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# WHITNEY'S THEOREM FOR OSCILLATING ON $\mathbb{R}$ FUNCTIONS

YU. KRYAKIN

ABSTRACT. We find the order of Whitney's constants for oscillating functions.

## 1 Introduction

Whitney's Theorem [7] is the difference analogue of the following Taylor's estimate:

If  $I := [0, 1]$  and

$$f^{(k)}(0) = 0, \quad k = 0, \dots, m-1, \quad (CTM)$$

then

$$\sup_{x \in I} |f(x)| \leq (m!)^{-1} \sup_{x \in I} |f^{(m)}(x)|. \quad (TM)$$

**Theorem W** [7, 2]. If

$$f^{(k/(m-1))} = 0, \quad k = 0, \dots, m-1, \quad (CIW)$$

then

$$\sup_{x \in I} |f(x)| \leq 3 \sup_{x, x+mh \in I} |\Delta_h^m f(x)|, \quad (IW)$$

where

$$\Delta_h^m f(x) := \sum_{j=0}^m (-1)^j \binom{m}{j} f(x + (j - m/2)h).$$

The inequality (IW) is a consequence of the following Whitney type inequality [2]:

If

$$\int_0^{k/m} f(t) dt = 0, \quad k = 1, \dots, m, \quad (CMW)$$

then

$$\sup_{x \in I} |f(x)| \leq W_m \sup_{x, x+mh \in I} |\Delta_h^m f(x)|, \quad (MW)$$

$$W_m \leq 2 + 1/e^2.$$

**The Main Conjecture** on Whitney's constants states that  $W_m = 1$ . This was proved for  $m \leq 8$  [8, 9].

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Historical remarks on this theme one can find in [4]. Note, that the First Sendov's Conjecture about the constant for the best approximation by algebraic polynomials [5] follows from the Main Conjecture. The Second Sendov's Conjecture [5] states that we can put in (IW) 2 instead of 3. At the time when Sendov's conjectures have been proposed the estimates of constants were quite rough. Three in the inequality (IW) is the result of serious efforts and the proof of (IW) is not simple [2].

In this note we prove the following Whitney type estimate:

Suppose that  $f$  is a locally integrable on  $\mathbb{R}$ . If

$$\int_{jh}^{(j+1)h} f(u) du = 0, \quad j \in \mathbb{Z}, \quad h > 0,$$

then

$$\|f\| := \sup_{x \in \mathbb{R}} |f(x)| \leq W_{2k}^* \omega_{2k}(f, h),$$

where

$$\omega_{2k}(f, h) := \sup_{|t| \leq h} \|\Delta_t^{2k} f(\cdot)\|.$$

We shall prove (in section 2) that the order of  $W_{2k}^*$  can be defined up to the log-factor. Namely,

$$\frac{1}{\binom{2k}{k}} \leq W_{2k}^* \leq \frac{1 + H_k}{\binom{2k}{k}}, \quad H_k := \sum_{j=1}^k 1/j. \quad (WK)$$

However, the question about exact constant  $W_{2k}^*$  is, probably, extremely difficult.

Particularly, we do not know the exact constant in the simplest case  $k = 1$ . The proof of the estimate (see Section 3)

$$1/2 + 3/37 < W_2^* < 1/2 + 1/8 \quad (W2)$$

is the evidence of the character of the difficulties that appear. We have a significant difference from the interval-case here (see (MW)), where the similar approach allows to obtain the sharp results:  $W_m = 1$  for  $m \leq 8$ . [8, 9]

We think that it is important to find the right order of constants  $W_{2k}^*$ . This is the question about the factor  $1 + H_k \asymp \ln k$ .

The main motivation for the study of the order of the constants  $W_{2k}^*$  is a possible connection of Whitney's type inequality with Jackson–Stechkin inequality

$$E_n(f) \leq J_k(\delta) \omega_k(f, \delta).$$

In the recent paper [3] the exact order (with respect to  $k$ ) of Jackson–Stechkin constants  $J_k(\delta)$  has been found for  $\delta > \pi/n$ . The case  $\delta = \pi/n$  is the most difficult not only for approximation in  $L^\infty$  and  $L^p$ ,  $p \geq 1$  [3]. Even in the case  $L^2$ , where sharp results were obtained [1, 6], the exact constant for this argument have not yet been found.

We hope that Whitney's type theorem we suggest may be useful to determine the order of the Jackson–Stechkin constants in this principal case.

On the other hand, note that the results of the paper [3] are motivated by Whitney's estimate (MW). Namely, this estimate was essential for creating the main tool of the paper [3] — Favard's inequality for differences.

## 2 Asymptotic estimate

**Theorem 1.** *If  $f$  is a locally integrable on  $\mathbb{R}$  and*

$$\int_{jh}^{(j+1)h} f(u) du = 0, \quad j \in \mathbb{Z}, \quad h > 0, \quad (CPW)$$

then

$$\|f\| \leq W_{2k}^* \omega_{2k}(f, h),$$

with

$$\frac{1}{\binom{2k}{k}} \leq W_{2k}^* \leq \frac{1 + H_k}{\binom{2k}{k}}, \quad H_k := \sum_{j=1}^k 1/j. \quad (PW)$$

*Proof.* The main tool for the proof is Steklov's function

$$F(x, y) := \frac{1}{y-x} \int_x^y f(u) du, \quad F(x, x) := f(x).$$

This function of two variables has a better smoothness than the function  $f$  in the following sense. Put

$$\Delta_{h_1, h_2}^{2k} F(x, y) := \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} F(x + (j-k)h_1, y + (j-k)h_2).$$

It is easy to check that

$$\Delta_{h_1, h_2}^{2k} F(x, y) = \int_0^1 \Delta_{h_1 + t(h_2 - h_1)}^{2k} f(x + t(y-x)) dt.$$

Note, that we can assume that  $h = 1$  and  $\omega_{2k}(f, h) = 1$ . The integral representation above implies that  $|\Delta_{h_1, h_2}^{2k} F(x, y)| \leq 1$ ,  $h_j \in [0, 1]$ .

The idea of the proof is simple. By the conditions  $F(j, k) = 0$  we have

$$f(j) = F(j, j) = \binom{2k}{k}^{-1} \Delta_{0,1}^{2k} F(j, j).$$

We get the lower estimate immediately. It is sufficient to consider the function equal to 0, except one point, where the function is equal to 1.

In the next part of the proof we shall show that when we pass from integer points to arbitrary ones we lose not greater than  $\ln k$ .

The proof is based on the following simple identity:

$$\Delta_{1,0}^{2k} F(0, x) = \Delta_1^{2k}(1/x) \int_0^x f(u) du = \Delta_1^{2k}(1/x) x F(0, x),$$

which we can rewrite in the form

$$\int_0^x f(u) du = \frac{(x-k) \cdots x \cdots (x+k)}{2k!} \Delta_{1,0}^{2k} F(0, x). \quad (E)$$

For the proof of (PW) we can suppose (by symmetry argument) that  $x \in (0, 1/2]$ .

Let us rewrite  $\Delta_{0,1}^{2k} F(x, x)$  in the form:

$$\Delta_{0,1}^{2k} F(x, x) = (-1)^k \binom{2k}{k} f(x) + R_k(x),$$

$$R_k(x) := \sum_{j=-k, j \neq 0}^k (-1)^{k-j} \binom{2k}{k+j} F(x, x+j).$$

We need to show that

$$|R_k(x)| \leq H_k.$$

Since

$$F(x, x+j) = \frac{1}{j} \left( \int_x^0 + \int_0^j + \int_j^{j+x} \right),$$

and the difference  $\Delta_{0,1}^{2k}$  is symmetric, and  $\int_0^j = 0$ , we have only the terms with  $\int_j^{j+x}$  in  $R_k(x)$ :

$$R_k(x) = \sum_{j=-k, j \neq 0}^k \frac{(-1)^{k-j}}{j} \binom{2k}{k+j} \int_j^{j+x}.$$

The identity (E) with the condition  $\omega_{2k}(f, 1) = 1$  give

$$\left| \int_0^x \right| \leq \binom{x+k}{2k} = x \prod_{j=1}^k (1 - x^2/j^2) \binom{2k}{k}^{-1} \leq \frac{1}{2} \binom{2k}{k}^{-1}, \quad x \in (0, 1/2].$$

Again, by symmetry

$$\left| \int_j^{j+x} \right| \leq \frac{1}{2} \binom{2k}{k}^{-1}$$

and the estimate for  $|R_k(x)|$  is proved.

*Remark 1.* It is clear that for the periodic functions with oscillation we get the exact values of constants (since  $R_k(x) = 0$ ).

### 3 The case $k = 1$

#### Theorem 2.

$$1/2 + 3/37 \leq W_2^* \leq 1/2 + 1/8.$$

*Proof. 1.* Prove the upper estimate at first. Note, that in the case  $k = 1$  we have the better estimate than for  $k \geq 2$  in the following sense:

$$|f(j/2)| = \left| \binom{2}{1}^{-1} \Delta_{0,1/2}^2 F(j/2, j/2) \right| \leq 1/2.$$

As before we can suppose that  $x \in (0, 1/2)$ .

**1.a.** If  $x \in (0, 1/4]$ , then

$$\Delta_{0,1}^2 F(x, x) = -2f(x) + \int_1^{1+x} - \int_{-1}^{-1+x}$$

The inequality (see (E))

$$\left| \int_0^x \right| \leq \left| \frac{(x-1)x(x+1)}{2} \right| \leq \frac{1}{8} \frac{15}{16} < 1/8,$$

implies

$$\left| \int_1^{1+x} \right| < 1/8, \quad \left| \int_{-1}^{-1+x} \right| < 1/8,$$

and

$$|f(x)| \leq |f(1/4)| < 1/2 + 1/8.$$

**1.b.** If  $x \in (1/4, 1/2]$ , then

$$\Delta_{0,1-x}^2 F(x, x) = -2f(x) - \frac{1}{1-x} \int_{-1}^{-1+2x}$$

and

$$\left| \frac{1}{1-x} \int_{-1}^{-1+2x} \right| \leq \left| \frac{(2x-1)2x(2x+1)}{2(1-x)} \right|_{x=1/4} = 1/4.$$

□

*Remark 2.* It is clear that we can prove a little better estimate by combining 1.a and 1.b. The equation

$$(1-x)x(1+x) = (2x-1)2x(2x+1)/(1-x)$$

has the root  $x_0 = 2 - \sqrt{3}$  and instead of upper estimate  $1/2 + 1/8 = 0.6250$  we have 0.6244.

**2.** The following example gives the lower estimate.

Consider the function, which is piecewise continuous, equals to 0 on  $(-\infty, -1)$  and on  $[2, \infty)$  and equals to  $1/2 + 3/37$  at the point  $1/4$ . On the intervals  $I_1 = [-1, -0.5)$ ,  $I_2 = [-0.5, 1)$  (without the point  $1/4$ ),  $I_3 = [1, 5/4)$ ,  $I_4 = [5/4, 3/2)$  and  $I_5 = [3/2, 2)$  the function is linear :

$$f(-1) = -12/37, f(-0.5-) = -6/37, f(-0.5) = 12/37, f(1-) = -6/37, f(1) = 12/37, f(5/4-) = 15/37, f(5/4) = -10/37, f(3/2) = -1/37, f(2-) = -7/37.$$

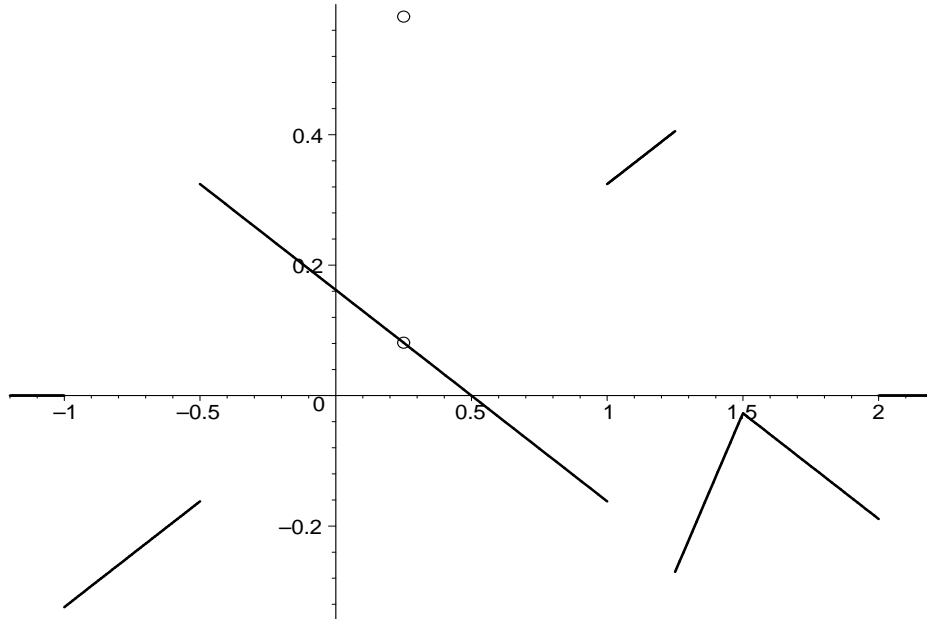


Figure 1.

The direct computation gives  $\omega_2(f, 1) \leq 1$  and  $f_j^{j+1} = 0$ ,  $j \in \mathbb{Z}$ . Note that we can obtain from this function by smoothing the example of continuous function for the lower estimate.  $\square$

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