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Whitney's Constants and Sendov's Conjectures

Yuri V. Kryakin

Dedicated to Professor Blagovest Sendov on the occasion of his seventieth birthday

In this paper we review the history and the current state of the Whitney's constants problem.

1. Introduction

Let C be a space of continuous functions f on $I := [0, 1]$ with a uniform norm

$$\|f\| := \max_{x \in I} |f(x)|.$$

For a function $f \in C$ denote the k -th difference with step h by

$$\Delta_h^k f(x) := \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} f(x + jh)$$

and k -th modulus of continuity by

$$\omega_k(f) := \sup_{x, x+kh \in I} |\Delta_h^k f(x)|.$$

Let \mathbf{P}_k be a space of algebraic polynomials of degree $\leq k$. Whitney constants are defined by

$$W_k := \sup_{f \in C \setminus \mathbf{P}_{k-1}} \inf_{p \in \mathbf{P}_{k-1}} \frac{\|f - p\|}{\omega_k(f)}.$$

Let $L_{k-1}(f, x)$ be the Lagrange polynomial of degree $\leq k-1$, which interpolates f at equidistant points $x_m := m/(k-1)$:

$$f(x_m) = L_{k-1}(f, x_m), \quad m = 0, \dots, k-1.$$

Whitney interpolation constants are defined by

$$W'_k := \sup_{f \in C \setminus \mathbf{P}_{k-1}} \frac{\|f - L_{k-1}(f, \cdot)\|}{\omega_k(f)} = \sup_{f \in C, f(x_m)=0} \frac{\|f\|}{\omega_k(f)}.$$

In this paper we are mainly interested in estimates of W_k and W'_k . Namely, we intend to describe the current situation with the following Sendov's conjectures [17].

First Sendov's conjecture : $W_k \leq 1$.

Second Sendov's conjecture : $W'_k \leq 2$.

2. Historical remarks

2.1. Results of Burkill, Whitney, Beurling and Brudnyi. It is clear, that $W_1 = 1/2$ and $W'_1 = 1$. This is the case of approximation of continuous function by constant. The results $W_2 = 1/2$ and $W'_2 = 1$ are due to H.Burkill [4] and H. Whitney [27].

Burkill's Lemma. *If $f \in C$ and $f(0) = f(1) = 0$, then $\|f\| \leq \omega_2(f)$.*

Proof. Suppose that $|f(a)| = \|f\|$ and $a \leq 1/2$. Then $f(a) = -1/2(f(0) - 2f(a) + f(2a)) + 1/2f(2a)$ and $\|f\| \leq |f(0) - 2f(a) + f(2a)| \leq \omega_2(f)$. In the case $a > 1/2$ we have the same conclusion by symmetry. ■

H.Burkill conjectured, that $W_k \leq W'_k < \infty$. This conjecture was proved in 1957 by H.Whitney [27]. The main ingredient of his proof is the following Lemma.

Whitney's Lemma. *Let $k, \nu \in N$, $k, \nu \geq 2$, $X = \{0, 1, \dots, \nu(k-1)\}$. Then for $f(s)$, $s \in X$ exist numbers*

$$a_i = a_i(s, \nu, k), \quad i = 0, \dots, \nu(k-1) - k,$$

$$b_j = b_j(s, \nu, k), \quad j = 0, \dots, k-1,$$

such, that

$$f(s) = \sum_{j=0}^{\nu(k-1)-k} a_j \Delta_1^k f(j) + \sum_{j=0}^{k-1} b_j f(j\nu).$$

Moreover, for $s = 1$ and for arbitrary positive $\varepsilon > 0$ there exists ν , such that

$$\sum_{j=1}^{k-1} b_j(s, \nu, k) < \varepsilon.$$

This Lemma may be used in various generalizations of Whitney inequality $W'_k \leq \infty$. For example, it was used in proofs of analogs of Whitney's estimate for functions in L^p [24], for functions on complex arcs [26, 15], for Chebyshevian approximations [28, 16]. But we can not obtain good estimates of W'_k and W_k on this way. We can not write any, may be bed estimate, for all k . By using the special identities Whitney proved the inequalities

$$\frac{8}{15} \leq W_3 \leq \frac{7}{10}, \quad \frac{16}{15} \leq W'_3 \leq \frac{14}{9}.$$

Whitney noted, that " the problem of finding the W_k , W'_k is probably extremely difficult ...".

Another proof of Whitney theorem with estimates $W_k \leq Ck^{2k}$ was obtained by Yu.Brudnyi [2]. Modified Brudnyi's proof with estimate $W_k \leq (k+1)k^k$ one can find in Sendov's paper [19].

On the other hand, the situation is not difficult for integrable on $[0, +\infty)$ functions. We have the following Whitney–Beurling identity [27]: For $A > 0$, $0 \leq y < x$

$$\begin{aligned} & 1/A \left\{ \int_0^A \Delta_h^k f(x) dh - \int_0^A \Delta_h^k f(y) dh \right\} \\ &= (-1)^k (f(x) - f(y)) + 1/A \left\{ \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} \left(\int_{y+jA}^{x+jA} f - \int_y^x f \right) \right\}. \end{aligned}$$

This identity implies (we may consider $A \gg 1$) estimate

$$|f(x) - f(y)| \leq 2 \sup_{x, h > 0} |\Delta_h^k f(x)|.$$

2.2. Results of Ivanov, Takev, Binev and Sendov. In papers [17,18,19] Sendov invited attention to the problem of Whitney's constants. In 1985, as the result of works [7, 1, 20] the following remarkable inequality was obtained [21, 22]:

$$W_k \leq \text{Const} \leq 6. \quad (1)$$

Using of the Ivanov–Takev integral operators [7]

$$\psi_i(f, x) := \frac{(-1)^{k-i}}{h \binom{k}{i}} \int_0^h \Delta_y^k f(x - iy) dy,$$

$$h = (k+1)^{-1}, \quad t \in [0, h], \quad x = ih + t, \quad i = 0, 1, \dots, k,$$

was essential for proving (1). This operators can be considered as the convenient tool for transplantation of Whitney–Beurling proof from $[0, \infty)$ to $I = [0, 1]$. Sendov deduced (1) from the following Lemma.

Sendov’s Lemma. *Let $f \in C$, $k \in N$. Then there exist $p \in \mathbf{P}_{k-1}$ such, that*

$$f(x) = p(x) + \psi_i(f, x) + \sum_{j=0}^k \frac{1}{h} \int_0^t \psi_j(f, jh + y) l'_{k,j} \left(\frac{x-y}{h} \right) dy,$$

where

$$l_{k,j}(x) := \prod_{m=0, m \neq j}^k (x - m)/(j - m).$$

Analysis of Sendov’s Lemma leads [9, 10] to the following

Modified Sendov’s Lemma. *Let $f \in C$, $k \in N$ and $\int_0^{i/k} f(t) dt = 0$, $i = 1, \dots, k$. Then for $x \in (0, 1/k]$ we have*

$$f(ix) = \varphi_i(f, x) + \int_{1/k}^x \sum_{j=1}^k \varphi_j(f, y) \cdot \frac{j}{i} \cdot \left[l_{k,j} \left(\frac{x}{y} i \right) \right]'_x dy,$$

where

$$\varphi_i(f, x) := \frac{(-1)^{k-i}}{\binom{k}{i}} \frac{1}{x} \int_0^x \Delta_y^k f(i(x-y)) dy.$$

Modified Sendov’s Lemma implies inequality

$$W_k \leq 3. \tag{2}$$

Inequality was (2) independently announced by Yu.Brudnyi [3], B.Sendov [23] and author [8].

Estimate

$$W'_k < \text{Const} < 36$$

was obtained by M.Takev [25]. Combination of Takev’s method with the modified Sendov’s Lemma gave the inequality [13]

$$W'_k < 5.$$

3. Recent developments

3.1. Estimates of W_k . In modified Sendov's Lemma we can replace $f(x)$ on $f(x) - q(x)$ and remove the condition $\int_0^{i/k} f = 0$, by special choosing of $q \in \mathbf{P}_{k-1}$:

$$\int_0^{i/k} f(t) - q(t) dt = 0, \quad i = 1, \dots, k.$$

It is natural to define the corresponding constants:

$$W_k^* := \sup_{f \in C \setminus \mathbf{P}_{k-1}} \frac{\|f - q\|}{\omega_k(f)}.$$

It is clear that $W_k \leq W_k^*$.

Theorem 1 [\[12, 5\]](#).

$$\begin{aligned} W_k^* &\leq 2 && \text{for } k \leq 82000. \\ W_k^* &\leq 2 + \exp(-2) && \text{for } k > 82000. \end{aligned}$$

To prove Theorem 1 we need an another modification of Whitney–Beurling idea. To this end put $F(x) := \int_0^x f(u) du$. The following identity one can check directly.

Lemma 1 [\[12\]](#). *If $m \in \{0, 1, \dots, k\}$, $x \in I$ and $\delta > 0$ are such, that $[x - m\delta, x + (k - m)\delta] \subset I$, then*

$$\begin{aligned} (-1)^{k-m} \binom{k}{m} f(x) &= \int_0^1 \Delta_{t\delta}^k f(x - m\delta t) dt \\ &\quad - (-1)^{k-m} \frac{1}{\delta} \binom{k}{m} (\sigma_{k-m} - \sigma_m) F(x) \\ &\quad - \frac{1}{\delta} \sum_{j=0, j \neq m}^k (-1)^{k-j} \binom{k}{j} \frac{1}{j-m} F(x + (j-m)\delta), \end{aligned}$$

where

$$\sigma_0 := 0, \quad \sigma_m := \sum_{j=1}^m \frac{1}{j}, \quad m = 1, 2, \dots$$

The estimates of F in Lemma 1 provide the following Zhuk–Natanson identity.

Lemma 2 [\[31, 12\]](#). *If $F(i/k) = 0$, $i = 1, \dots, k$, then*

$$F(x) = A_k(x) \int_0^1 \Delta_{t/k}^k f(x(1-t)) dt, \quad x \in I,$$

where

$$A_k(x) := \frac{k^k}{k!} x \left(x - \frac{1}{k}\right) \left(x - \frac{2}{k}\right) \dots \left(x - \frac{k}{k}\right) = x(-1)^k \prod_{j=1}^k \left(1 - \frac{kx}{j}\right).$$

By combining Lemma 1 and Lemma 2 one can obtain inequality $W_k^* < \text{Const}$. For relative small $k < 1000$ one can use PC for the estimates of constants. But for $k \gg 1$ we need something else. The next Shevchuk's Lemma is the appropriate tool for the estimates for large k .

Lemma 3 [5]. *Let $g := f - q$. Suppose that $\omega_k(g) \leq 1$, $m < k/2$, $x \in [m/k, (m+1)/k]$, $\delta := (1-x)/(k-m)$. Then*

$$\begin{aligned} & \binom{k}{m} |g(x)| \\ & \leq 1 + (k\delta)^k - (-1)^{k-m} \binom{k}{m} A'_k(x) + \frac{2}{\delta} \sum_{j=0}^{m-1} \binom{k}{j} \frac{1}{m-j} (|A_k(x + \delta(j-m))|). \end{aligned}$$

Note, that modified Sendov's Lemma implies inequality $|g(x)| \leq 1$ for $x \in [1/k, 1-1/k]$. So, to prove Theorem 1 we need only

Lemma 4 [5]. *For $x \in [0, 1/k]$ we have*

$$(1-x)^k - (-1)^k A'_k(x) \leq 1 + \frac{1}{e^2},$$

and

$$(1-x)^k - (-1)^k A'_k(x) \leq 1, \quad k \leq 82000.$$

Remark. Theorem 1 corrects an arithmetical mistake in [12], where it was claimed that $W_k^* \leq 2$ for all k .

Theorem 2 [11, 14, 29].

$$W_k^* = 1, \quad k \leq 7.$$

It is not hard to prove that $W_k^* \geq 1, k \geq 1$ (see [10]). Example may be constructed by smoothing the function $f(x) = 0, x \neq 0, f(1) = 1$. Inequality

$W_1^* \leq 1$ is trivial. With intention to make idea of the proof as clear as possible, consider the simple case $k = 2$. Put

$$G(x, y) := \frac{1}{y-x} \int_x^y g(t) dt, \quad G(x, x) := g(x)$$

and

$$\Delta_{h_1, h_2}^k G(x, y) := \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} G(x + jh_1, y + jh_2).$$

It easy to check, that

$$\Delta_{h_1, h_2}^k G(x, y) = \int_0^1 \Delta_{h_1 + t(h_2 - h_1)}^k g(x + t(y - x)) dt.$$

We need to prove that if $\int_0^{i/2} g = \int_0^{i/2} (f - g) = 0$, $i = 1, 2$ and $\omega_2(g) \leq 1$ then $|g(x)| \leq 1$, or in other notation: if $G(0, j/2) = 0$, $j = 1, 2$ and $|\Delta_{h_1, h_2}^2 G(x, y)| \leq 1$, then we have $|g(x)| \leq 1$.

Suppose that $\max |g(y)| = g(x)$ and $x < 1/3$ (the case of $x \in [1/3, 1/2]$ is more simple; in the case $\max |g(y)| = -g(x)$, we can consider the function $g_1 = -g$). We have identities:

$$g(x) = \Delta_{(1-x)/2, x}^2 G(x, x) - \frac{x}{1-3x} (6G(0, 2x) - G(2x, 3x) - 2G(1/2, 1/2 + x/2))$$

and

$$g(x) = -\frac{1}{2} \Delta_{x, 0}^2 G(0, x) + G(0, 2x).$$

First identity is a global estimate (with big step $h = (1-x)/2$). Second identity is a local estimate (with step $h = x$). Now from the local estimate we deduce

$$G(0, 2x) \geq g(x) - 1/2.$$

Combining last inequality with the global estimate we find that

$$\begin{aligned} 1 &\geq \left| \Delta_{(1-x)/2, x}^2 G(x, x) \right| \\ &\geq \left| g(x) + \frac{x}{1-3x} (6G(0, 2x) - G(2x, 3x) - 2G(1/2, 1/2 + x/2)) \right| \\ &\geq \left| g(x) + \frac{x}{1-3x} (6(g(x) - 1/2) - g(x) - 2g(x)) \right| \end{aligned}$$

or

$$g(x) \leq 1.$$

The proof of Theorem 2 for $2 < k < 8$ is not simple. The main idea of proofs, presented in [14,29], is due to H. Whitney [27]. Put $G(u) := \int_0^u g(t) dt$ and suppose, that $G(i/k) = 0$, $i = 1, \dots, k$. Let $\max |g(y)| = g(x) > \omega_k(f)$. Consider the identity

$$\begin{aligned} & (-1)^k \int_0^1 \Delta_{th+(1-t)\alpha x/2}^k g(x) dt - g(x) \\ &= \frac{1}{h - \alpha x/2} \left(\sum_{j=1}^k \frac{(-1)^j \binom{k}{j}}{j} G(x + jh) - \sum_{j=1}^k \frac{(-1)^j \binom{k}{j}}{j} G(x + j \frac{\alpha x}{2}) \right), \\ & \quad h = (1-x)/k, \quad \alpha : 0 < \frac{\alpha x}{2} < h. \end{aligned}$$

Since the left hand part of this identity is non-positive then

$$M_\alpha(x) := -\frac{1}{x} \sum_{j=1}^k \frac{(-1)^j \binom{k}{j}}{j} G(x + j \frac{\alpha x}{2}) \leq -\frac{1}{x} \sum_{j=1}^k \frac{(-1)^j \binom{k}{j}}{j} G(x + jh).$$

Lemma 2 implies (see [14, 29])

$$M_\alpha(x) \leq \sigma_k - 1.$$

The kernel of the proof is the identity

$$Ag(x) = \sum a_i g_i + \sum b_j \Delta_j^k + \sum c_l M_{\alpha(l)}(x), \quad (3)$$

where $a_i, b_j \in R$, $c_k \in R_+$, $g_i := G(\frac{x(i-1)}{2}, \frac{xi}{2})$, $\Delta_j^k :=$ means of finite differences. We will use (3) for x near origin (difficult case of Theorem 2). From other x we can use some modification of (3) (see [14, 29]). We can suppose, that $|\Delta_j^k| \leq 1$. To prove Theorem 2 it is sufficient to construct identity (3) with constraint

$$A > \sum |a_i| + \sum |b_j| + (\sigma_k - 1) \sum c_l.$$

Identity for $k = 2$, $x \in [0, 1/3]$.

$$g(x) = \frac{1}{12}(g_5 + g_6) - \frac{1}{2} \Delta_{x,0}^2 G(0, x) + \frac{1}{3} M_2(x).$$

Identity for $k = 3$, $x \in [0, 1/6]$, $g_0 := g(0)$.

$$Ag(x) = \sum_{i=0}^{12} a_i g_i + \sum_{j=1}^4 b_j \Delta_j^3 + c M_2(x),$$

$$\begin{aligned}\Delta_1^3 &= \Delta_{x/2,0}^3 G(x/2, x), & \Delta_2^3 &= \Delta_{x/2,0}^3 G(0, x), \\ \Delta_3^3 &= \Delta_{x,x/2}^3 G(0, x/2), & \Delta_4^3 &= \Delta_{2x,3x/2} G(0, 0). \\ A &= 396/7, & c &= 12,\end{aligned}$$

$$\mathbf{a} = [4/3, 0, 0, 0, 148/7, 5/7, 5/7, 0, 0, 0, -4/9, -4/9, -4/9],$$

$$\mathbf{b} = [22/7, -22/7, 88/7, 4/3].$$

We have $396/7 = A > \sum |a_i| + \sum |b_j| + (\sigma_3 - 1) \cdot 12 = 388/7$.

Identity for $k = 4$, $x \in [0, 1/12]$.

$$Ag(x) = \sum_{i=1}^{22} a_i g_i + \sum_{j=1}^{12} b_j \Delta_j^4 + \sum_{l=1}^4 c_l M_{(l+1)}(x),$$

$$\begin{aligned}\Delta_1^4 &= \Delta_{\frac{x}{2},0}^4 G(0, x), & \Delta_2^4 &= \Delta_{x,\frac{x}{2}}^4 G(0, \frac{x}{2}), & \Delta_3^4 &= \Delta_{\frac{x}{2},\frac{x}{2}}^4 G(0, \frac{x}{2}), \\ \Delta_4^4 &= \Delta_{x,x}^4 G(0, \frac{x}{2}), & \Delta_5^4 &= \Delta_{\frac{3x}{2},\frac{3x}{2}}^4 G(0, \frac{x}{2}), & \Delta_6^4 &= \Delta_{\frac{x}{2},\frac{x}{2}}^4 G(\frac{x}{2}, x), \\ \Delta_7^4 &= \Delta_{x,x}^4 G(\frac{x}{2}, x), & \Delta_8^4 &= \Delta_{\frac{3x}{2},\frac{3x}{2}}^4 G(\frac{x}{2}, x), & \Delta_9^4 &= \Delta_{x,x}^4 G(3x, \frac{7x}{2}),\end{aligned}$$

$$\Delta_{10}^4 = \Delta_{\frac{x}{2},\frac{x}{2}}^4 G(\frac{9x}{2}, 5x), \quad \Delta_{11}^4 = \Delta_{\frac{3x}{2},\frac{3x}{2}}^4 G(\frac{9x}{2}, 5x), \quad \Delta_{12}^4 = \Delta_{\frac{x}{2},\frac{x}{2}}^4 G(\frac{15x}{2}, 8x).$$

Coefficients a_i, b_j, c_l , for case $k = 4$ and identities for x , which are separated from the intervals endpoints one can find in [14]. Appropriate identities for $k = 5, 6, 7$ were constructed by O.Zhelnov [29].

3.2. Estimates of W_k' . The method, proposed by M.Takev [25], intermediate approximation by polynomials $q: \int_0^{i/k} (f - q) = 0, i = 1, \dots, k$, and estimates of Lemma 3 led to the following Theorem.

Theorem 3 [5].

$$W_k' \leq 3.$$

Since the inequality

$$|f(x) - L_{k-1}(f, x)| \leq \omega_k(f), \quad x \in [1/k, 1 - 1/k],$$

is known (see, for example, estimates in [13]), we only need to prove that

$$|f(x) - L_{k-1}(f, x)| \leq 3\omega_k(f), \quad x \in [0, 1/k].$$

By using the notation $g(x) := f(x) - q(x)$ we get

$$\begin{aligned} & |f(x) - L_{k-1}(f; x)| \leq |f(x) - q(x) - L_{k-1}(f, x) + q(x)| \\ & \leq |f(x) - q(x)| + |L_{k-1}(f - q, x)| = |g(x)| + \left| \sum_{m=0}^{k-1} g(x_m) l_{k-1,m}((k-1)x) \right|. \end{aligned}$$

To estimate the value of $|g(x)|$ for $x \in [0, 1/k)$ and in points x_m , $m = 0, \dots, k-1$ we shall use Lemma 3.

For $x \in [0, 1/k)$ we have the inequality

$$|g(x)| \leq 1 + (1-x)^k - (-1)^k A'_k(x).$$

To estimate

$$|L_{k-1}(g, x)| = \left| \sum_{m=0}^{k-1} g(x_m) l_{k-1,m}((k-1)x) \right|, \quad x_m = \frac{m}{k-1},$$

we may use the following delicate Lemma 5.

Lemma 5 [5]. *Suppose that $f \in C$, $\omega_k(f) \leq 1$. Then for each $m = 0, \dots, k-1$ we have*

$$|g(x_m)| \leq \binom{k-1}{m}^{-1} + 2(k-1) \sigma_{k-1} |A_k(x_m)|.$$

The proof of Lemma 5 is the most technical part of paper [5]. One can use Lemma 5 to deduce Lemma 6.

Lemma 6 [5]. *Let $f \in C$, $k > 7$, $\omega_k(f) \leq 1$, $x \in [0, 1/k)$. Then*

$$|f(x) - L_{k-1}(f, x)| \leq 2 + e(k-1) \sigma_{k-1} |A_{k-1}(x)|.$$

Since

$$e(k-1) \sigma_{k-1} |A_{k-1}(x)| \leq 1$$

we have Theorem 3 for $k > 7$.

For $k \leq 7$ second Sendov's conjecture follows from Theorem 2. It was proved by Danilenko ($k = 4$) and O. Zhelnov ($k = 5, 6, 7$).

Theorem 4 [6, 30].

$$W'_k \leq 2, \quad k = 4, 5, 6, 7.$$

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