt)\sin\biggl(\frac{(k-2s)\pi}{2x}+t\biggr)\,dt$$
$$=(-1)^s\biggl(\int_0^{\pi/(2x)}+\int_{-\pi/(2x)}^0\biggr).$$

Making a change of the variable in the last integral and bearing in mind that $\xi(t)\cos t$ is an even function we see that

$$J_s=(-1)^s\int_0^{\pi/(2x)}\xi(xt)\cos(xt)\biggl(\sin\biggl(\frac{(k-2s)\pi}{2x}+t\biggr)+\sin\biggl(\frac{(k-2s)\pi}{2x}-t\biggr)\,dt\biggr)$$
$$=(-1)^s2\sin\biggl(\frac{(k-2s)\pi}{2x}\biggr)\int_0^{\pi/(2x)}\xi(xt)\cos(xt)\cos t\,dt.$$

Finally, we obtain

$$I_1+I_2=2\cos\biggl(\frac{k\pi}{2x}\biggr)\int_0^{\pi/(2x)}\xi(xt)\cos(xt)r_l(x,t)\,dt,$$

where

$$r_l(x,t)=\sin\biggl(\frac{k\pi}{2x}-t\biggr)+2\cos t\biggl(-\sin\biggl(\frac{(k-2)\pi}{2x}\biggr)$$
$$+\sin\biggl(\frac{(k-4)\pi}{2x}\biggr)+\cdots+(-1)^l\sin\biggl(\frac{\pi}{2x}\biggr)\biggr).$$

We now calculate the last sum. Setting $v:=\pi/(2x)$ we obtain

$$(-1)^l(\sin v-\sin(3v)+\cdots+(-1)^l\sin((2l-3)v)+(-1)^{l+1}\sin((2l-1)v))$$
$$=\frac{(-1)^l}{2\sin(2v)}(2\sin(2v))\bigl(\sin v-\sin(3v)+\cdots+(-1)^l\sin((2l-3)v)$$
$$+(-1)^{l+1}\sin((2l-1)v)\bigr)=\frac{(-1)^l}{2\sin(2v)}\bigl((\cos v-\cos(3v))-(\cos v-\cos(5v))$$
$$+(\cos(3v)-\cos(7v))-(\cos(5v)-\cos(9v))$$
$$+\cdots+(-1)^l\bigl(\cos((2l-5)v)-\cos((2l-1)v)\bigr)$$
$$+(-1)^{l+1}\bigl(\cos((2l-3)v)-\cos((2l+1)v)\bigr)\bigr)$$
$$=\frac{(-1)^l}{2\sin(2v)}\bigl((-1)^l\bigl(\cos((2l+1)v)-\cos((2l-1)v)\bigr)\bigr)$$
$$=-\frac{\sin(2lv)\sin v}{\sin(2v)}=-\frac{\sin(2lv)}{2\cos v}\,.$$

As a result,

$$r_l(x,t)=\sin\biggl(\frac{k\pi}{2x}-t\biggr)-\cos t\frac{\sin(l\pi/x)}{\cos(\pi/(2x))}\,. \tag{4.1}$$

We now find a convenient representation for the function $r_l(x,t)$.

Lemma 10. *Let $k = 2l + 1$, $l \in \mathbb{Z}_+$. Then for $x > 1$ one has*

$$r_l(x,t) = \begin{cases} \sin\left(\dfrac{\pi}{2x} - t\right), & l = 0; \\ \cos\left(\dfrac{k\pi}{2x}\right)\cos t\left(\tan\dfrac{\pi}{2x} - \tan t\right), & l \in \mathbb{N}, \end{cases}$$

where the function $r_l(x,t)$ is defined by equality (4.1).

Proof. For $l = 0$ the assertion is trivial. We shall prove it for $l \in \mathbb{N}$. We have

$$\begin{aligned} r_l(x,t) &= \sin\left(\frac{k\pi}{2x}\right)\cos t - \cos\left(\frac{k\pi}{2x}\right)\sin t - \frac{\sin(l\pi/x)}{\cos(\pi/(2x))}\cos t \\ &= \frac{\cos t}{\cos(\pi/(2x))}\left(\cos\frac{\pi}{2x}\sin\frac{k\pi}{2x} - \sin\frac{l\pi}{x}\right) - \cos\left(\frac{k\pi}{2x}\right)\sin t. \end{aligned}$$

We now consider the expression in the parentheses:

$$\begin{aligned} &\cos\frac{\pi}{2x}\sin\frac{(l+1/2)\pi}{x} - \sin\frac{l\pi}{x} = \cos\frac{\pi}{2x}\sin\frac{\pi}{2x}\cos\frac{l\pi}{x} + \cos^2\frac{\pi}{2x}\sin\frac{l\pi}{x} - \sin\frac{l\pi}{x} \\ &\quad = \cos\frac{\pi}{2x}\sin\frac{\pi}{2x}\cos\frac{l\pi}{x} - \sin^2\frac{\pi}{2x}\sin\frac{l\pi}{x} = \sin\frac{\pi}{2x}\left(\cos\frac{\pi}{2x}\cos\frac{l\pi}{x} - \sin\frac{\pi}{2x}\sin\frac{l\pi}{x}\right) \\ &\quad = \sin\frac{\pi}{2x}\cos\frac{k\pi}{2x}. \end{aligned}$$

Hence

$$r_l(x,t) = \cos\frac{k\pi}{2x}\left(\frac{\cos t}{\cos(\pi/(2x))}\sin\frac{\pi}{2x} - \sin t\right) = \cos\frac{k\pi}{2x}\cos t\left(\tan\frac{\pi}{2x} - \tan t\right).$$

Definition 4.1. For $\xi(\,\cdot\,) \in \Upsilon$ we define an auxiliary function $R_\xi\colon [1,\infty) \to \mathbb{R}$ on intervals of the form $x \in [2l, 2l+1/2]$, $l \in \mathbb{N}$, by the equality

$$\begin{aligned} R_\xi(x) = 2\cos\frac{(2l+1/2)\pi}{x}\int_\pi^{(2l+1/2)\pi/x}\xi(xt)\cos(xt)\sin\left(t - \frac{(2l+1/2)\pi}{x}\right)dt \\ + \int_{(4l+1-x)\pi/x}^{(2l+1)\pi/x}\xi(xt)\cos(xt)\sin t\,dt, \end{aligned} \tag{4.2}$$

on intervals of the form $x \in [2l+1/2, 2l+1]$, $l \in \mathbb{N}$, by the equality

$$R_\xi(x) = \int_\pi^{(2l+1)\pi/x}\xi(xt)\cos(xt)\sin t\,dt, \tag{4.3}$$

on intervals of the form $x \in [2l+1, 2l+3/2]$, $l \in \mathbb{Z}_+$, by the equality

$$R_\xi(x) = -\int_{(2l+1)\pi/x}^{\pi}\xi(xt)\cos(xt)\sin t\,dt, \tag{4.4}$$

and on intervals of the form $x \in [2l+3/2, 2l+2]$, $l \in \mathbb{Z}_+$, by the equality

$$\begin{aligned} R_\xi(x) = 2\cos\frac{(2l+3/2)\pi}{x}\int_{(2l+3/2)\pi/x}^{\pi}\xi(xt)\cos(xt)\sin\left(\frac{(2l+3/2)\pi}{x} - t\right)dt \\ - \int_{(2l+1)\pi/x}^{(4l+3-x)\pi/x}\xi(xt)\cos(xt)\sin t\,dt. \end{aligned} \tag{4.5}$$

Lemma 11. *Let* $\xi(\,\cdot\,) \in \Upsilon$. *Then*

(1) $R_\xi(x) = 0$ *for* $x = 2l+1$, $l \in \mathbb{Z}_+$;

(2) $R_\xi(x) \geqslant 0$ *for* $x \in (2l+1, 2l+3)$, $l \in \mathbb{Z}_+$.

Proof. Since $R_\xi(x)$ is defined on $[1, +\infty)$ by several formulae, we shall prove the lemma separately on each of the corresponding subsets. We set

$$[1, +\infty) = A_1 \cup A_2 \cup A_3 \cup A_4,$$

where $A_1 = \bigcup_{l\in\mathbb{N}}[2l, 2l+1/2]$, $A_2 = \bigcup_{l\in\mathbb{N}}[2l+1/2, 2l+1]$, $A_3 = \bigcup_{l\in\mathbb{Z}_+}[2l+1, 2l+3/2]$, $A_4 = \bigcup_{l\in\mathbb{Z}_+}[2l+3/2, 2l+2]$.

Let $x \in A_1$, so that there exists $l \in \mathbb{N}$ such that $x \in [2l, 2l+1/2]$. We shall prove that each integral in (4.2) is non-negative. Indeed, in the first integral the set of integration with respect to t is the interval $[\pi, (2l+1/2)\pi/x]$. Consequently, $xt \in [2l\pi, (2l+1/2)\pi]$, and therefore $\cos(xt) \geqslant 0$, $\sin(t - (2l+1/2)\pi/x) \leqslant 0$, and also $\cos((2l+1/2)\pi/x) \leqslant 0$ because $(2l+1/2)\pi/x \in [\pi, \pi + \pi/(4l)]$. It now follows by (4.2) that the first term of the sum is non-negative. For the second integral the set of integration with respect to t is $[(4l+1-x)\pi/x, (2l+1)\pi/x]$. Consequently, $xt \in [(2l+1/2)\pi, (2l+1)\pi]$, therefore $\cos(xt) \leqslant 0$, $\sin t \leqslant 0$, and we conclude from (4.2) that

$$R_\xi(x) \geqslant 0, \qquad x \in A_1.$$

Now let $x \in A_2$, so that there exists $l \in \mathbb{N}$ such that $x \in [2l+1/2, 2l+1]$. We shall prove that the integral defined by (4.3) is non-negative. Actually, the set of integration with respect to t is the interval $[\pi, (2l+1)\pi/x]$. We have $xt \in [(2l+1/2)\pi, (2l+1)\pi]$. Hence $\cos(xt) \leqslant 0$, $\sin t \leqslant 0$, and therefore it follows by (4.3) that

$$R_\xi(x) = \begin{cases} \geqslant 0, & x \in A_2; \\ 0, & x = 2l+1, \ l \in \mathbb{N}. \end{cases}$$

Let $x \in A_3$, so that there exists $l \in \mathbb{Z}_+$ such that $x \in [2l+1, 2l+3/2]$. We shall show that the expression in (4.4) is non-negative. In fact, the set of integration with respect to t is the interval $[(2l+1)\pi/x, \pi]$. We have $xt \in [(2l+1)\pi, (2l+3/2)\pi]$; hence $\cos(xt) \leqslant 0$, $\sin t \geqslant 0$; it therefore follows from (4.4) that

$$R_\xi(x) = \begin{cases} \geqslant 0, & x \in A_3; \\ 0, & x = 2l+1, \ l \in \mathbb{Z}_+. \end{cases}$$

Finally let $x \in A_4$, so that there exists $l \in \mathbb{Z}_+$ such that $x \in [2l+3/2, 2l+2]$. We shall prove that each term in formula (4.5) is non-negative. In fact, for the first integral the set of integration with respect to t is the interval $[(2l+3/2)\pi/x, \pi]$. Consequently, $xt \in [(2l+3/2)\pi, (2l+2)\pi]$, therefore $\cos(xt) \geqslant 0$ and $\sin((2l+3/2)\pi/x - t) \leqslant 0$. Since $(2l+3/2)\pi/x \in [\pi - \pi/(4(l+1)), \pi]$, it follows that $\cos((2l+3/2)\pi/x) \leqslant 0$. Hence the first term of the sum in (4.5) is non-negative. For the second integral the set of integration with respect to t is the interval $t \in [(2l+1)\pi/x, (4l+3-x)\pi/x]$. Consequently, $xt \in [(2l+1)\pi, (2l+3/2)\pi]$; hence $\cos(xt) \leqslant 0$, $\sin t \geqslant 0$, and (4.5) yields

$$R_\xi(x) \geqslant 0, \qquad x \in A_4.$$

Since the above-considered intervals cover the entire half-axis $[1, +\infty)$, the proof of the lemma is now complete.

Recall that for $\xi(\,\cdot\,) \in \Upsilon$ we denote by $h_\xi(x)$ the following expression (see (1.12)): $(-1)\displaystyle\int_0^\pi \cos(xt)\xi(xt)\sin t\,dt$. We now formulate the basic lemma.

Lemma 12 (the integral representation). *The representations*

$$h_\xi(x) = 2\cos\biggl(\frac{\pi}{2x}\biggr)\int_0^{\pi/(2x)} \xi(xt)\cos(xt)\sin\biggl(\frac{\pi}{2x} - t\biggr)\,dt + R_\xi(x), \qquad x \in [1,2],$$

and

$$h_\xi(x) = 2\cos^2\biggl(\frac{(2l+1)\pi}{2x}\biggr)\int_0^{\pi/(2x)} \xi(xt)\cos(xt)\cos t\biggl(\tan\frac{\pi}{2x} - \tan t\biggr)\,dt + R_\xi(x),$$
$$x \in [2l, 2l+2], \quad l \in \mathbb{N}$$

hold for each $\xi(\,\cdot\,) \in \Upsilon$.

Proof. We split the proof into 4 parts.

(1) Let $x \in [2l, 2l+1/2]$, $l \in \mathbb{N}$. Then $(2l+1)\pi/x \geqslant (4l+1-x)\pi/x \geqslant (2l+1/2)\pi/x \geqslant \pi$. We represent the integral in the following form:

$$-\int_0^\pi \xi(xt)\cos(xt)\sin t\,dt$$
$$= -\int_0^{(2l+1)\pi/x} + \int_\pi^{(2l+1/2)\pi/x} + \int_{(2l+1/2)\pi/x}^{(4l+1-x)\pi/x} + \int_{(4l+1-x)\pi/x}^{(2l+1)\pi/x}.$$

In the penultimate integral we make the change of variable $t = (4l+1)\pi/x - z$, where z is a new variable; then replacing z by t we obtain

$$\int_{(2l+1/2)\pi/x}^{(4l+1-x)\pi/x} \xi(xt)\cos(xt)\sin t\,dt$$
$$= -\int_\pi^{(2l+1/2)\pi/x} \xi(xt)\cos(xt)\sin\biggl(\frac{(4l+1)\pi}{x} - t\biggr)\,dt.$$

We can continue as follows:

$$-\int_0^\pi \xi(xt)\cos(xt)\sin t\,dt = -\int_0^{(2l+1)\pi/x} + \int_{(4l+1-x)\pi/x}^{(2l+1)\pi/x}$$
$$+ \int_\pi^{(2l+1/2)\pi/x} \xi(xt)\cos(xt)\biggl(\sin t - \sin\biggl(\frac{(4l+1)\pi}{x} - t\biggr)\biggr)\,dt$$
$$= -\int_0^{(2l+1)\pi/x} + \int_{(4l+1-x)\pi/x}^{(2l+1)\pi/x}$$
$$+ 2\cos\biggl(\frac{(2l+1/2)\pi}{x}\biggr)\int_\pi^{(2l+1/2)\pi/x} \xi(xt)\cos(xt)\sin\biggl(t - \frac{(2l+1/2)\pi}{x}\biggr)\,dt$$
$$= -\int_0^{(2l+1)\pi/x} + R_\xi(x).$$

Using Lemmas 9 and 10 we obtain the required representation for $x \in [2l, 2l+1/2]$, $l \in \mathbb{N}$.

(2) For $x \in [2l+1/2, 2l+1]$, $l \in \mathbb{N}$, we use the relation

$$h_\xi(x) = -\int_0^\pi \xi(xt)\cos(xt)\sin t\,dt = -\int_0^{(2l+1)\pi/x} + \int_\pi^{(2l+1)\pi/x}$$
$$= -\int_0^{(2l+1)\pi/x} + R_\xi(x).$$

Using Lemmas 9 and 10 we obtain the required representation for x lining in $[2l+1/2, 2l+1]$, $l \in \mathbb{N}$.

(3) For $x \in [2l+1, 2l+3/2]$, $l \in \mathbb{Z}_+$, we use the following relation:

$$h_\xi(x) = -\int_0^\pi \xi(xt)\cos(xt)\sin t\,dt = -\int_0^{(2l+1)\pi/x} - \int_{(2l+1)\pi/x}^{\pi}$$
$$= -\int_0^{(2l+1)\pi/x} + R_\xi(x).$$

Again, Lemmas 9 and 10 produce the required representation for x lying in $[2l+1, 2l+3/2]$, $l \in \mathbb{Z}_+$.

(4) Now let $x \in [2l+3/2, 2l+2]$, $l \in \mathbb{Z}_+$. Note that for such x we have $(2l+1)\pi/x \leqslant (4l+3-x)\pi/x \leqslant (2l+3/2)\pi/x \leqslant \pi$. Hence we can represent the integral in question as follows:

$$-\int_0^\pi \xi(xt)\cos(xt)\sin t\,dt$$
$$= -\int_0^{(2l+1)\pi/x} - \int_{(2l+1)\pi/x}^{(4l+3-x)\pi/x} - \int_{(4l+3-x)\pi/x}^{(2l+3/2)\pi/x} - \int_{(2l+3/2)\pi/x}^{\pi}.$$

In the penultimate integral we make the change of variable $t = (4l+3)\pi/x - z$, where z is a new variable; then replacing z by t we obtain

$$-\int_{(4l+3-x)\pi/x}^{(2l+3/2)\pi/x} \xi(xt)\cos(xt)\sin t\,dt = \int_{(2l+3/2)\pi/x}^{\pi} \xi(xt)\cos(xt)\sin\biggl(\frac{(4l+3)\pi}{x} - t\biggr)\,dt.$$

We can now continue as follows:

$$-\int_0^\pi \xi(xt)\cos(xt)\sin t\,dt = -\int_0^{(2l+1)\pi/x} - \int_{(2l+1)\pi/x}^{(4l+3-x)\pi/x}$$
$$+ \int_{(2l+3/2)\pi/x}^{\pi} \xi(xt)\cos(xt)\biggl(\sin\biggl(\frac{(4l+3)\pi}{x} - t\biggr) - \sin t\biggr)\,dt$$
$$= -\int_0^{(2l+1)\pi/x} - \int_{(2l+1)\pi/x}^{(4l+3-x)\pi/x}$$
$$+ 2\cos\biggl(\frac{(2l+3/2)\pi}{x}\biggr)\int_{(2l+3/2)\pi/x}^{\pi} \xi(xt)\cos(xt)\sin\biggl(\frac{(2l+3/2)\pi}{x} - t\biggr)\,dt$$
$$= -\int_0^{(2l+1)\pi/x} + R_\xi(x).$$

On the basis of Lemmas 9 and 10 we now obtain the required representation also for $x \in [2l+3/2, 2l+2]$, $l \in \mathbb{Z}_+$.

Lemmas 11 and 12 yield the following result.

Lemma 13. *Let $\xi \in \Upsilon$, $\xi(x) \not\equiv 0$. Then*

$$h_\xi(x) = \begin{cases} > 0, & x \in (2l+1, 2l+3),\ l \in \mathbb{Z}_+; \\ = 0, & x = 2l+1,\ l \in \mathbb{Z}_+. \end{cases}$$

Proof. We set (for $l \in \mathbb{N}$)

$$g_\xi(x) = \begin{cases} 2\cos\biggl(\dfrac{\pi}{2x}\biggr) \displaystyle\int_0^{\pi/(2x)} \xi(xt)\cos(xt)\sin\biggl(\dfrac{\pi}{2x} - t\biggr)\,dt & \text{if } 1 \leqslant x \leqslant 2; \\ 2\cos^2\biggl(\dfrac{(2l+1)\pi}{2x}\biggr) \displaystyle\int_0^{\pi/(2x)} \xi(xt)\cos(xt) & \\ \qquad \times \cos t\biggl(\tan\dfrac{\pi}{2x} - \tan t\biggr)\,dt & \text{if } 2l \leqslant x \leqslant 2l+2. \end{cases}$$

By Lemma 12 we obtain $h_\xi(x) = g_\xi(x) + R_\xi(x)$, and it follows from Lemma 11 that $R_\xi(x) = 0$ if x is odd and $R_\xi(x) \geqslant 0$ if $x \in (2l+1, 2l+3)$, $l \in \mathbb{Z}_+$.

By the definition of the class Υ it follows, in particular, that $\xi(\,\cdot\,)$ and also $g_\xi(x)$ is non-negative, and therefore is non-negative for all $x \geqslant 1$. Moreover, it is clear that $g_\xi(2l+1) = 0$ for all $l \in \mathbb{Z}_+$.

Hence $h_\xi(x) > 0$ for $\xi(x) \not\equiv 0$, $x \in (2l+1, 2l+3)$, $l \in \mathbb{Z}_+$, and $h_\xi(2l+1) = 0$ for all $l \in \mathbb{Z}_+$. This completes the proof.

Example 4.1. Let $\xi(x) \equiv 1$. Then

$$h_1(x) = -\int_0^\pi \cos(xt)\sin t\,dt = \frac{1+\cos(\pi x)}{x^2-1}\,.$$

In fact, taking the equality $-\cos(xt)\sin t = (1/2)(\sin((x-1)t) - \sin((x+1)t))$ into account and integrating it with respect to t from 0 to π we obtain

$$h_1(x) = \frac{1}{2}\biggl(\frac{\cos((x+1)t)}{x+1} - \frac{\cos((x-1)t)}{x-1}\biggr)\bigg|_{t=0}^{\pi} = \frac{1+\cos(\pi x)}{x^2-1}\,.$$

This example shows that it is impossible to replace in Lemma 13 the range $[1, +\infty)$, of the variable x by a larger set because $h_1(x) < 0$ for $x \in (-1, 1)$.

Completion of the proof of Theorem 4. It follows from the definition of $\Psi(\mathbf{b})$ that ψ is an even function in the class $C_0^+(\mathbb{T})$. Hence by Corollary 1.1 we obtain

$$\overline{\varkappa}_\psi(n,\delta) \geqslant \overline{\varkappa}_\psi(n,\pi) = J_\psi^{-1/2} \tag{4.6}$$

for all $\delta > 0$, where $J_\psi = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\psi(x)\,dx$. It also follows from the definition of the class $\Psi(\mathbf{b})$ that for the function ψ we have the following representation:

$$\int_0^\pi \psi(xt)\frac{\sin t}{2}\,dt = J_\psi + \sum_k h_{\xi_k}(b_k x),$$

where $\xi_k(\,\cdot\,) \in \Upsilon$. It follows by Lemma 13 that $h_{\xi_k}(b_k x) \geqslant 0$ for all $b_k x \geqslant 1$. Consequently, $\displaystyle\int_0^\pi \psi(xt)\frac{\sin t}{2}\,dt \geqslant J_\psi$ for all $x \geqslant \widehat{b}$. It follows by Lemma 5 that

$$\overline{\varkappa}_\psi(n,\delta) \leqslant \biggl(\max_{\mu\in B\mathcal{M}^+([0,\pi])}\ \inf_{x\geqslant\widehat{b}} \int_0^\pi \psi(xt)\,d\mu(t)\biggr)^{-1/2}$$

for all $\delta \geqslant \widehat{b}\pi/n$.

Since the function $w(t) = (\sin t)/2$ obviously satisfies the conditions

$$\int_0^\pi w(t)\,dt = 1, \qquad w(t) \geqslant 0 \text{ for all } t \in [0,\pi],$$

setting $d\mu(t) = w(t)\,dt$ we obtain the inequalities

$$\max_{\mu\in B\mathcal{M}^+([0,\pi])}\ \inf_{x\geqslant\widehat{b}} \int_0^\pi \psi(xt)\,d\mu(t) \geqslant \inf_{x\geqslant\widehat{b}} \int_0^\pi \psi(xt)\frac{\sin t}{2}\,dt \geqslant J_\psi,$$

and therefore $\overline{\varkappa}_\psi(n,\delta) \leqslant J_\psi^{-1/2}$ for $\delta \geqslant \widehat{b}\pi/n$. The last estimate and inequalities (4.6) yield the required result.

§ 5. Proof of Corollary 1.7

For the proof of Corollary 1.7 it is sufficient to verify that the function $\psi_{a,r}(x) = 2^r\prod_{k=0}^{r-1}(1-\cos(a^k x))$ belongs to the class $\Psi(\mathbf{b})$ for some finite set $\mathbf{b}$ of numbers the smallest of which is equal to 1. The property $\psi_{a,r}(x) \in C_0^+(\mathbb{T})$ and the evenness of the function $\psi_{a,r}$ are obvious. Hence to fulfill the required condition on $\psi_{a,r}(\,\cdot\,)$ one must produce functions $\xi_k(\,\cdot\,) = \xi_{k,a,r}(\,\cdot\,) \in \Upsilon$ and quantities $b_k = b_{k,a,r} \geqslant 1$ such that

$$\int_0^\pi \psi_{a,r}(xt)\frac{\sin t}{2}\,dt = J_{\psi_{a,r}} + \sum_k h_{\xi_k}(b_k x) \quad \text{for all } x \geqslant 1.$$

We point out first that by the orthogonality of the trigonometric system we obtain $J_{\psi_{a,r}} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \psi_{a,r}(x)\,dx = 2^r$ for $r, a \in \mathbb{N}$, $a \geqslant 2$. Hence for the proof of the corollary it is sufficient to establish the relation

$$\int_0^\pi \psi_{a,r}(xt)\frac{\sin t}{2}\,dt = 2^r + \sum_k h_{\xi_k}(b_k x). \tag{5.1}$$

In the present section we prove that the function

$$\xi_{0,a,r}(x) \equiv 2^{r-1},$$
$$\xi_{k,a,r}(x) = 2^{r-1}\prod_{s=1}^{k}(1-\cos(a^s x)), \qquad k = 1,\dots,r-1,$$

and the quantities $b_k = a^{r-1-k}$, $k = 0,\dots,r-1$, satisfy relation (5.1) (and the condition $\xi_{k,a,r} \in \Upsilon$).

Definition 5.1. For a fixed positive integer r and positive even a let us introduce the following functions:

$$\begin{aligned}
&\widetilde{\xi}_{0,a}(x) \equiv 1,\\
&\widetilde{\xi}_{1,a}(x) = 1-\cos(ax),\\
&\widetilde{\xi}_{2,a}(x) = (1-\cos(ax))(1-\cos(a^2x)),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\widetilde{\xi}_{r-1,a}(x) = (1-\cos(ax))(1-\cos(a^2x))\cdots(1-\cos(a^{r-1}x)),
\end{aligned}$$

which differ from the functions $\xi_{k,a,r}(x)$ only by the coefficient 2^{r-1}.

Let $r \in \mathbb{N}$, $k = 1,2,\dots,r-1$, $a = 2m$, $m \in \mathbb{N}$. We shall prove the following important properties of the functions $\xi_{k,a,r}(x)$ and $\widetilde{\xi}_{k,a}(x)$:

$$\xi_{k,a,r}(x),\ \widetilde{\xi}_{k,a}(x) \in \Upsilon; \tag{5.2}$$

$$\widetilde{\xi}_{k,a}(a^{r-1-k}x)(1-\cos(a^r x)) = \widetilde{\xi}_{k+1,a}(a^{r-1-k}x); \tag{5.3}$$

$$\widetilde{\xi}_{1,a}(a^{r-1}x) = 1-\cos(a^r x). \tag{5.4}$$

Proof. We shall verify property (5.2). The following properties are direct consequences of the definitions of $\xi_{k,a,r}(x)$ and $\widetilde{\xi}_{k,a}(x)$:

(1) $\xi_{k,a,r}(x) \geqslant 0$, $\widetilde{\xi}_{k,a}(x) \geqslant 0$;
(2) $\xi_{k,a,r}(x)$, $\widetilde{\xi}_{k,a}(x) \in C(\mathbb{R})$;
(3) $\xi_{k,a,r}(x)$, $\widetilde{\xi}_{k,a}(x)$ are even functions;
(4) $\xi_{k,a,r}(x)$ and $\widetilde{\xi}_{k,a}(x)$ are π-periodic because a is a positive even integer.

Consequently, $\xi_{k,a,r}(x)$, $\widetilde{\xi}_{k,a}(x) \in \Upsilon$.

Property (5.3) is a direct consequence of the definitions. Indeed, by the definition of $\widetilde{\xi}_{k,a}(x)$ it follows that

$$\widetilde{\xi}_{k,a}(a^{r-k-1}x) = \prod_{l=1}^{k}\bigl(1-\cos(a^l \cdot a^{r-k-1}x)\bigr),$$

$$\widetilde{\xi}_{k+1,a}(a^{r-k-1}x) = \prod_{l=1}^{k+1}\bigl(1-\cos(a^l \cdot a^{r-k-1}x)\bigr).$$

Thus, multiplying the first equality by $(1-\cos(a^r x))$ we obtain the second.

Property (5.4) is obvious.

Lemma 14. *Let* $r \in \mathbb{N}$, $a = 2m$, $m \in \mathbb{N}$. *Then*

$$\begin{aligned}
&(1-\cos x)(1-\cos(ax))(1-\cos(a^2x))\cdots\bigl(1-\cos(a^{r-1}x)\bigr)\\
&\qquad = 1 - \sum_{k=1}^{r-1}\cos(a^{r-1-k}x)\widetilde{\xi}_{k,a}(a^{r-1-k}x) - \cos(a^{r-1}x).
\end{aligned} \tag{5.5}$$

Proof. For $r \in \mathbb{N}$, $r \geqslant 1$, we prove (5.5) by induction on r. For $r=1$ the relation is trivial since it becomes $1-\cos x = 1-\cos x$.

Assume that equality (5.5) holds for $r=r_0$. We shall prove it for $r=r_0+1$. We have

$$\begin{aligned}
&\prod_{k=0}^{r_0-1}(1-\cos(a^k x))(1-\cos(a^{r_0}x))\\
&\qquad=\biggl(1-\sum_{k=1}^{r_0-1}\cos(a^{r_0-1-k}x)\widetilde{\xi}_{k,a}(a^{r_0-1-k}x)-\cos(a^{r_0-1}x)\biggr)(1-\cos(a^{r_0}x))\\
&\qquad=1-\cos(a^{r_0}x)-\sum_{k=1}^{r_0-1}\cos(a^{r_0-1-k}x)\widetilde{\xi}_{k,a}(a^{r_0-1-k}x)(1-\cos(a^{r_0}x))\\
&\qquad\quad-\cos(a^{r_0-1}x)(1-\cos(a^{r_0}x)).
\end{aligned}\tag{5.6}$$

Now, using equalities (5.3) and (5.4) we obtain

$$\begin{aligned}
\prod_{k=0}^{r_0}(1-\cos(a^k x))&=1-\cos(a^{r_0}x)-\sum_{k=1}^{r_0-1}\cos(a^{r_0-(k+1)}x)\widetilde{\xi}_{k+1,a}(a^{r_0-(k+1)}x)\\
&\quad-\cos(a^{r_0-1}x)\widetilde{\xi}_{1,a}(a^{r_0-1}x)\\
&=1-\cos(a^{r_0}x)-\sum_{k=2}^{r_0}\cos(a^{r_0-k}x)\widetilde{\xi}_{k,a}(a^{r_0-k}x)-\cos(a^{r_0-1}x)\widetilde{\xi}_{1,a}(a^{r_0-1}x)\\
&=1-\sum_{k=1}^{r_0}\cos(a^{r_0-k}x)\widetilde{\xi}_{k,a}(a^{r_0-k}x)-\cos(a^{r_0}x).
\end{aligned}\tag{5.7}$$

The proof of Lemma 14 is complete.

Lemma 15 (a representation for $\psi_{a,r}(x)$). *Let* $r \in \mathbb{N}$, $a=2m$, $m \in \mathbb{N}$, *and let* $\psi_{a,r}(x) = 4^r\sin^2(x/2)\sin^2(ax/2)\cdots\sin^2(a^{r-1}x/2)\cdots$. *Then*

$$\int_0^\pi \psi_{a,r}(xt)\frac{\sin t}{2}\,dt = 2^r + 2^{r-1}\sum_{k=0}^{r-1} h_{\widetilde{\xi}_{k,a}}(a^{r-1-k}x).$$

Proof. Formula (1.9) yields

$$2^{-r}\psi_{a,r}(x) = (1-\cos x)(1-\cos(ax))(1-\cos(a^2x))\cdots(1-\cos(a^{r-1}x)).$$

Hence it follows by Lemma 14 that

$$2^{-r}\psi_{a,r}(xt) = 1-\sum_{k=1}^{r-1}\cos(a^{r-1-k}xt)\widetilde{\xi}_{k,a}(a^{r-1-k}xt)-\cos(a^{r-1}xt).$$

Integrating this equality with respect to t with weight $\sin t/2$ from 0 to π and taking into account the definition of the functions $h_{\widetilde{\xi}_{k,a,r}}(x)$ we obtain the required result.

Completion of the proof of Corollary 1.7. By Lemma 15 we obtain

$$\int_0^\pi \psi_{a,r}(xt)\frac{\sin t}{2}\,dt = 2^r + \sum_{k=0}^{r-1} h_{\xi_k}(a^{r-1-k}x),$$

where $\xi_k(\,\cdot\,) = \xi_{k,a,r}(\,\cdot\,) \in \Upsilon$ and $a^{r-1-k} \geqslant 1$. Now, by Lemma 13,

$$\int_0^\pi \psi_{a,r}(xt)\frac{\sin t}{2}\,dt \geqslant 2^r, \qquad x \geqslant 1.$$

This completes the proof of Corollary 1.7.

Remark 10. In the proof of Corollary 1.7 we in fact reduce the problem of finding an upper bound for the sharp constant to the proof of the inequality

$$\inf_{x\geqslant 1} \int_0^\pi \psi_{a,r}(xt)\nu(t)\,dt \geqslant 2^r,$$

where $\nu(t) = \sin t/2$ is a weight function. We point out, however, that this inequality remains valid, for instance, in the case of the weight $\nu(t) = 6t(\pi - t)/(\pi^3)$.

For an illustration of this observation we produce a convenient representation of the integral $\displaystyle\int_0^\pi \psi_{a,r}(xt)\frac{6}{\pi^3}\,t(\pi - t)\,dt$ in the case $a = r = 2$:

$$\begin{aligned}\int_0^\pi 4^2 \sin^2\biggl(\frac{xt}{2}\biggr)\sin^2(xt)\frac{6t(\pi - t)}{\pi^3}\,dt &= 4 + \frac{4}{9} \\ &\quad\times \frac{1}{(\pi x)^3}\Bigl(\pi x\bigl(24 - 12\cos^3(\pi x) + 27\cos^2(\pi x) + 36\cos(\pi x)\bigr) \\ &\qquad + \sin(\pi x)\bigl(8\cos^2(\pi x) - 27\cos(\pi x) - 56\bigr)\Bigr).\end{aligned}$$

Next, if we set $g(y) = 24 - 12y^3 + 27y^2 + 36y$, $\widetilde{g}(y) = 8y^2 - 27y - 56$, and $f(x) = \pi x g(\cos(\pi x)) + \sin(\pi x)\widetilde{g}(\cos(\pi x))$, then using elementary arguments we prove the inequality $f(x) \geqslant 0$ separately for $1 \leqslant x \leqslant 2$ and for $2 \leqslant x < \infty$. Hence

$$\int_0^\pi \psi_{2,2}(xt)\frac{6t(\pi - t)}{\pi^3}\,dt = 4 + \frac{4}{9}\,\frac{f(x)}{(\pi x)^3} \geqslant 4, \qquad x \geqslant 1.$$

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