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CONSTRUCTIVE APPROXIMATION

Generalized Hyperinterpolation on the Sphere and the Newman–Shapiro Operators

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Abstract. Hyperinterpolation on the sphere, as introduced by Sloan in 1995, is a constructive approximation method which is favorable in comparison with interpolation, but still lacking in pointwise convergence in the uniform norm. For this reason we combine the idea of hyperinterpolation and of summation in a concept of generalized hyperinterpolation. This is no longer projectory, but convergent if a matrix method A is used which satisfies some assumptions. Especially we study A partial sums which are defined by some singular integral used by Newman and Shapiro in 1964 to derive a Jackson-type inequality on the sphere. We could prove in 1999 that this inequality is realized even by the corresponding discrete operators, which are generalized hyperinterpolation operators. In view of this result the Newman–Shapiro operators themselves gain new attention. Actually, in their case, A furnishes second-order approximation, which is best possible for positive operators. As an application we discuss a method for tomography, which reconstructs smooth and nonsmooth components at their adequate rate of convergence. However, it is an open question how second-order results can be obtained in the discrete case, this means in generalized hyperinterpolation itself, if results of this kind are possible at all.

1. Introduction

In the constructive approximation of continuous spherical functions in $\mathbf{R}^r$, $r \in \mathbf{N}\backslash\{1, 2\}$, complexity is a serious concern. For this reason, interpolatory projections are of great interest, as their evaluation is comparatively cheap, at least if it is based on zonal polynomials. The use of a polynomial projection depends on the size of the uniform norm (Lebesgue constant). Unfortunately, even the minimal projection norms grow unreasonably quickly, and interpolation cannot behave better. It is even unknown whether the minimal projection order can be attained by interpolatory projections. For this reason hyperinterpolation, introduced by Sloan [18], is an important progress as it guarantees the minimal projection norm order [19], [20], [15]. But it is well known that by no means convergence can be gained by projections for all continuous spherical functions. This is also valid for hyperinterpolatory projections.

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In this work we combine the idea of hyperinterpolation with the concept of summation in what we call *generalized hyperinterpolation*. The resulting operators are positive and pointwise convergent in the uniform norm to the identity, but no longer projections. Apart from a constant factor, their evaluation costs are comparable with the costs of hyperinterpolation or interpolation itself. As an example we discuss generalized hyperinterpolation based on Cesàro's method, where we have to suppose that the index is sufficiently large in comparison with the space dimension. The process is convergent in the uniform norm, but in the general case the order of convergence is poor.

A better, and in some sense best possible choice is to use what we call Newman–Shapiro sums. They are based on some particular singular integrals, introduced by Newman and Shapiro in 1964 [8]. We prove that the integral kernels define a positive infinite subdiagonal matrix A and hence a matrix summation method by which the Newman–Shapiro sums occur as A partial sums from the Laplace series. The corresponding generalized hyperinterpolation operators are obtained more directly if the singular integrals are evaluated by means of positive quadrature rules of a certain degree of exactness. In [16] we proved that they inherit from the original Newman–Shapiro operators the important property that the approximation error can be estimated uniformly by means of the modulus of continuity of the first order, such that a Jackson-type inequality on the sphere is realized by discrete operators. This is surprising as the nodes of the quadrature rules must show some irregularity in their distribution. A corresponding result with respect to the modulus of continuity of the second order is desirable, but seems to be possible at best if strong assumptions are imposed upon the node configurations. However, we prove such a second-order result for the Newman–Shapiro operators themselves. It follows that in their case the approximation error can be estimated uniformly by means of the first or of the second derivative and at the corresponding approximation order, provided the approximated function is once or twice continuously differentiable, respectively. But it is well known that for positive operators a corresponding third-order result cannot hold simultaneously, which does not contradict with more optimistic results with respect to proximum operators, see Pawelke [9], [10] and others [2], [17], but which are in general nonlinear, nonpositive and hardly constructive. In this sense our result is best possible.

As mentioned above, the Newman–Shapiro operators can be interpreted as A partial sums of the Laplace series. But we can use the corresponding matrix summation method independently of its original meaning, for instance in tomography, and this as follows. In 1997 we have shown [13], that it is favourable to represent a density function, which is defined on the unit ball and which is to be reconstructed from its image under Radon's transform, from the beginning by a A-series of some polynomial projections, and to regain it from the corresponding image series. This defines a constructive reconstruction method as the A partial sums of the density function itself can be evaluated easily from its Radon image by the help of a positive quadrature rule. So we get a generalized hyperinterpolation procedure on the ball. One of the main results of this paper is that if A is the Newman–Shapiro matrix in two additional dimensions, and if the density function is once or twice continuously differentiable, then it is reconstructed at the first or second approximation order, respectively.

2. Orthogonal Projection and Hyperinterpolation

Let $S^{r-1} := \{x \in \mathbf{R}^r : |x| = 1\}$ denote the unit sphere in $\mathbf{R}^r$ where $r \in \mathbf{N}\backslash\{1, 2\}$. We shall use the spaces of spherical polynomials

$$\mathbf{P}^r_\mu := \mathbf{P}^r_\mu(S^{r-1}), \qquad \overset{*}{\mathbf{P}}{}^r_\mu := \overset{*}{\mathbf{P}}{}^r_\mu(S^{r-1}), \qquad \overset{*}{\mathbf{H}}{}^r_\mu := \overset{*}{\mathbf{H}}{}^r_\mu(S^{r-1}),$$

i.e., the spaces of restrictions onto S^{r-1} of the r-variate polynomials, homogeneous polynomials, or homogeneous harmonic polynomials of degree $\mu \in \mathbf{N}_0$, respectively.

The space $C(S^{r-1})$ of all continuous spherical functions is provided by the maximum norm $\|\cdot\|_\infty$ and by the inner product

$$\langle F, G\rangle := \int_{S^{r-1}} F(x)G(x)\, d\omega(x), \tag{2.1}$$

where ω is the invariant measure on S^{r-1}, which induces the norm $\|\cdot\|_2$. The corresponding reproducing kernel function of $\overset{*}{\mathbf{H}}{}^r_\mu$ is given by $G_\mu(xy)$, $x, y \in S^{r-1}$, where xy is the Euclidean inner product in $\mathbf{R}^r$ and where

$$G_\mu = \frac{2\mu + r - 2}{(r-2)\omega_{r-1}} \cdot C_\mu^{(r-2)/2}. \tag{2.2}$$

$\omega_{r-1} = \langle 1, 1\rangle$ is the measure of the unit sphere, and C^λ_μ denotes the Gegenbauer polynomial of degree μ and index λ, see [11], for example.

Note that the spaces $\overset{*}{\mathbf{H}}{}^r_\mu$ are orthogonal with respect to $\langle\ ,\ \rangle$. For later use, we normalize G_μ by the definition

$$\tilde{G}_\mu := G_\mu / G_\mu(1) = C_\mu^{(r-2)/2} / C_\mu^{(r-2)/2}(1),$$

such that $\tilde{G}_\mu(1) = 1$. The first normalized polynomials are

$$\tilde{G}_0 = 1, \qquad \tilde{G}_1(\xi) = \xi, \qquad \tilde{G}_2(\xi) = \frac{1}{r-1}(r\xi^2 - 1). \tag{2.3}$$

By the use of the reproducing kernel function, the orthogonal projections

$$\Omega_\nu : C(S^{r-1}) \to \overset{*}{\mathbf{H}}{}^r_\nu, \tag{2.4}$$

where $\nu \in \mathbf{N}_0$, can be represented in the form

$$(\Omega_\nu F)(t) = \int_{S^{r-1}} F(x)G_\nu(tx)\, d\omega(x) \tag{2.5}$$

where $F \in C(S^{r-1})$, $t \in S^{r-1}$. Similarly, the reproducing kernel of $\mathbf{P}^r_\mu$ is given by

$$\Gamma_\mu(xy) = \sum_{\nu=0}^{\mu} G_\nu(xy), \tag{2.6}$$

and the *orthogonal projection* $\Pi_\mu : C(S^{r-1}) \to \mathbf{P}_\mu^r$ has the representation

$$(\Pi_\mu F)(t) = \int_{S^{r-1}} F(x)\Gamma_\mu(tx)\, d\omega(x), \tag{2.7}$$

see [11]. We remark that it follows from (2.5)–(2.7) that

$$\Pi_\mu = \sum_{\nu=0}^{\mu} \Omega_\nu \tag{2.8}$$

holds for $\mu \in \mathbf{N}_0$, i.e., the Π_μ's are the partial-sum operators of the *Laplace series*.

By a result of Daugavet [3], Π_μ coincides with the minimal projection with respect to the uniform norm, which is defined for arbitrary bounded linear operators $L : C(S^{r-1}) \to C(S^{r-1})$ by

$$\|L\|_\infty := \sup\{\|LF\|_\infty : F \in C(S^{r-1}), \|F\|_\infty \le 1\}.$$

Unfortunately,

$$\|\Pi_\mu\|_\infty \sim \frac{2}{\pi^{3/2}} \cdot \frac{\Gamma\left(\frac{r-1}{4}\right)\Gamma\left(\frac{r}{4}\right)}{\Gamma\left(\frac{r-1}{2}\right)^2} \cdot \mu^{(r-2)/2}, \qquad \mu \to \infty, \tag{2.9}$$

holds, [3], which, in view of the Theorem of Banach–Steinhaus, prohibits pointwise convergence.

In what follows we assume that $Q = \{Q_\mu\}_{\mu \in \mathbf{N}0}$ is a sequence of quadrature rules of the form

$$Q_\mu F = \sum_{j=1}^{M} A_j F(t_j), \tag{2.10}$$

where $F \in C(S^{r-1})$, with nodes $t_j \in S^{r-1}$ and positive weights A_j. The number M of nodes occurring with Q_μ will also be denoted by $|Q_\mu|$. In this section, we assume that the rules are exact of degree 2μ, i.e.,

$$Q_\mu F = \int_{S^{r-1}} F(x)\, d\omega(x) \qquad \text{for all} \quad F \in \mathbf{P}_{2\mu}^r. \tag{2.11}$$

(For simplicity, we do not indicate the dependence of M and of the weights on μ.)

Hyperinterpolation now arises if the quadrature rule Q_μ is used in the evaluation of the orthogonal projection, this means of the integral occurring in (2.7). The resulting *hyperinterpolation operators*

$$L_\mu : C(S^{r-1}) \to \mathbf{P}_\mu^r, \tag{2.12}$$

where $\mu \in \mathbf{N}$, are defined by

$$(L_\mu F)(t) := \sum_{j=0}^{M} A_j F(t_j)\Gamma_\mu(t_j t) \tag{2.13}$$

for $F \in C(S^{r-1})$ and $t \in S^{r-1}$.

If the quadrature Q_μ is *interpolatory*, i.e., if $|Q_\mu| = \dim(\mathbf{P}^r_\mu)$ holds, then it is in view of (2.11) a *Gauss-quadrature*, and L_μ coincides with the corresponding *interpolatory projection*. So far, hyperinterpolation is a generalization of interpolation. But as Gauss-quadratures do not exist for $(r, \mu) \geq (3, 3)$ by a result of [1], the situation described does not occur in the relevant cases.

Because of (2.11), L_μ is, like Π_μ, a projection operator, however, with the possibility of an evaluation at low costs, which it is sharing with interpolatory operators. Naturally, the norms $\|L_\mu\|_\infty$ grow at least like the minimal projection norms, see (2.9). But it has been proved by [20] ($r = 3$, some regularity condition on Q) and by [15] (r arbitrary, no additional assumption on Q) that, in addition to $a_\mu > 0$, a constant $b_\mu > 0$ exists such that

$$a_\mu \mu^{(r-2)/2} < \|L_\mu\|_\infty < b_\mu \mu^{(r-2)/2} \tag{2.14}$$

holds for arbitrary $\mu \in \mathbf{N}$, and this independently of the choice of the sequence of positively weighted quadrature rules which is used.

In spite of this best-order result, pointwise convergence cannot be attained by hyperinterpolation. For this reason, we generalize the concept of hyperinterpolation in the following section.

3. Generalized Hyperinterpolation

In this and in the following sections we assume that the sequence Q of quadrature rules (2.10) satisfies the

Weak Assumptions on *Q*:

$$\begin{aligned} &A_j > 0, \qquad j = 1, \ldots, M, \\ &Q_\mu F = \int_{S^{r-1}} F(x)\, d\omega(x) \qquad \text{for all} \quad F \in \mathbf{P}^r_{\mu+1}. \end{aligned} \tag{3.1}$$

The assumption (3.1) is "weak" in comparison with (2.11).

Next we assume that $A = (a_{\mu,\nu})_{\mu,\nu\in\mathbf{N}_0}$ is some infinite real matrix which satisfies the

Assumptions on *A*:

(i) $a_{\mu,\nu} = 0$ *for* $\nu > \mu$ (*subdiagonality*);
(ii) $\lim_{\mu\to\infty} a_{\mu,\nu} = 1$ *for* $\nu \in \{0, 1\}$ (*Korovkin assumption*);
(iii) $K_\mu(\xi) \geq 0$ *for* $-1 \leq \xi \leq +1$ (*positivity*);

where

$$K_\mu := \sum_{\nu=0}^{\mu} a_{\mu,\nu} G_\nu, \qquad \mu \in \mathbf{N}_0. \tag{3.2}$$

Note that the kernels K_μ depend on r, which we do not indicate. By their help we define the "partial-sum" operators

$$\Lambda_\mu := \sum_{\nu=0}^{\mu} a_{\mu,\nu}\Omega_\nu, \tag{3.3}$$

with the representation (*A partial sums* of F):

$$(\Lambda_\mu F)(t) = \int_{S^{r-1}} F(x) K_\mu(tx)\, d\omega(x), \tag{3.4}$$

for $F \in C(S^{r-1})$, $t \in S^{r-1}$, see (2.5), (3.2). The definitions are such that the following theorem holds:

Theorem 1. *If the Assumptions on A hold, then the sequence* $\{\Lambda_\mu\}_{\mu\in\mathbf{N}_0}$ *is a sequence of positive linear maps*

$$\Lambda_\mu : C(S^{r-1}) \to \mathbf{P}_\mu^r, \tag{3.5}$$

which converges pointwise to the identity in $\|\cdot\|_\infty$.

Proof. Obviously, it follows from (i) that Λ_μ is a map (3.5). Because of (iii) the operators are positive. Moreover, the spaces $\overset{*}{\mathbf{H}}{}_\nu^r$ are orthogonal. So we get $\Omega_\mu G_\nu(e\,\cdot) = \delta_{\mu,\nu} G_\nu(e\,\cdot)$ for $e \in S^{r-1}$ and hence

$$\Lambda_\mu G_\nu(e\,\cdot) = a_{\mu,\nu} G_\nu(e\,\cdot). \tag{3.6}$$

In addition, assumption (ii) implies

$$\lim_{\mu\to\infty} \Lambda_\mu G_\nu(e\cdot) = G_\nu(e\cdot) \qquad \text{for} \quad \nu \in \{0, 1\} \tag{3.7}$$

in $\|\cdot\|_\infty$. In view of (2.3) this is equivalent to

$$\lim_{\mu\to\infty} \Lambda_\mu 1 = 1, \qquad \lim_{\mu\to\infty} \Lambda_\mu (e\cdot) = (e\cdot).$$

But the functions $1, x_1, \ldots, x_r$ form a *Korovkin* set, see [7, p. 7], and the statement of the theorem follows from the *Theorem of Bohman and Korovkin*. ■

We remark that (3.6) allows us to regain the defining matrix A from the operators Λ_μ by

$$a_{\mu,\kappa} = (\Lambda_\mu \tilde{G}_\kappa(e\,\cdot))(e) \qquad \text{for} \quad e \in S^{r-1}. \tag{3.8}$$

Generalized Hyperinterpolation. *Generalized* hyperinterpolation now arises if a quadrature rule (2.10), which satisfies the *Weak Assumptions on Q*, is used in the evaluation of the integral on the right side of (3.4). This means that the *generalized* hyperinterpolation operators

$$L_\mu : C(S^{r-1}) \to \mathbf{P}_\mu^r, \tag{3.9}$$

where $\mu \in \mathbf{N}_0$, are defined by

$$(L_\mu F)(t) := \sum_{j=0}^{M} A_j F(t_j) K_\mu(t_j t) \tag{3.10}$$

for $F \in C(S^{r-1})$ and $t \in S^{r-1}$. Note that the evaluation of generalized hyperinterpolation and of hyperinterpolation cause the same cost.

Theorem 2. *Let the matrix A satisfy the Assumptions on A while the quadrature sequence Q satisfies the Weak Assumption on Q. Then the operators L_μ, $\mu \in \mathbf{N}_0$, are positive and they converge pointwise to the identity in $\|\cdot\|_\infty$.*

Proof. Obviously, by the assumptions, the operators L_μ are positive and, as the quadratures Q_μ are exact on $\mathbf{P}^r_{\mu+1}$, we get

$$L_\mu F = \Lambda_\mu F \qquad \text{for all} \quad F \in \mathbf{P}^r_1. \tag{3.11}$$

This implies convergence on $\mathbf{P}^r_1$. But $\mathbf{P}^r_1$ contains a Korovkin set. So pointwise convergence follows again from the *Theorem of Bohman and Korovkin*. ∎

4. Example: Generalized Hyperinterpolation Based on Cesàro Partial Sums

Let $A = A^{(k)}$ be defined by the Cesàro method (C, k), where $k \geq r - 1$, which implies $k \geq 1$. Then the kernel function takes the form

$$K_\mu = K_\mu^{(k)} = \frac{(1)_\mu}{(k+1)_\mu} \sum_{\nu=0}^{\mu} \frac{(k+1)_{\mu-\nu}}{(1)_{\mu-\nu}} \cdot G_\nu, \tag{4.1}$$

where $(p)_q$ is *Pochhammer's symbol*. Especially, we obtain

$$a_{\mu 0} = 1, \qquad a_{\mu 1} = 1 - \frac{k}{\mu + k}. \tag{4.2}$$

We write the corresponding operators Λ_μ and L_μ as $\Lambda_\mu^{(k)}$ and $L_\mu^{(k)}$, respectively. Then the following theorem holds:

Theorem 3. *Assume that the Weak Assumptions on Q are satisfied and let $k \geq r - 1$. Then $L_\mu^{(k)}$ is a positive operator for all $\mu \in \mathbf{N}_0$, and*

$$\lim_{\mu \to \infty} L_\mu^{(k)} F = F$$

holds in $\|\cdot\|_\infty$ for all $F \in C(S^{r-1})$.

Proof. Obviously, $A = A^{(k)}$ satisfies (i), and (ii) holds because of (4.2), while (iii) is valid by a result of Kogbetliantz [6]. For a shorter proof, see [12]. So the statement follows from Theorem 2. ∎

It is worthwhile mentioning the following. Using (3.11), i.e.,

$$L_\mu^{(k)} F = \Lambda_\mu^{(k)} F \qquad \text{for all} \quad F \in \mathbf{P}_1^r,$$

we get, from (3.6), (4.2) in view of (2.3):

$$F - L_\mu^{(k)} F = 0 \qquad \text{for all} \quad F \in \overset{*}{\mathbf{H}}{}_0^r,$$

$$F - L_\mu^{(k)} F = \frac{k}{\mu + k} \cdot F \qquad \text{for all} \quad F \in \overset{*}{\mathbf{H}}{}_1^r.$$

Especially, we get

$$\|F - \Lambda_\mu F\|_\infty = \|F - L_\mu F\|_\infty \geq \frac{k}{k+1} \|F\|_\infty \cdot \frac{1}{\mu} \tag{4.3}$$

for all $F \in \overset{*}{\mathbf{H}}{}_1^r$, saying that the order of convergence is, in general, not better than $1/\mu$, which is poor. For the corresponding saturation class, see [2].

So far, there is a need for generalized hyperinterpolation operators which act more precisely on smooth functions.

5. The Newman–Shapiro Summation Method

Let $\nu \in \mathbf{N}_0$. Following Newman and Shapiro [8], we define the univariate kernel polynomials

$$K_{2\nu+1}(\xi) := K_{2\nu}(\xi) := g_{\nu+1} \left[\frac{G_{\nu+1}(\xi)}{\xi - \eta_{\nu+1}} \right]^2, \tag{5.1}$$

where

$$\eta_{\nu+1} = \cos \chi_{\nu+1}, \qquad 0 < \chi_{\nu+1} \leq \pi/2, \tag{5.2}$$

is the largest root of $G_{\nu+1}$ and where the constant $g_{\nu+1}$ is defined by

$$g_{\nu+1}^{-1} = \int_{S^{r-1}} \left[\frac{G_{\nu+1}(tx)}{tx - \eta_{\nu+1}} \right]^2 d\omega(x), \tag{5.3}$$

where $t \in S^{r-1}$ is arbitrary. K_μ is a polynomial of degree μ and hence of form (3.2), so defining a subdiagonal infinite matrix A. Vice versa, the linear operators Λ_μ can be defined as above in (3.3). Especially, we get

$$(\Lambda_{2\nu} F)(x) = \int_{S^{r-1}} F(t) K_{2\nu}(tx)\, d\omega(t) \tag{5.4}$$

for $F \in C(S^{r-1})$, $x \in S^{r-1}$. Newman and Shapiro used these operators in providing a Jackson-type inequality for spherical functions. So we call them the *Newman–Shapiro operators*, and the summation method defined by A, the *Newman–Shapiro method*. We shall see later that the corresponding generalized hyperinterpolation operators, $L_{2\nu}$, inherit this important property thanks to the special choice of the kernel function. For this reason we investigate thoroughly these operators in what follows. First, we note that

$$\Lambda_{2\nu} 1 = 1 \tag{5.5}$$

because of (5.3), and following [16, Equation 4.12] (misspelled, ω_{r-1} has to be replaced by ω_{r-1}^{-1}):

$$g_{\nu+1} = (r-2)\omega_{r-1} \cdot \frac{1-\eta_{\nu+1}^2}{(2\nu+r)^2} \cdot \binom{\nu+r-2}{r-3}^{-1} \tag{5.6}$$

is valid. This yields:

Lemma 4. *Let* $I_{\nu j} := \int_{S^{r-1}} [tx - \eta_{\nu+1}]^j K_{2\nu}(tx)\, d\omega(x)$ *for* $j \in \mathbf{N}_0$, $t \in S^{r-1}$. *Then*

$$I_{\nu 0} = 1, \qquad I_{\nu 1} = 0, \qquad I_{\nu 2} = \frac{1}{2\nu + r} \cdot (1 - \eta_{\nu+1}^2). \tag{5.7}$$

Proof. For $I_{\nu 0}$, this follows from the definition of the kernel. Next we get, using orthogonality,

$$I_{\nu 1} = g_{\nu+1} \int_{S^{r-1}} \frac{G_{\nu+1}(tx)}{tx - \eta_{\nu+1}} \cdot G_{\nu+1}(tx)\, d\omega(x) = 0. \tag{5.8}$$

Note that the first factor of the integrand is a polynomial of degree ν in x. Finally, we obtain

$$I_{\nu 2} = g_{\nu+1} \int_{S^{r-1}} G_{\nu+1}^2(tx)\, d\omega(x) = g_{\nu+1} G_{\nu+1}(1)$$

by the reproducing property of $G_{\nu+1}$. But from (2.2), we get

$$G_{\nu+1}(1) = \frac{2\nu + r}{(r-2)\omega_{r-1}} \binom{\nu+r-2}{r-3}. \tag{5.9}$$

Together with (5.6) this yields the statement on $I_{\nu 2}$.

Theorem 5. *The matrix A defined by the Newman–Shapiro operators is nonnegative, where*

$$a_{2\nu+1,\kappa} = a_{2\nu,\kappa} > 0 \qquad for \quad \kappa = 0, \ldots, 2\nu,$$

where $\nu \in \mathbf{N}_0$. *The first three columns are given by*

$$\begin{aligned} a_{2\nu,0} &= 1, \\ a_{2\nu,1} &= 1 - 2\sin^2 \frac{\chi_{\nu+1}}{2}, \\ a_{2\nu,2} &= 1 - \frac{r}{r-1} \cdot \frac{2\nu + r - 1}{2\nu + r} \cdot \sin^2 \chi_{2\nu+1}, \end{aligned} \tag{5.10}$$

where

$$\chi_1 = \frac{\pi}{2}, \qquad \chi_\nu \sim \frac{1}{\nu} \cdot j_{(r-3)/2} \qquad as \quad \nu \to \infty, \tag{5.11}$$

where $j_{(r-3)/2}$ is the lowest positive zero of the Bessel function $J_{(r-3)/2}$, approximately in the meaning of asymptotical equality. Moreover, a constant d_r exists, depending on r only, such that, for $\kappa = 0, 1, \ldots, 2\nu,\ \nu \in \mathbf{N}$:

$$|a_{2\nu,\kappa} - 1| \le d_r \cdot \left(\frac{\kappa}{\nu}\right)^2. \tag{5.12}$$

Proof. By the reproducing property of G_κ we get $\|G_\kappa(t\,\cdot)\|_2^2 = G_\kappa(1)$ for $\kappa \in \mathbf{N}_0$, $t \in S^{r-1}$. So the formula of Christoffel–Darboux takes the form

$$\frac{k_\nu}{k_{\nu+1}} \cdot \frac{G_{\nu+1}(\xi)G_\nu(\eta) - G_\nu(\xi)G_{\nu+1}(\eta)}{(\xi-\eta)\sqrt{G_\nu(1)G_{\nu+1}(1)}} = \sum_{\kappa=0}^{\nu} \frac{G_\kappa(\xi)G_\kappa(\eta)}{G_\kappa(1)}, \tag{5.13}$$

where k_ν is the leading coefficient of $G_\nu/\sqrt{G_\nu(1)}$ and hence positive. Note that

$$G_\kappa(\eta_{\nu+1}) > 0 \qquad \text{for} \quad \kappa = 0, \ldots, \nu,$$

holds by the interlacing property of the roots of the Gegenbauer polynomials. So, putting $\eta := \eta_{\nu+1}$, such that $G_{\nu+1}(\eta)$ vanishes, (5.13) can be brought to the form

$$\begin{aligned} \frac{G_{\nu+1}(\xi)}{\xi - \eta_{\nu+1}} &= \sum_{\kappa=0}^{\nu} c_{\nu\kappa} G_\kappa(\xi), \\ c_{\nu\kappa} &= \frac{k_{\nu+1}}{k_\nu} \cdot \sqrt{\frac{G_{\nu+1}(1)}{G_\nu(1)}} \cdot \frac{\tilde{G}_\kappa(\eta_{\nu+1})}{\tilde{G}_\nu(\eta_{\nu+1})}, \end{aligned} \tag{5.14}$$

where $k_\nu\sqrt{G_\nu(1)}$ is the leading coefficient of G_ν and is well known from (2.2) and [11, Equation 3.3]. Using this, we obtain

$$c_{\nu\kappa} = \frac{2\nu + r}{\nu + 1} \cdot \frac{\tilde{G}_\kappa(\eta_{\nu+1})}{\tilde{G}_\nu(\eta_{\nu+1})} > 0, \tag{5.15}$$

where $\kappa = 0, 1, \ldots, \nu$. Now we get from (5.14) and the definition of the kernel

$$K_{2\nu} = g_{\nu+1} \sum_{\iota=0}^{\nu} \sum_{\kappa=0}^{\nu} c_{\nu\iota} G_\iota(\xi) G_\kappa(\xi) c_{\nu\kappa}, \tag{5.16}$$

where we can insert, using the linearization formulas of Rogers and Ramanujan [5],

$$\begin{aligned} G_\iota G_\kappa = {} & \frac{(\iota+\lambda)(\kappa+\lambda)}{\omega_{r-1}\lambda^2} \times \sum_{k=0}^{\min\{\iota,\kappa\}} \frac{(\lambda)_k}{(1)_k} \\ & \cdot \frac{(\lambda)_{\iota-k}}{(1)_{\iota-k}} \cdot \frac{(\lambda)_{\kappa-k}}{(1)_{\kappa-k}} \cdot \frac{(1)_{\iota+\kappa-2k}}{(2\lambda)_{\iota+\kappa-2k}} \cdot \frac{(2\lambda)_{\iota+\kappa-k}}{(\lambda+1)_{\iota+\kappa-k}} \cdot G_{\iota+\kappa-2k}, \end{aligned} \tag{5.17}$$

with the abbreviation $\lambda := (r-2)/2$. Especially, every product $G_\iota G_\kappa$ is a positive linear combination of the polynomials

$$G_{\iota+\kappa-2k}, \qquad k = 0, 1, \ldots, \min\{\iota, \kappa\}.$$

As the index set $\{\iota + \kappa \mid 0 \le \iota, \kappa \le \nu\}$ covers the set $\{0, 1, \ldots, 2\nu\}$, this implies that all coefficients $a_{\mu\kappa}$ in the representation (3.2) of the kernel are positive, and we are able to calculate them straightforwardly from (5.16) together with (5.3), (5.15), and (5.17).

From (3.8) we get, for even $\mu = 2\nu$ and with a fixed $e \in S^{r-1}$, by a Taylor series expansion of $\tilde{G}_\kappa$ about $\eta_{\nu+1}$:

$$a_{\mu\kappa} = \int_{S^{r-1}} \tilde{G}_\kappa(ex) K_\mu(ex)\, d\omega(x) = \sum_{\iota=0}^{\kappa} \frac{1}{\iota!} \cdot \tilde{G}_\kappa^{(\iota)}(\eta_{\nu+1}) \cdot I_{\nu\iota}.$$

In view of (2.3), this yields

$$\begin{aligned} a_{\mu 0} &= I_{\nu 0}, \\ a_{\mu 1} &= I_{\nu 1} + \eta_{\nu+1} I_{\nu 0}, \\ a_{\mu 2} &= \frac{r}{r-1}\left\{ I_{\nu 2} + 2\eta_{\nu+1} I_{\nu 1} + \left(\eta_{\nu+1}^2 - \frac{1}{r}\right) I_{\nu 0}\right\}, \end{aligned}$$

and using (5.7), we obtain (5.10).

Equation (5.11) is well known [22]. Finally we get, again from (3.8), and with $\mu = 2\nu$ and a fixed $t \in S^{r-1}$:

$$\begin{aligned} a_{\mu\kappa} - \tilde{G}_\kappa(\eta_{\nu+1}) &= \int_{S^{r-1}} (\tilde{G}_\kappa(tx) - \tilde{G}_\kappa(\eta_{\nu+1})) \cdot K_\mu(tx) \cdot d\omega(x) \\ &= g_{\nu+1} \int_{S^{r-1}} \frac{\tilde{G}_\kappa(tx) - \tilde{G}_\kappa(\eta_{\nu+1})}{tx - \eta_{\nu+1}} \cdot \frac{G_{\nu+1}^2(tx)}{tx - \eta_{\nu+1}} \cdot d\omega(x). \end{aligned}$$

For every $x \in S^{r-1}$, the first factor of the integrand can be represented as $\tilde{G}'_\kappa(\tau_x)$, $-1 \le \tau_x \le +1$, and because of $\max\{|\tilde{G}_\kappa(\xi)|\colon -1 \le \xi \le +1\} = \tilde{G}_\kappa(1) = 1$ we get, by the help of the Markov inequality,

$$|a_{\mu\kappa} - \tilde{G}_\kappa(\eta_{\nu+1})| \le g_{\nu+1} \cdot \kappa^2 \cdot \int_{S^{r-1}} \frac{G_{\nu+1}^2(tx)}{|tx - \eta_{\nu+1}|} \cdot d\omega(x). \tag{5.18}$$

In addition, we get from (5.8), i.e., by orthogonality,

$$\begin{aligned} &\int_{\chi_{\nu+1}}^{\pi} \frac{G_{\nu+1}^2(\cos\varphi)}{\cos\chi_{\nu+1} - \cos\varphi} \cdot (\sin\varphi)^{r-2}\, d\varphi \\ &\qquad = \int_0^{\chi_{\nu+1}} \frac{G_{\nu+1}^2(\cos\varphi)}{\cos\varphi - \cos\chi_{\nu+1}} \cdot (\sin\varphi)^{r-2}\, d\varphi \end{aligned} \tag{5.19}$$

and, hence,

$$\begin{aligned} \int_{S^{r-1}} \frac{G_{\nu+1}^2(tx)}{|tx - \eta_{\nu+1}|} \cdot d\omega(x) &= 2\omega_{r-2} \int_0^{\chi_{\nu+1}} \frac{G_{\nu+1}^2(\cos\varphi)}{\cos\varphi - \cos\chi_{\nu+1}} \cdot (\sin\varphi)^{r-2}\, d\varphi \\ &\le 2\omega_{r-2} \frac{G_{\nu+1}^2(1)}{1 - \cos\chi_{\nu+1}} \cdot \frac{\chi_{\nu+1}^{r-1}}{r-1}. \end{aligned} \tag{5.20}$$

Here we used that $G^2_{\nu+1}(\xi)/(\xi - \eta_{\nu+1})$ is a polynomial whose $2\nu + 1$ roots are all located in the interval $-1 < \xi \le \eta_{\nu+1} = \cos\chi_{\nu+1}$, such that it is positive and attaining its maximum value in the interval $\eta_{\nu+1} < \xi \le +1$ at $\xi = 1$.

Finally, we use the inequality

$$|a_{\mu\kappa} - 1| \le |1 - \tilde{G}_\kappa(\eta_{\nu+1})| + |a_{\mu\kappa} - \tilde{G}_\kappa(\eta_{\nu+1})|.$$

Again, by using the Markov inequality, we get

$$|1 - \tilde{G}_\kappa(\eta_{\nu+1})| = |\tilde{G}_\kappa(1) - \tilde{G}_\kappa(\eta_{\nu+1})| \le \kappa^2(1 - \eta_{\nu+1}).$$

Together with (5.18) and (5.20) this yields

$$|a_{\mu\kappa} - 1| \le \kappa^2 \left\{1 - \cos\chi_{\nu+1} + \frac{2\omega_{r-2}}{r-1} \cdot g_{\nu+1} \frac{G^2_{\nu+1}(1)}{1 - \cos\chi_{\nu+1}} \cdot \chi^{r-1}_{\nu+1}\right\}.$$

Because of (5.6), (5.9), and (5.11) the asymptotics of the right side are known as $\nu \to \infty$, which finally yields the existence of a constant d_r such that (5.12) holds. This finishes the proof. ■

From (3.6) and Theorem 5 we get

$$\begin{aligned}
\|G_0(e\,\cdot) - \Lambda_{2\nu} G_0(e\,\cdot)\|_\infty &= 0,\\
\|G_1(e\,\cdot) - \Lambda_{2\nu} G_1(e\,\cdot)\|_\infty &= 2\sin^2\frac{\chi_{\nu+1}}{2}\|G_1(e\,\cdot)\|_\infty,\\
\|G_2(e\,\cdot) - \Lambda_{2\nu} G_2(e\,\cdot)\|_\infty &= \frac{r}{r-1}\sin^2\chi_{\nu+1}\|G_2(e\,\cdot)\|_\infty + O\left(\frac{1}{\nu^3}\right),
\end{aligned}$$

as $\nu \to \infty$, saying that with some constant

$$\|F - \Lambda_{2\nu} F\|_\infty \le \text{const} \cdot \frac{1}{(\nu+1)^2} \cdot \|F\|_\infty \qquad \text{for all} \quad F \in \mathbf{P}^r_2 \tag{5.21}$$

holds. But no better order of convergence is valid in general. For the full saturation problem, see [2] again.

Next we define the modulus of continuity of the first order for functions $F \in C(S^{r-1})$ by

$$\omega_1(F, \varphi) := \max\{|F(x) - F(y)|\colon\ xy \ge \cos\varphi\}, \qquad 0 \le \varphi \le \pi. \tag{5.22}$$

(A confusion with the measure of S^1 or, later, of S^2 is excluded by the appearance of arguments.) Note that we use in the definition of $\omega_1(F, \varphi)$ the geodetic distance $\arccos(xy)$ in S^{r-1} instead of the Euclidean distance $|x - y|$ in $\mathbf{R}^r$, with the advantage that it coincides with the trigonometric distance along every main circle. Asymptotically, they are all equal if x approaches y.

As indicated, Newman and Shapiro used the operators (5.4) in order to prove a Jackson-type inequality with respect to this modulus, see [8, p. 216]. In [16] we could transfer their result to the discretized Newman–Shapiro operators, which are generalized hyperinterpolation operators, as we now know.

Theorem 6. *Let A be defined by the Newman–Shapiro operators. Then A satisfies the Assumptions on A, and a constant k exists such that the following holds: If Q is a sequence of quadrature rules which satisfies the Weak Assumptions on Q, then*

$$\|F - L_{2\nu}F\|_\infty \leq k \cdot \omega_1\left(F, \frac{1}{\nu}\right)$$

holds for arbitrary $\nu \in \mathbf{N}$ and $F \in C(S^{r-1})$.

Proof. The Assumptions on A are satisfied because of Theorem 5. We proved the remaining in [16]. ■

Remarks. The constant k does not depend on the choice of Q. The proof is based on the fact that positive weights guarantee some regularity in the distribution of the weights (not necessarily of the nodes themselves). The proof of Theorem 6 depends on the special choice of the kernel, which allows us, in the estimation of the error at a fixed point $x \in C(S^{r-1})$, to reflect the influence of the outer nodes of some small cap with center x to the influence of the inner ones by an equality which corresponds with (5.19).

6. Simultaneous Approximation of Smooth and Nonsmooth Functions

Theorem 6 says that generalized hyperinterpolation, if defined by discretization of the Newman–Shapiro operators, creates approximations which are comparable with the Newman–Shapiro approximations themselves up to the first order. For higher orders we have no similar results. Maybe very special choices of the quadrature rules could make them possible. Anticipating this, we investigate here, and in the following section Newman–Shapiro approximations themselves, and this again in a more general concept. We want to discuss, finally, their use in tomography, hoping that this will also hold for discretized versions, i.e., for special generalized hyperinterpolation processes.

In tomography, often density functions on B^r occur which contain smooth and nonsmooth components. If the density function is expanded by its orthogonal projections, then a summation method is needed which acts adequately on both of them. If the problem is lifted to the unit sphere S^{r+1} in $\mathbf{R}^{r+2}$, then a corresponding summation method is needed for the $(r+2)$-dimensional *Laplace series*. For details, we refer to [13].

That a problem is hidden here we learn from the univariate case. Actually, if $P^\star_\mu$ is the operator of best algebraic approximation of degree μ on $C[-1,+1]$, then Jackson's theory says that

$$\|f - P^\star_\mu f\|_\infty = o(\mu^{-j}) \tag{6.1}$$

holds for arbitrary $j \in \mathbf{N}$ as $\mu \to \infty$, provided f is so smooth that $f^{(j)} \in C[-1,+1]$ is valid. Likewise, inequalities of form

$$\|f - L_\mu f\|_\infty \leq k_j \|f^{(j)}\|_\infty \cdot \mu^{-j}, \qquad k_j > 0, \tag{6.2}$$

where $\mu \in \mathbf{N}$, $f \in C^j[-1,+1]$ hold where the L_μ form a sequence of linear operators. But they cannot hold in the case of positive operators simultaneously for $j = 1, 2$, and

3, see G. G. Lorentz [7]. So, the best possible which can be reached by a sequence of positive linear operators, is that (6.2) is valid simultaneously for the orders $j = 1$ and $j = 2$.

A similar result holds for 2π-periodic functions and hence for spherical functions in $r = 2$ dimensions. So we may restrict ourselves again to the case $r \geq 3$ in what follows:

Smooth Functions on S^{r-1}. Let $F : S^{r-1} \to \mathbf{R}$ be a spherical function. F is called *j-times continuously differentiable*, $j \in \mathbf{N}$, if all restrictions $F_{u,v}$ of F onto a main circle, i.e., if all functions defined by

$$F_{u,v}(\varphi) := F(u \cos\varphi + v \sin\varphi), \qquad \varphi \in \mathbf{R},$$

where $u, v \in S^{r-1}$, $u \perp v$, are j-times continuously differentiable. Obviously, in this case, every function $F_{u,v}^{(j)}$ is a 2π-periodic continuous function which is provided with the maximum norm $\|F_{u,v}^{(j)}\|_{2\pi}$. So we may define the numbers

$$\|F^{(j)}\|_\infty := \sup\{\|F_{u,v}^{(j)}\|_{2\pi} : u, v \in S^{r-1}, u \perp v\}$$

and the linear spaces

$$C^j(S^{r-1}) := \{F : S^{r-1} \to \mathbf{R} : \|F^{(j)}\|_\infty < \infty\}.$$

Obviously, the elements of such a space are j-times differentiable at every point $x \in S^{r-1}$ in the direction of every tangent. The definitions belong to the *inner geometry* of the sphere, but if F is defined in a neighborhood of S^{r-1} and has continuous partial derivatives up to the order j, then its restriction onto S^{r-1} belongs to $C^j(S^{r-1})$.

Next we define the modulus of continuity of the second order for functions $F \in S^{r-1}$ as follows: by its univariate definition

$$\omega_2(F_{u,v}, \varphi) = \max\{|F_{u,v}(\alpha + \varphi) - 2F_{u,v}(\alpha) + F_{u,v}(\alpha - \varphi)| : \alpha \in \mathbf{R}\}$$

holds for $\varphi \geq 0$. So we may define, in the multivariate case,

$$\omega_2(F, \varphi) := \max\{\omega_2(F_{u,v}, \varphi) : u, v \in S^{r-1}, u \perp v\},$$

again for $\varphi \geq 0$. It is obvious that, as in the univariate case,

$$\omega_2(F, \varphi) \leq 2\omega_1(F, \varphi) \tag{6.3}$$

holds. In addition, we get

$$\omega_2(F, \varphi) \leq (\mu\varphi + 1)^2 \omega_2\left(F, \frac{1}{\mu}\right) \qquad \text{for all} \quad \mu \in \mathbf{N}, \tag{6.4}$$

from the univariate version of this inequality [7]. And, finally, the moduli of the first and second order inherit the property

$$\omega_j(F, \varphi) \leq \|F^{(j)}\|_\infty \cdot \varphi^j \qquad \text{for all} \quad F \in C^j(S^{r-1}), \tag{6.5}$$

where $\varphi \geq 0$, $j \in \{1, 2\}$, from their univariate versions [7].

Newman–Shapiro Operators. In what follows we investigate the Newman–Shapiro operators again, Λ_μ as defined in Section 5, claiming that they have the property to act adequately on smooth and nonsmooth functions and this simultaneously in the following, extended sense.

Theorem 7. *Let Λ_μ, $\mu \in \mathbf{N}_0$, be the Newman–Shapiro operators. Then the following holds with some positive constants k_1, k_2, and for $j \in \{1, 2\}$, $\mu \in \mathbf{N}$:*

$$C(S^{r-1}) \ni F \quad \Longrightarrow \quad \|F - \Lambda_\mu F\|_\infty \;\le\; k_j \cdot \omega_j\left(F, \frac{1}{\mu}\right), \tag{6.6}$$

$$C^j(S^{r-1}) \ni F \quad \Longrightarrow \quad \|F - \Lambda_\mu F\|_\infty \;\le\; k_j \cdot \|F^{(j)}\|_\infty \cdot \mu^{-j}. \tag{6.7}$$

Proof. Equation (6.6) implies (6.7) because of (6.5). Therefore it suffices, in view of (6.3), to prove (6.6) for $j = 2$ and $\mu = 2\nu$, $\nu \in \mathbf{N}$, only.

So we assume this and let $F \in C(S^{r-1})$, $x \in S^{r-1}$. Using (5.5), we get

$$\begin{aligned}(\Lambda_{2\nu} F - F)(x) &= \int_{S^{r-1}} [F(t) - F(x)] K_{2\nu}(tx) \cdot d\omega(t) \\ &= \int_{-1}^{+1} K_{2\nu}(\xi) \int_{\substack{u \perp x \\ |u|=1}} [F(\xi x + \sqrt{1-\xi^2}u) - F(x)]\, d\omega(u) \\ &\quad \times (1-\xi^2)^{(r-3)/2}\, d\xi.\end{aligned}$$

In the last equation we may replace u by $-u$ and we obtain, likewise,

$$\begin{aligned}&(\Lambda_{2\nu} F - F)(x) \\ &\quad = \int_{-1}^{+1} K_{2\nu}(\xi) \int_{\substack{u \perp x \\ |u|=1}} [F(\xi x - \sqrt{1-\xi^2}u) - F(x)]\, d\omega(u) (1-\xi^2)^{(r-3)/2}\, d\xi.\end{aligned}$$

Averaging the last two equations and substituting $\xi = \cos\varphi$, we get

$$\begin{aligned}&(\Lambda_{2\nu} F - F)(x) \\ &\quad = \tfrac{1}{2} \cdot \int_0^\pi K_{2\nu}(\cos\varphi) \int_{\substack{u \perp x \\ |u|=1}} [F_{xu}(\varphi) - 2F_{xu}(0) + F_{xu}(-\varphi)]\, d\omega(u) (\sin\varphi)^{r-2}\, d\varphi\end{aligned}$$

and, hence, as $x \in S^{r-1}$ was arbitrary,

$$\|F - \Lambda_{2\nu} F\|_\infty \le \tfrac{1}{2} \cdot \omega_{r-2} \cdot \int_0^\pi K_{2\nu}(\cos\varphi) \omega_2(F, \varphi) (\sin\varphi)^{r-2}\, d\varphi.$$

Now we use (6.4) and get

$$\begin{aligned}\|F - \Lambda_{2\nu} F\|_\infty \;&\le\; \tfrac{1}{2} I_\nu \cdot \omega_2\left(F, \frac{1}{\mu}\right), \\ I_\nu \;&:=\; \omega_{r-2} \int_0^\pi (\mu\varphi + 1)^2 K_{2\nu}(\cos\varphi) (\sin\varphi)^{r-2}\, d\varphi.\end{aligned} \tag{6.8}$$

We have to show that the I_ν are uniformly bounded. To this end, we write

$$I_\nu = A_\nu + B_\nu, \tag{6.9}$$

where

$$A_\nu := \omega_{r-2} \int_0^{\chi_{\nu+1}} (\mu\varphi + 1)^2 K_{2\nu}(\cos\varphi)(\sin\varphi)^{r-2}\, d\varphi.$$

Because of (5.11) there is a constant $c_1 > 1$ such that

$$2\kappa\chi_{\kappa+1} + 1 \le c_1 \qquad \text{for all} \quad \kappa \in \mathbf{N}_0. \tag{6.10}$$

This yields, with a fixed $t \in S^{r-1}$:

$$A_\nu \le c_1^2 \cdot \omega_{r-2} \int_0^{\pi} K_{2\nu}(\cos\varphi)(\sin\varphi)^{r-2}\, d\varphi = c_1^2 \int_{S^{r-1}} K_{2\nu}(tx)\, d\omega(x) = c_1^2 \tag{6.11}$$

because of (5.1) and (5.3). Next we write $B_\nu = I_\nu - A_\nu$ in the form

$$B_\nu = \omega_{r-2} \int_{\chi_{\nu+1}}^{\pi} [\mu(\varphi - \chi_{\nu+1}) + \mu\chi_{\nu+1} + 1]^2 K_{2\nu}(\cos\varphi)(\sin\varphi)^{r-2}\, d\varphi.$$

We split it as follows:

$$B_\nu = B_{\nu 0} + B_{\nu 1} + B_{\nu 2}, \tag{6.12}$$

$$B_{\nu 2} = \mu^2 g_{\nu+1}\omega_{r-2} \int_{\chi_{\nu+1}}^{\pi} \left(\frac{\varphi - \chi_{\nu+1}}{\cos\chi_{\nu+1} - \cos\varphi} \right)^2 G_{\nu+1}^2(\cos\varphi)(\sin\varphi)^{r-2}\, d\varphi, \tag{6.13}$$

$$B_{\nu 1} = 2\mu(\mu\chi_{\nu+1} + 1) g_{\nu+1}\omega_{r-2} \times \int_{\chi_{\nu+1}}^{\pi} \frac{\varphi - \chi_{\nu+1}}{\cos\chi_{\nu+1} - \cos\varphi} \cdot \frac{G_{\nu+1}^2(\cos\varphi)}{\cos\chi_{\nu+1} - \cos\varphi} (\sin\varphi)^{r-2}\, d\varphi, \tag{6.14}$$

$$B_{\nu 0} = (\mu\chi_{\nu+1} + 1)^2 \omega_{r-2} \int_{\chi_{\nu+1}}^{\pi} K_\mu(\cos\varphi)(\sin\varphi)^{r-2}\, d\varphi. \tag{6.15}$$

Because of (6.10) we get, again with an arbitrary $t \in S^{r-1}$:

$$B_{\nu 0} < c_1^2 \int_{S^{r-1}} K_\mu(tx)\, dx = c_1^2. \tag{6.16}$$

In the interval $\chi_{\nu+1} < \varphi \le \pi$, we get

$$\frac{\varphi - \chi_{\nu+1}}{\cos\chi_{\nu+1} - \cos\varphi} = \frac{\dfrac{\varphi - \chi_{\nu+1}}{2}}{\sin\dfrac{\varphi - \chi_{\nu+1}}{2}} \cdot \frac{1}{\sin\dfrac{\varphi + \chi_{\nu+1}}{2}} < \frac{\pi}{2} \cdot \frac{1}{\sin\dfrac{\varphi + \chi_{\nu+1}}{2}}.$$

It is well known that $0 < \eta_{\nu+1} < 1$ holds for $\nu \in \mathbf{N}$, which corresponds with $0 < \chi_{\nu+1} < \pi/2$. Together with the convexity of the sin-function in $[0, \pi]$ this yields

$$\begin{aligned} \sin\frac{\varphi + \chi_{\nu+1}}{2} &\ge \min\left\{ \sin\chi_{\nu+1}, \sin\frac{\pi + \chi_{\nu+1}}{2} \right\} \\ &= \min\left\{ \sin\chi_{\nu+1}, \sin\frac{\pi - \chi_{\nu+1}}{2} \right\} \\ &\ge \min\{\sin\chi_{\nu+1}, \tfrac{1}{2}\sin(\pi - \chi_{\nu+1})\} = \tfrac{1}{2}\sin\chi_{\nu+1}. \end{aligned} \tag{6.17}$$

So we obtain

$$\frac{\varphi - \chi_{\nu+1}}{\cos\chi_{\nu+1} - \cos\varphi} \le \frac{\pi}{\sin\chi_{\nu+1}}, \qquad \chi_{\nu+1} < \varphi \le \pi, \tag{6.18}$$

and (6.14), (6.10), (5.19), and (5.20) yield

$$\begin{aligned} B_{\nu 1} &\le 4\pi\omega_{r-2}c_1 \frac{\nu g_{\nu+1}}{\sin\chi_{\nu+1}} \int_{\chi_{\nu+1}}^{\pi} \frac{G_{\nu+1}^2(\cos\varphi)}{\cos\chi_{\nu+1} - \cos\varphi} (\sin\varphi)^{r-2}\, d\varphi \\ &\le 4\pi\omega_{r-2}c_1 \frac{\nu g_{\nu+1}}{\sin\chi_{\nu+1}} \cdot \frac{G_{\nu+1}^2(1)}{1-\cos\chi_{\nu+1}} \cdot \frac{\chi_{\nu+1}^{r-1}}{r-1}. \end{aligned}$$

Because of (5.6), (5.9), and (5.11) we know the asymptotics of all the factors which appear. So we find that $B_{\nu 1}$ is bounded by some constant c_2:

$$B_{\nu 1} \le c_2. \tag{6.19}$$

It is left to estimate (6.13). In every arbitrary interval, $\chi < \varphi \le \pi$, $\chi > 0$:

$$\frac{(\varphi-\chi)^2}{\cos\chi - \cos\varphi} = 2\left(\frac{\frac{\varphi-\chi}{2}}{\sin\frac{\varphi-\chi}{2}}\right)^2 \cdot f(\varphi) < \frac{\pi^2}{2}\cdot f(\varphi)$$

holds if $f(\varphi)$ is defined by

$$f(\varphi) := \frac{\sin\frac{\varphi-\chi}{2}}{\sin\frac{\varphi+\chi}{2}}, \qquad \chi < \varphi \le \pi.$$

$f(\varphi)$ is monotonically increasing. So we obtain $f(\varphi) \le f(\pi) = 1$ and hence

$$\frac{(\varphi-\chi)^2}{\cos\chi - \cos\varphi} < \frac{\pi^2}{2}.$$

So we obtain with $\chi := \chi_{\nu+1}$ from (6.13), again, using (5.19) and (5.20):

$$\begin{aligned} B_{\nu 2} &\le 2\pi^2\nu^2 g_{\nu+1}\omega_{r-2} \int_{\chi_{\nu+1}}^{\pi} \frac{G_{\nu+1}^2(\cos\varphi)}{\cos\chi_{\nu+1} - \cos\varphi} (\sin\varphi)^{r-2}\, d\varphi \\ &\le 2\pi^2\omega_{r-2}\nu^2 g_{\nu+1} \cdot \frac{G_{\nu+1}^2(1)}{1-\cos\chi_{\nu+1}} \cdot \frac{\chi_{\nu+1}^{r-1}}{r-1}, \end{aligned}$$

and by the asymptotics of the right side we can find a constant c_3 such that

$$B_{\nu 1} \le c_3 \tag{6.20}$$

is valid for $\nu \in \mathbf{N}$. Finally, we obtain from (6.8), (6.9), (6.12), together with (6.16), (6.19), and (6.20) that (6.6) holds with the constant $k_2 := 2c_1^2 + c_2 + c_3$, and the theorem is proved. ∎

Remark. Pawelke [10] proved inequalities of the form

$$\|F - P_\mu^\star F\|_\infty \le M_k \cdot \omega_1 \left(\Delta^k F, \frac{1}{\mu}\right) \cdot \mu^{-2k}, \qquad \mu \in \mathbf{N},$$

where $P_\mu^\star F$ is a proximum to F in $\mathbf{P}_\mu^r$, Δ is the Laplace–Beltrami operator, and $k \in \mathbf{N}$ is arbitrary. This does not contradict with our not-so-optimistic statements, as $P_\mu^\star$ is nonlinear and nonpositive and is not constructive.

7. Application to Tomography

Let $R_0 : C(B^r) \to C(Z^r),\ Z^r := [-1, +1] \times S^{r-1}$, be Radon's transform, i.e., let

$$(R_0 F)(s, t) := \int_{\substack{v \perp t \\ v^2 \le 1 - s^2}} F(st + v)\, dv \tag{7.1}$$

for $F \in C(B^r)$, $(s, t) \in Z^r$. We assume that the infinite matrix A satisfies the Assumptions on A in $r + 2$ instead of r dimensions, i.e., with the kernel (3.2) defined by the G_ν which belong to the spaces $\overset{*}{\mathbf{H}}{}_\nu^{r+2}$ instead of $\overset{*}{\mathbf{H}}{}_\nu^{r}$.

We want to discuss a reconstruction method, presented in [13], with respect to its approximation error. To this end, we begin by summarizing this method.

Let us consider the isometry

$$C(B^r) \ni F \quad \mapsto \quad \bar{F} \in \bar{C}(S^{r+1}),$$

$$\bar{F}(\bar{x}) := F(x) \qquad \text{for} \qquad \bar{x} = \begin{pmatrix} x \\ x_{r+1} \\ x_{r+2} \end{pmatrix} \in S^{r+1},$$

where $\bar{C}(S^{r+1})$ is the subspace of $C(S^{r+1})$ of functions which do not depend on x_{r+1} and x_{r+2}.

For $F_1, F_2 \in C(B^r)$ we get the following identity:

$$\langle \bar{F}_1, \bar{F}_2 \rangle_{r+2} := \int_{S^{r+1}} \bar{F}_1(\bar{x}) \bar{F}_2(\bar{x})\, d\omega(\bar{x}) = \omega_1 \int_{B^r} F_1(x) F_2(x)\, dx =: (F_1, F_2)_r. \tag{7.2}$$

The spaces $\overset{*}{\mathbf{H}}{}_\nu^{r+2}$ of spherical harmonics of degree ν are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_{r+2}$, and so are the subspaces

$$\bar{\mathbf{H}}_\nu^{r+2} := \overset{*}{\mathbf{H}}{}_\nu^{r+2} \cap \bar{C}(S^{r+1}), \qquad \nu \in \mathbf{N}_0.$$

By our isometry, $\bar{\mathbf{H}}_\nu^{r+2}$ is the image space of some subspace V_ν^r of $C(B^r)$ which is thoroughly described in [13]. Note that

$$\bigoplus_{\nu=0}^{\infty} V_\nu^r$$

is the orthogonal decomposition of the full space of polynomials in r variables over B^r with respect to the inner product $(\cdot, \cdot)_r$, see [13, p. 339].

Next let F_ν be the orthogonal projection of F in $\boldsymbol{V}_\nu^r$ with respect to $(\cdot,\cdot)_r$. Then $\bar{F}_\nu$ is the orthogonal projection of $\bar{F}$ in $\bar{\mathbf{H}}_\nu^{r+2}$ with respect to $\langle\cdot,\cdot\rangle_{r+2}$, see [13, Theorem 2]. So, if we define

$$\Lambda_\mu F := \sum_{\nu=0}^{\mu} a_{\mu\nu} F_\nu, \qquad \bar{\Lambda}_\mu \bar{F} := \sum_{\nu=0}^{\mu} a_{\mu\nu} \bar{F}_\nu,$$

then we get by our isometry

$$\|F - \Lambda_\mu F\|_{B^r} = \|\bar{F} - \bar{\Lambda}_\mu \bar{F}\|_{S^{r+1}}, \tag{7.3}$$

where we use the uniform norm in both cases.

By our assumptions on A the right side in (7.3) tends to zero for $\mu \to \infty$ by Theorem 1. So, we get likewise

$$F = \lim_{\mu\to\infty} \Lambda_\mu F \tag{7.4}$$

in the uniform norm. On the other hand, the $R_0 F_\nu$ are well known by the results of Davison and Grünbaum [4] and by ourselves [13]. This allows us to reconstruct $\Lambda_\mu F$ from $R_0(\Lambda_\mu F) = \sum_{\nu=0}^{\mu} a_{\mu\nu} R_0 F_\nu$. The result is the reconstruction formula

$$(\Lambda_\mu F)(x) = \int_{Z^r} (R_0 F)(s,t) K_\mu^A(s, tx)\, d(s,t), \qquad x \in B^r, \tag{7.5}$$

where

$$K_\mu^A(s,\sigma) := \sum_{\nu=0}^{\mu} a_{\mu\nu} \lambda_\nu C_\nu^{r/2}(s) C_\nu^{r/2}(\sigma), \qquad \mu \in \mathbf{N}_0, \tag{7.6}$$

$$\lambda_\nu := \frac{2\nu + r}{\omega_{r-1}^2}, \qquad \nu \in \mathbf{N}_0, \tag{7.7}$$

see [13] again.

Example 1. If $\bar{\Lambda}_\mu$ is defined by the (C,k) *partial sums* of index $k \geq r+1$ (instead of $r-1$), then the Assumptions on A are satisfied and convergence follows from Theorem 1. Extensive numerical experiments confirm this theoretical result and show the importance of the choice of the index k, where however it does not pay to choose k greater than necessary. Even in areas where F is arbitrarily smooth the convergence is only moderate because of μ^{-1} saturation, see (4.3).

Example 2. Next we define $\bar{\Lambda}_\mu$ by the Newman–Shapiro sums in $r+2$ dimensions. If F is in $C^j(B^r)$, then $\bar{F}$ is in $C^j(S^{r+1})$, and Theorem 7 yields, together with (7.3),

$$\|F - \Lambda_\mu F\|_\infty \leq \bar{k}_j \|\bar{F}^{(j)}\|_\infty \cdot \mu^{-j}, \qquad \mu \in \mathbf{N}, \quad j \in \{1,2\}, \tag{7.8}$$

with $\bar{k}_j$ being now the k_j belonging to the $(r+2)$-dimensional problem. But as mentioned above, (7.8) cannot hold for positive operators for $j = 1, 2$, and 3. In this sense, the reconstruction method based on the Newman–Shapiro sums is optimal. It reconstructs continuously differentiable components of a function by an $\mathcal{O}(\mu^{-1})$ rate of convergence, and reconstructs at least twice continuously differentiable components by the optimal convergence rate $\mathcal{O}(\mu^{-2})$. So, our method is recovering the structure of a density function more rapidly in smooth areas than in nonsmooth areas, so increasing the contrast between the different components. This is just what is wanted in practice.

Remark 1. By lifting our problem by exactly two dimensions we could combine our knowledge about the action of R_0 on the subspaces V_ν^r with our knowledge on the summation of the Laplace series in $\mathbf{R}^{r+2}$. This procedure can also be understood from the fact that the homogeneous polynomials in $\mathbf{R}^r$ and the homogeneous harmonic polynomials in $\mathbf{R}^{r+2}$ of degree μ, can be generated by the same Gegenbauer polynomial $C_\mu^{r/2} = C_\mu^{[(r+2)-2]/2}$.

Remark 2. Formula (7.5) can be evaluated by means of a positively weighted quadrature on Z^r, for instance, by a product of a Gauss quadrature on $[-1, +1]$ and a product Gauss quadrature on S^{r-1}, both of degree μ. This means that an approximation to $\Lambda_\mu F$ is obtained from $G = R_0 F$ by the use of a discrete operator $L_\mu : C(Z^r) \to \mathbf{P}_\mu^r(B^r)$ of the form

$$(L_\mu G)(x) = \sum_{j=1}^{M} A_j G(s_j, t_j) K_\mu^A(s_j, t_j x), \qquad x \in B^r,$$

with positive weights A_j again. The analogy to generalized hyperinterpolation on the sphere, see (2.13), is obvious, so we may call L_μ again a generalized hyperinterpolation operator, but now on $C(Z^r)$ with image in $\mathbf{P}_\mu^r(B^r)$.

Naturally, for fixed F we can get estimates like (7.8) where Λ_μ is replaced by L_μ, by increasing the accuracy of the quadratures which are dependent on μ. The hope is that this result can be obtained independently of F for special sequences of quadrature rules at a number of nodes which grows at the order of the dimension of $\mathbf{P}_\mu^1([-1, 1]) \otimes \mathbf{P}_\mu^r(S^{r-1})$, which is $\mathcal{O}(\mu^r)$.

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