MR0340901 (49 #5651) 41A25 (42A08)
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Comparison theorems for a generalized modulus of continuity.
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In previous work [*Smoothing and approximation of functions*, second edition, Van Nostrand Reinhold, New York, 1969; Bull. Amer. Math. Soc. **74** (1968), 500–504; MR0225074 (37 #671); Acta Math. **120** (1968), 279–292; MR0239332 (39 #689)] the second author introduced a generalized modulus of continuity of a function $f\in L^p=L^p(\mathbf{R}^n)$, $l\le p\le\infty$, in the following way. Let $f\in L^p$ and $\sigma\in M^1=M^1(\mathbf{R}^n)$ (the Banach algebra of bounded complex Radon measures on $\mathbf{R}^n$) and let a lie in the interval $]0,\infty[$. The function $\omega_{\sigma,p}(f;a)=\sup_{0<b\le a}D_{\sigma,p}(f;b)$ is called the (σ,p)-modulus of f, where $D_{\sigma,p}(f;a)=\|f*\sigma_{(a)}\|_p$ and $\sigma_{(a)}$ denotes the measure in M^1 defined by the continuous linear functional $\int f(at)\,d\sigma(t)$ for $f\in C_0(\mathbf{R}^n)$. The authors show that the comparison theorems in the papers cited above are not sharp for values of p other than 1 and ∞, and proceed to prove the following sharp generalisations to L^p of the two basic comparison theorems of the second paper. Let $\overline{p}=\min(p,2)$ if $1\le p<\infty$ and $\overline{p}=1$ if $p=\infty$. (1) If $\sigma,\tau\in M^1$ and the Fourier transform $\hat\sigma$ of σ satisfies a certain Tauberian condition then for $f\in L^p$, $1\le p\le\infty$, and $a>0$ the inequality $\omega_{\tau,p}(f;a)\le A\int_0^{Ba}\omega_{\sigma,p}(f;v)^{\overline{p}}\,dv/v$ holds, where A and B depend only on σ, τ and p if $\hat\tau$ divides $\hat\sigma$ in $\hat M^1$ in some neighborhood of the origin. (2) If $\sigma,\tau\in M^1$ and P is a positive homogeneous function of degree $r>0$ then for $f\in L^p$, $1\le p\le\infty$, and $a>0$ the inequality

$$\omega_{\tau,p}(f;a)\le A\int_0^\infty[\min(1,(a/v)^r)\omega_{\sigma,p}(f;v)]^{\overline{p}}\,dv/v$$

holds, where A depends only on σ, τ and p holds if there exist $F,G\in\hat M^1$ with $F(x)=P(x)$ and $G(x)\cdot P(x)=\hat\tau(x)$ for all x in some neighborhood of the origin. The main steps in the proofs of these theorems are proofs of inequalities of Littlewood-Paley type for non-periodic functions and a technique which goes back to A. Zygmund [Univ. Nac. Tucumán Rev. Ser. A **7** (1950), 259–269; MR0042479 (13,118f)] in the periodic case. The proofs of the Littlewood-Paley inequalities are based on a partition of unity suggested by J. Peetre's treatment ["Thoughts about Besov spaces", mimeographed lecture notes, Lund Univ., Lund, 1966] of Besov spaces. Another interesting consequence of the inequalities is proved and is in its nature a sort of an "embedding theorem" related to Sobolev: If $\sigma\in M^1$ and $\hat\sigma$ satisfies a certain Tauberian condition then for $f\in L^p$, $1\le p\le\infty$, $D^\alpha f\in L^p$ for each α with $|\alpha|\le S$ if

$$\int_0^1[a^{-s}\omega_{\sigma,p}(f;a)]^{\overline{p}}\,da/a<\infty.$$

Finally, some concrete applications of the comparison theorems are given, which are not included in the results of the second paper cited above. {The comparison theorems can also be found in the book of the second author [*Topics in approximation theory*, Lecture Notes in Math., Vol. 187, Springer, Berlin, 1971].}

Reviewed by *B. Dreseler*