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The Taikov Functional in the Space of Algebraic Polynomials on the Multidimensional Euclidean Sphere

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Abstract—We discuss three related extremal problems on the set $\mathscr{P}_{n,m}$ of algebraic polynomials of given degree n on the unit sphere $\mathbb{S}^{m-1}$ of Euclidean space $\mathbb{R}^m$ of dimension $m \ge 2$. (1) The norm of the functional $F(h) = F_h P_n = \int_{\mathbb{C}(h)} P_n(x)\,dx$, which is equal to the integral over the spherical cap $\mathbb{C}(h)$ of angular radius $\arccos h$, $-1 < h < 1$, on the set $\mathscr{P}_{n,m}$ with the norm of the space $L(\mathbb{S}^{m-1})$ of summable functions on the sphere. (2) The best approximation in $L_\infty(\mathbb{S}^{m-1})$ of the characteristic function χ_h of the cap $\mathbb{C}(h)$ by the subspace $\mathscr{P}^{\perp}_{n,m}$ of functions from $L_\infty(\mathbb{S}^{m-1})$ that are orthogonal to the space of polynomials $\mathscr{P}_{n,m}$. (3) The best approximation in the space $L(\mathbb{S}^{m-1})$ of the function χ_h by the space of polynomials $\mathscr{P}_{n,m}$. We present the solution of all three problems for the value $h = t(n,m)$ which is the largest root of the polynomial in a single variable of degree $n+1$ least deviating from zero in the space L_1^ϕ on the interval $(-1,1)$ with ultraspheric weight $\phi(t) = (1-t^2)^\alpha$, $\alpha = (m-3)/2$.

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1. INTRODUCTION

Suppose that $\mathbb{R}^m$, $m \ge 2$, is the Euclidean space with inner product

$$xy = \sum_{k=1}^{m} x_k y_k, \qquad x = (x_1, \dots, x_m), \quad y = (y_1, \dots, y_m),$$

and norm $|x| = \sqrt{xx}$. In the space $\mathbb{R}^m$, consider the ball $\mathbb{B}^m(r) = \{x \in \mathbb{R}^m : |x| \le r\}$ and the sphere $\mathbb{S}^{m-1}(r) = \{x \in \mathbb{R}^m : |x| = r\}$ of radius r, $r > 0$, centered at the origin of coordinates; we denote the unit ball and the sphere ($r = 1$) by $\mathbb{B}^m$ and $\mathbb{S}^{m-1}$, respectively. Using numbers h, $-1 < h < 1$, and points $e \in \mathbb{S}^{m-1}$, we define the set

$$\mathbb{C}(h,e) = \{x = (x_1, \dots, x_m) \in \mathbb{S}^{m-1} : xe \ge h\} \subset \mathbb{S}^{m-1},$$

which is the spherical cap centered at the point e of angular (spherical) radius $\theta = \arccos h$. In what follows, we often use the spherical cap

$$\mathbb{C}(h) = \mathbb{C}(h, e_m) = \{x = (x_1, \dots, x_m) \in \mathbb{S}^{m-1} : x_m \ge h\}$$

centered at the "north pole" $e_m = (0, \dots, 0, 1)$ of the sphere.

Let us make a few remarks concerning the integrals (considered in the present paper) over the following sets: the ball $\mathbb{B}^m(r) = \{x \in \mathbb{R}^m : |x| \le r\}$ of radius r centered at the origin of coordinates of the space $\mathbb{R}^m$, the sphere $\mathbb{S}^{m-1}(r) = \{x \in \mathbb{R}^m : |x| = r\}$ of radius r of the space or, more generally,

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the ball $\mathbb{B}^k(r)$ and the sphere $\mathbb{S}^{k-1}(r)$ of some linear subspace of dimension k, $2 \le k \le m$, in particular, the unit ball $\mathbb{B}^m = \mathbb{B}^m(1)$, and the unit sphere $\mathbb{S}^{m-1} = \mathbb{S}^{m-1}(1)$. On each of these sets $\mathbb{H}$, we consider the classical Lebesgue measure (of the corresponding dimension). For a measurable subset $E \subset \mathbb{H}$, the symbol $|E|$ denotes the (corresponding) measure of the set E. Suppose that $L(E)$ is the space of functions measurable and summable on E; for a function $f \in L(E)$, its (Lebesgue) integral over the set E is written as $\int_E f(x)\,dx$. Incidentally, in what follows, the integrals in most cases can be defined in the sense of Riemann (with respect to the corresponding Jordan measure). It is assumed that the space $L(E) = L_1(E)$ is endowed with the norm $\|f\|_{L(E)} = \int_E |f(x)|\,dx$. On the set E, we shall also consider the space $L_\infty(E)$ measurable with respect to essentially bounded functions with the norm

$$\|f\|_{L_\infty(E)} = \operatorname{ess\,sup}\{|f(x)| : x \in E\}.$$

Denote by $\mathscr{P}_{n,m}$ the set of algebraic polynomials

$$P_n(x) = \sum_{\substack{|\alpha|=\alpha_1+\dots+\alpha_m \le n,\\ \alpha=(\alpha_1,\dots,\alpha_m)\in\mathbb{Z}_+^m}} c_\alpha x^\alpha, \qquad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}, \quad x = (x_1,\dots,x_m) \in \mathbb{R}^m, \tag{1}$$

of degree (at most) n in m (real) variables with real coefficients c_α. In the space $\mathscr{P}_{n,m}$, we define the linear functional $F = F(h)$, $-1 < h < 1$, by the formula

$$F(h)P_n = \int_{\mathbb{C}(h)} P_n(x)\,dx, \qquad P_n \in \mathscr{P}_{n,m}. \tag{2}$$

In the present paper, we are concerned with the norm $\nu_{n,m}(h)$ of the functional (2) on the space $\mathscr{P}_{n,m}$ endowed with the norm of the space $L(\mathbb{S}^{m-1})$:

$$\nu_{n,m}(h) = \sup\{|F(h)P_n| : P_n \in \mathscr{P}_{n,m},\ \|P_n\|_{L(\mathbb{S}^{m-1})} \le 1\}; \tag{3}$$

note that $\nu_{n,m}$ is the least constant in the inequality

$$|F(h)P_n| \le \nu_{n,m}(h)\|P_n\|_{L(\mathbb{S}^{m-1})}, \qquad P_n \in \mathscr{P}_{n,m}. \tag{4}$$

Problems of type (2)–(4) were first studied by Taikov [1][2] on the set $\mathscr{T}_n$ of trigonometric polynomials

$$f_n(t) = a_0 + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt)$$

of degree (at most) n with real coefficients in connection with the study of the best constant $c(n)$ in inequalities of Jackson–Nikol'skii type

$$\|f_n\|_{C_{2\pi}} \le c(n)\|f_n\|_{L_{2\pi}}, \qquad f_n \in \mathscr{T}_n,$$

between the $C_{2\pi}$ and $L_{2\pi}$ norms,

$$\|f_n\|_{C_{2\pi}} = \max\{|f_n(x)| : x \in \mathbb{R}\}, \qquad \|f_n\|_{L_{2\pi}} = \frac{1}{\pi}\int_0^{2\pi} |f_n(x)|\,dx,$$

of trigonometric polynomials of a given degree. Taikov calculated the norm $\widetilde{c}(n)$ of the functional

$$Ff_n = \frac{1}{\pi}\int_{-\pi/(2(n+1))}^{\pi/(2(n+1))} f_n(t)\,dt, \qquad f_n \in \mathscr{T}_n,$$

on the subspace $\mathscr{T}_n$ with the norm of the space $L_{2\pi}$, i.e., he found the least constant $\widetilde{c}(n)$ in the inequality

$$|Ff_n| \le \widetilde{c}(n)\|f_n\|_{L_{2\pi}}, \qquad f_n \in \mathscr{T}_n. \tag{5}$$

Namely, he proved that *for $n \ge 1$, the following statements are valid*:

1) *the relation $\widetilde{c}(n) = 1/2$ holds;*

2) *the polynomial*

$$\widetilde{f}_n(t)=\frac{\cos(n+1)t}{\cos t-\cos(\pi/(2(n+1)))}$$

is extremal in inequality (5).

Taikov obtained this result (as well as a number of other results) by invoking the dual problem [1], [2]. In this paper, we use a similar approach to determine the best constant in inequality (4)

The functional (2) can be defined via the characteristic function

$$\chi_h(x)=\begin{cases}1, & x\in\mathbb{C}(h),\\ 0, & x\notin\mathbb{C}(h),\end{cases} \tag{6}$$

of the spherical cap $\mathbb{C}(h)$; namely,

$$F(h)P_n=\int_{\mathbb{S}^{m-1}}\chi_h(x)P_n(x)\,dx, \qquad P_n\in\mathscr{P}_{n,m}.$$

Suppose that $\mathscr{P}^{\perp}=\mathscr{P}^{\perp}_{n,m}$ is the subspace of functions from $L_\infty(\mathbb{S}^{m-1})$, orthogonal to the space of polynomials $\mathscr{P}_{n,m}$, i.e., the set of all functions $\varphi\in L_\infty(\mathbb{S}^{m-1})$ possessing the property

$$\int_{\mathbb{S}^{m-1}}\varphi(t)P_n(t)\,dt=0, \qquad P_n\in\mathscr{P}_{n,m}.$$

Consider the best approximation

$$\omega_{n,m}(h)=\omega(\chi_h,\mathscr{P}^{\perp}_{n,m})=\inf\{\|\chi_h-\varphi\|_{L_\infty(\mathbb{S}^{m-1})}:\varphi\in\mathscr{P}^{\perp}_{n,m}\} \tag{7}$$

in $L_\infty(\mathbb{S}^{m-1})$ of the function χ_h by the subspace $\mathscr{P}^{\perp}=\mathscr{P}^{\perp}_{n,m}$.

Suppose that q_{n+1} is an algebraic polynomial with leading coefficient 1 least deviating from zero in the space $L_1^{\phi}(-1,1)$ of functions summable on the interval $(-1,1)$ with ultraspheric weight

$$\phi(t)=\phi^{(\alpha)}(t)=(1-t^2)^\alpha, \qquad \alpha=\alpha(m)=\frac{m-3}{2}.$$

It is well known that all $n+1$ zeros of the polynomial q_{n+1} are simple and lie on $(-1,1)$; suppose that $t(n,m)$ is the largest of them. Set

$$g_n^*(t)=c\,\frac{q_{n+1}(t)}{t-t(n,m)},$$

where the constant c is chosen from the condition that the norm $\|g_n^*\|_{L(\mathbb{S}^{m-1})}$ of the polynomial $g_n^*(x_m)$ in the space $L(\mathbb{S}^{m-1})$ on the sphere is 1. The following is one of the main statements in the present paper

Theorem. *For all $m\ge 3$, $n\ge 0$, and $h=t(n,m)$ the following assertions are valid*:

1) *the following relations hold*:

$$\nu_{n,m}(t(n,m))=\omega_{n,m}(t(n,m))=\frac{1}{2};$$

2) *the polynomial g_n^* and the function $\varphi_n^*=(1/2)\operatorname{sign}q_{n+1}$ as the zonal harmonics of one variable $t=x_m$, $x=(x_1,\dots,x_m)\in\mathbb{S}^{m-1}$, are extremal in problems* (2)–(4) *and* (7), *respectively.*

The results of the present paper were announced by the author in [3] and [4].

2. GENERAL CONSIDERATIONS: THE DUAL PROBLEM

2.1. Suppose that E is a measurable subset (of nonzero measure) of the space $\mathbb{R}^m$, $m \ge 1$, or the sphere $\mathbb{S}^{m-1}(r)$, $m \ge 2$. Suppose that v is a measurable, nonnegative function almost everywhere different from zero on E. Denote by $L_1^v = L_1^v(E)$ the (linear) space of measurable (on E) functions f for which the product $f \cdot v$ is summable on E; this is a Banach space with respect to the norm (see, for example, [5, Chap. 3, Sec. 4])

$$\|f\|_{L_1^v(E)} = \int_E |f(x)|v(x)\,dx, \qquad f \in L_1^v(E).$$

It is well known (see, for example, [5, Chap. 4, Sec. 8, Theorem 5]) that the space $L_\infty(E)$ of measurable essentially bounded functions f on E with norm

$$\|f\|_{L_\infty(E)} = \operatorname{ess\,sup}\{|f(x)| : x \in E\} = \inf\{M : |f(x)| \le M \text{ a.e. on } E\}, \qquad f \in L_\infty(E),$$

is adjoint to the space $L_1^v(E)$. Moreover, linear bounded functionals y^* on $L_1^v(E)$ are of the form

$$y^* f = \int_E f(t)y(t)v(t)\,dt, \qquad f \in L_1^v(E),$$

where $y \in L_\infty(E)$ and $\|y^*\|_{(L_1^v(E))^*} = \|y\|_{L_\infty(E)}$.

Suppose that $\psi \in L_\infty(E)$ is a particular measurable, bounded function on E and $\mathscr{F}$ is a linear (not necessarily closed) subspace of the space $L_1^v(E)$. On the set $\mathscr{F}$, consider the (linear) functional

$$\Psi f = \int_E \psi(t)f(t)v(t)\,dt, \qquad f \in \mathscr{F}. \tag{8}$$

Denote by $c(\Psi, \mathscr{F})$ the best (least possible) constant in the inequality

$$|\Psi f| \le c(\Psi, \mathscr{F})\|f\|_{L_1^v(E)}, \qquad f \in \mathscr{F}; \tag{9}$$

the constant $c(\Psi, \mathscr{F})$ can be regarded as the norm of the functional (8) on the subspace $\mathscr{F} \subset L_1^v(E)$.

Denote by $\mathscr{F}^\perp = \mathscr{F}^\perp(v)$ the subspace of functions from $L_\infty(E)$ orthogonal to the space $\mathscr{F}$ (with weight v) i.e., the set of all functions $\varphi \in L_\infty(E)$ possessing the property

$$\int_E \varphi(t)f(t)v(t)\,dt = 0, \qquad f \in \mathscr{F}. \tag{10}$$

Consider the best approximation

$$\omega(\psi, \mathscr{F}^\perp) = \inf\{\|\psi - \varphi\|_{L_\infty(E)} : \varphi \in \mathscr{F}^\perp\} \tag{11}$$

in $L_\infty(E)$ of the function ψ by the subspace $\mathscr{F}^\perp \subset L_\infty(E)$.

The following statement is an analog of the corresponding statement of Taikov [1], [2] and can be proved by using the same scheme. A measurable function f on the set E is said to be not identically zero ($f \not\equiv 0$) if the set $E(f \ne 0) = \{t \in E : f(t) \ne 0\}$ of points from E with nonzero values of the function has a nonzero measure.

Lemma 1. *Under the assumptions given above, the following assertions are valid*:

1) *The relation*

$$c(\Psi, \mathscr{F}) = \omega(\psi, \mathscr{F}^\perp) \tag{12}$$

holds.

2) *In problem* (11), *there exists an extremal function* $\varphi^* \in \mathscr{F}^\perp$. *If the subspace* $\mathscr{F}$ *is finite-dimensional, then there is also an extremal function* $f^* \in \mathscr{F}$ *in problem* (9).

3) *The function* $\widetilde{f} \in \mathscr{F}$, $\widetilde{f} \not\equiv 0$, *and the function* $\widetilde{\varphi} \in \mathscr{F}^\perp$ *are extremal in problems* (9) *and* (11), *respectively, if and only if there exists a constant* A *such that the following two conditions hold*:

$$|\psi - \widetilde{\varphi}| \le |A| \qquad \text{a.e. on } E, \tag{13}$$

$$\psi - \widetilde{\varphi} = A \operatorname{sign} \widetilde{f} \qquad \text{a.e. on } E(\widetilde{f} \ne 0). \tag{14}$$

Moreover, necessarily, $|A| = c(\Psi, \mathscr{F}) = \omega(\psi, \mathscr{F}^\perp)$.

Proof. For any function $\varphi \in \mathscr{F}^\perp$, in view of (10), the following formula holds along with (8):

$$\Psi f = \int_E f(t)\{\psi(t) - \varphi(t)\}v(t)\,dt, \qquad f \in \mathscr{F}. \tag{15}$$

Therefore, we have

$$|\Psi f| \le \int_E |f(t)| \cdot |\psi(t) - \varphi(t)| v(t)\,dt \le \|\psi - \varphi\|_{L_\infty(E)} \int_E |f(t)| v(t)\,dt = \|\psi - \varphi\|_{L_\infty(E)} \cdot \|f\|_{L_1^v(E)}.$$

Since $c(\Psi, \mathscr{F})$ is the least constant in inequality (9), this implies the inequality

$$c(\Psi, \mathscr{F}) \le \|\psi - \varphi\|_{L_\infty(E)}$$

and hence also the inequality $c(\Psi, \mathscr{F}) \le \omega(\psi, \mathscr{F}^\perp)$.

Let us verify the reverse inequality. As was already noted, the constant $c(\Psi, \mathscr{F})$ can be regarded as the norm of the functional (8) on the subspace $\mathscr{F} \subset L_1^v$. By the Hahn–Banach theorem, this functional can be extended to some functional $\overline{\Psi}$ on the whole space $L_1^v(E)$ with the norm preserved. The functional $\overline{\Psi}$ has the representation

$$\overline{\Psi} f = \int_E \overline{\varphi}(t) f(t) v(t)\,dt, \qquad f \in L_1^v(E), \tag{16}$$

in which $\overline{\varphi} \in L_\infty(E)$; moreover,

$$c(\Psi, \mathscr{F}) = \|\overline{\Psi}\|_{(L_1^v(E))^*} = \|\overline{\varphi}\|_{L_\infty(E)}. \tag{17}$$

Since $f \in \mathscr{F}$ satisfies the relation $\Psi f = \overline{\Psi} f$, it follows from (8) and (16) that

$$\int_E (\psi(t) - \overline{\varphi}(t)) f(t) v(t)\,dt = 0, \qquad f \in \mathscr{F};$$

this implies that the function $\varphi^* = \psi - \overline{\varphi}$ is orthogonal to the space $\mathscr{F}$. Thus, by (17) we have $c(\Psi, \mathscr{F}) = \|\psi - \varphi^*\|_{L_\infty(E)}$, where $\varphi^* \in \mathscr{F}^\perp$. In turn, this implies the inequality $\omega(\psi, \mathscr{F}^\perp) \le c(\Psi, \mathscr{F})$. Thus, the assertion (12) is proved.

Simultaneously with (12), we have proved the existence of an extremal function φ^* in problem (11). If problem (9) is finite-dimensional, then, as is well known, there exists an extremal function $f^* \in \mathscr{F}$ in this problem.

It remains to prove the third assertion of the lemma. Suppose that the functions $\widetilde{f} \in \mathscr{F}$ and $\widetilde{\varphi} \in \mathscr{F}^\perp$ are extremal in problems (9) and (11); this implies that the following relations hold:

$$|\Psi \widetilde{f}| = c(\Psi, \mathscr{F}) \|\widetilde{f}\|_{L_1^v(E)}, \qquad \|\psi - \widetilde{\varphi}\|_{L_\infty(E)} = \omega(\psi, \mathscr{F}^\perp).$$

We can assume that $\Psi \widetilde{f} \ge 0$ (if this were not so, then, instead of the function $\widetilde{f}$, we would take the function $(-\widetilde{f})$, which is also extremal in (9); moreover, $\Psi(-\widetilde{f}) = -\Psi \widetilde{f} \ge 0$). Set $\overline{\varphi} = \psi - \widetilde{\varphi}$. By (15), we have

$$\begin{aligned}\Psi \widetilde{f} &= \int_E (\psi(t) - \widetilde{\varphi}(t)) \widetilde{f}(t) v(t)\,dt \\ &\le \|\psi - \widetilde{\varphi}\|_{L_\infty(E)} \|\widetilde{f}\|_{L_1^v(E)} = \omega(\psi, \mathscr{F}^\perp) \|\widetilde{f}\|_{L_1^v(E)} = c(\Psi, \mathscr{F}) \|\widetilde{f}\|_{L_1^v(E)}.\end{aligned}$$

By the assumption $\Psi \widetilde{f} = c(\Psi, \mathscr{F}) \|\widetilde{f}\|_{L_1^v(E)}$, the last inequality becomes an equality. Therefore, the relation $\overline{\varphi}(t) = c(\Psi, \mathscr{F}) \operatorname{sign} \widetilde{f}(t)$ holds almost everywhere on the set $E(\widetilde{f} \ne 0)$, i.e., property (14) with the constant $A = c(\Psi, \mathscr{F})$ is valid. Inequality (13) is obvious.

Let us prove the converse statement. Suppose that the functions $\widetilde{f} \in \mathscr{F}$, $\widetilde{\varphi} \in \mathscr{F}^\perp$, and the constant A satisfy conditions (13) and (14); moreover, $\widetilde{f} \not\equiv 0$. Then, for any polynomial $f \in \mathscr{F}$, we can write

$$\Psi f = \int_E (\psi(t) - \widetilde{\varphi}(t)) f(t) v(t)\,dt = A \int_E (\operatorname{sign} \widetilde{f}(t)) f(t) v(t)\,dt.$$

This implies the inequality

$$|\Psi f| \le |A| \|f\|_{L_1^v(E)}, \qquad f \in \mathscr{F},$$

which, as is seen from the proof, becomes an equality on the function $\widetilde{f}$. Since $\widetilde{f} \not\equiv 0$, we have $|A| = c(\Psi, \mathscr{F})$ and the function $\widetilde{f}$ is extremal. Further, by the assumptions (13), (14), we have

$$\|\psi - \widetilde{\varphi}\|_{L_\infty(E)} \le |A| = c(\Psi, \mathscr{F}),$$

and hence we see that the function $\widetilde{\varphi}$ is extremal in problem (11). The lemma is proved. □

Remark. The equality $c(\Psi, \mathscr{F}) = 0$ or, equivalently, the equality $\omega(\psi, \mathscr{F}^\perp) = 0$ holds if and only if $\psi \in \mathscr{F}^\perp$.

Indeed, suppose that $\psi \in \mathscr{F}^\perp$. Then

$$\Psi f = \int_E \psi(t) f(t) v(t)\, dt = 0, \qquad f \in \mathscr{F}.$$

Therefore, $c(\Psi, \mathscr{F}) = 0$ as the norm of this functional. Conversely, suppose that $c(\Psi, \mathscr{F}) = 0$. Then, by (12), we also have

$$\omega(\psi, \mathscr{F}^\perp) = \inf\{\|\psi - \varphi\|_{L_\infty(E)} : \varphi \in \mathscr{F}^\perp\} = 0.$$

Therefore, the function ψ almost everywhere coincides with the extremal function $\varphi^* \in \mathscr{F}^\perp$ of problem (11), i.e., $\psi \in \mathscr{F}^\perp$.

2.2. Problem (8), (9) on the unit sphere $\mathbb{S}^{m-1}$ in the space $L = L_1(\mathbb{S}^{m-1})$ with unit weight $v \equiv 1$ on the set $\mathscr{F} = \mathscr{P}_{n,m}$ is of interest to us. Suppose that ψ is a measurable bounded function on the sphere, i.e., $\psi \in L_\infty = L_\infty(\mathbb{S}^{m-1})$. On the set $\mathscr{P}_{n,m}$ of algebraic polynomials, we define the linear functional $F = F(\psi)$ by the formula

$$F(\psi) P_n = \int_{\mathbb{S}^{m-1}} \psi(x) P_n(x)\, dx, \qquad P_n \in \mathscr{P}_{n,m}. \tag{18}$$

Further, we denote by $c_{n,m}(\psi)$ the least constant in the inequality

$$|F(\psi) P_n| \le c_{n,m}(\psi) \|P_n\|_{L(\mathbb{S}^{m-1})}, \qquad P_n \in \mathscr{P}_{n,m}; \tag{19}$$

thus, $c_{n,m}(\psi)$ is the norm of the functional (18) on the space $\mathscr{P}_{n,m}$ with the norm of the space $L(\mathbb{S}^{m-1})$.

Suppose that $\mathscr{P}^\perp = \mathscr{P}^\perp_{n,m}$ is the subspace of functions from $L_\infty(\mathbb{S}^{m-1})$ orthogonal to the space of polynomials $\mathscr{P}_{n,m}$, i.e., the set of all functions $\varphi \in L_\infty(\mathbb{S}^{m-1})$ possessing the property

$$\int_{\mathbb{S}^{m-1}} \varphi(t) P(t)\, dt = 0, \qquad P \in \mathscr{P}_{n,m}. \tag{20}$$

Consider the best approximation

$$\omega_{n,m}(\psi) = \omega(\psi, \mathscr{P}^\perp_{n,m}) = \inf\{\|\psi - \varphi\|_{L_\infty(\mathbb{S}^{m-1})} : \varphi \in \mathscr{P}^\perp_{n,m}\} \tag{21}$$

in $L_\infty(\mathbb{S}^{m-1})$ of the function ψ by the subspace $\mathscr{P}^\perp = \mathscr{P}^\perp_{n,m}$. By Lemma 1, we can, in particular, establish the relation

$$c_{n,m}(\psi) = \omega(\psi, \mathscr{P}^\perp_{n,m}).$$

Note that problem (2)–(4) is a particular case of problem (18), (19). Namely, in the case under consideration, the function $\psi = \psi_h$ coincides with the characteristic function χ_h of the spherical cap $\mathbb{C}(h)$ defined in (6).

3. REDUCTION TO ONE-DIMENSIONAL PROBLEMS

In this section, for $m \ge 3$ (and certain conditions on the function ψ), problem (18) (19) is reduced to a one-dimensional problem on the closed interval. Below, in our arguments, we often use the well-known formula (see, for example, [6, Chap. 18, Sec. 5, item 676])

$$|\mathbb{S}^{m-1}(r)| = |\mathbb{S}^{m-1}|r^{m-1}.$$

For convenience, we split subsequent arguments into several sections.

3.1. On the unit sphere $\mathbb{S}^{m-1}$, consider the polar coordinate system

$$\begin{aligned} x_1 &= \sin\theta_{m-1}\cdots\sin\theta_2\sin\theta_1,\\ x_2 &= \sin\theta_{m-1}\cdots\sin\theta_2\cos\theta_1,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ x_{m-1} &= \sin\theta_{m-1}\cos\theta_{m-2},\\ x_m &= \cos\theta_{m-1}; \end{aligned} \tag{22}$$

$$0 \le \theta_1 < 2\pi, \qquad 0 \le \theta_k \le \pi, \quad 2 \le k \le m-1. \tag{23}$$

Write the parametrization (22) (23) in the form

$$x = X(\Theta) = X_m(\Theta) = (x_1(\Theta), \dots, x_m(\Theta)),$$
$$\Theta = \Theta_{m-1} = (\theta_1, \dots, \theta_{m-1}) \in \Pi^{m-1} = [0, 2\pi) \times [0, \pi]^{m-2};$$

here by $x_k(\Theta)$ we denote the right-hand side of the kth row in (22).

Throughout this section, we assume that $m \ge 3$. Using the polar change of variables (22), (23), we can express the integral of a function $f \in L(\mathbb{S}^{m-1})$ over the unit sphere as (see, for example, [7, Chap. 9, Sec. 1])

$$\int_{\mathbb{S}^{m-1}} f(x)\,dx = \int_{\Pi^{m-1}} f(X(\Theta))J_{m-1}(\Theta)\,d\Theta, \tag{24}$$

$$J_{m-1}(\Theta) = \sin^{m-2}\theta_{m-1}\sin^{m-3}\theta_{m-2}\cdots\sin\theta_2. \tag{25}$$

More generally, for any $r > 0$, the function $f \in L(\mathbb{S}^{m-1}(r))$ satisfies the formula

$$\int_{\mathbb{S}^{m-1}(r)} f(x)\,dx = r^{m-1}\int_{\mathbb{S}^{m-1}} f(rx)\,dx = r^{m-1}\int_{\Pi^{m-1}} f(rX(\Theta))J_{m-1}(\Theta)\,d\Theta. \tag{26}$$

We write the right integral in (24) as an iterated one:

$$\int_{\Pi^{m-1}} f(X(\Theta))J_{m-1}(\Theta)\,d\Theta = \int_0^\pi I(\theta_{m-1})\,d\theta_{m-1},$$
$$I(\theta_{m-1}) = \int_{\Pi^{m-2}} f(X(\Theta', \theta_{m-1}))J_{m-1}(\Theta', \theta_{m-1})\,d\Theta';$$

here $\Theta' = \Theta_{m-2} = (\theta_1, \dots, \theta_{m-2}) \in \Pi^{m-2}$, and hence $\Theta = (\Theta', \theta_{m-1})$. Let us transform the integral $I(\theta_{m-1})$. By formulas (22), the point

$$x = X_m(\Theta) = X_m(\Theta', \theta_{m-1}) \in \mathbb{S}^{m-1}$$

can be expressed as

$$x = (\sin\theta_{m-1}\,x', \cos\theta_{m-1}), \qquad x' = X_{m-1}(\Theta') \in \mathbb{S}^{m-2}.$$

In addition, it follows from (25) that

$$J_{m-1}(\Theta) = \sin^{m-2}\theta_{m-1}J_{m-2}(\Theta').$$

Therefore,

$$I(\theta_{m-1}) = \sin^{m-2}\theta_{m-1}\int_{\Pi^{m-2}} f(\sin\theta_{m-1}\cdot X_{m-1}(\Theta'), \cos\theta_{m-1}))J_{m-2}(\Theta')\,d\Theta'.$$

Further, by (24) and (26), we have

$$I(\theta_{m-1}) = \sin^{m-2}\theta_{m-1}\int_{\mathbb{S}^{m-2}} f(\sin\theta_{m-1}\cdot x', \cos\theta_{m-1})\,dx'.$$

Thus, the integral over the unit sphere satisfies the formula

$$\int_{\mathbb{S}^{m-1}} f(x)\,dx = \int_0^\pi \sin^{m-2}\theta \int_{\mathbb{S}^{m-2}} f(\sin\theta\cdot x', \cos\theta)\,dx'\,d\theta. \tag{27}$$

Let us replace $p=\cos\theta$ in the outer integral on the right-hand side of (27). Then

$$\sin\theta = \sqrt{1-p^2}, \qquad \theta = \arccos p, \qquad d\theta = -\frac{dp}{\sqrt{1-p^2}}.$$

As a result, we obtain

$$\int_{\mathbb{S}^{m-1}} f(x)\,dx = \int_{-1}^1 (1-p^2)^{(m-3)/2}\int_{\mathbb{S}^{m-2}} f(\sqrt{1-p^2}\,x', p)\,dx'\,dp. \tag{28}$$

Write the last formula as

$$\int_{\mathbb{S}^{m-1}} f(x)\,dx = |\mathbb{S}^{m-2}|\int_{-1}^1 g(p)(1-p^2)^{(m-3)/2}\,dp, \tag{29}$$

where

$$g(p) = \frac{1}{|\mathbb{S}^{m-2}|}\int_{\mathbb{S}^{m-2}} f(\sqrt{1-p^2}\,x', p)\,dx'. \tag{30}$$

For a real number p, denote by $\Lambda(p)$ the hyperplane of points $x=(x_1,\dots,x_{m-1},p)\in\mathbb{R}^m$; we shall write the points $x=(x_1,\dots,x_{m-1},p)\in\Lambda(p)$ in the form

$$x = (x_1,\dots,x_{m-1},p) = (\widetilde{x}, p), \qquad \widetilde{x} = (x_1,\dots,x_{m-1})\in\mathbb{R}^{m-1}.$$

For $p\in(-1,1)$, the section of the sphere $\mathbb{S}^{m-1}$ by the hyperplane $\Lambda(p)$ is the $(m-2)$-dimensional sphere of radius $a=a(p)=\sqrt{1-p^2}$ centered at the point pe_m, parallel to the coordinate space $\mathbb{R}^{m-1}$ of points $\widetilde{x}=(x_1,\dots,x_{m-1})$; we identify this sphere with the sphere $\mathbb{S}^{m-2}(a)\subset\mathbb{R}^{m-1}$. The function (30) can be regarded as the averaging $g=Sf$ of the function f over the sections of the sphere by the hyperplanes. The averaging operator S defined by (30) is a linear bounded operator from the space $L(\mathbb{S}^{m-1})$ to the space $L_1^\phi(-1,1)$ with ultraspheric weight

$$\phi(t) = \phi^{(\alpha)}(t) = (1-t^2)^\alpha, \qquad \alpha = \alpha(m) = \frac{m-3}{2}. \tag{31}$$

Moreover, by (29), (30), we have

$$\begin{aligned}
|\mathbb{S}^{m-2}|\cdot\|Sf\|_{L_1^\phi(-1,1)} &= |\mathbb{S}^{m-2}|\cdot\|g\|_{L_1^\phi(-1,1)} \\
&= |\mathbb{S}^{m-2}|\cdot\int_{-1}^1 |g(p)|(1-p^2)^{(m-3)/2}\,dp \\
&\overset{(30)}{=} |\mathbb{S}^{m-2}|\int_{-1}^1\left|\frac{1}{|\mathbb{S}^{m-2}|}\int_{\mathbb{S}^{m-2}} f(\sqrt{1-p^2}\,x', p)\,dx'\right|(1-p^2)^{(m-3)/2}\,dp \\
&\le \int_{-1}^1 (1-p^2)^{(m-3)/2}\int_{\mathbb{S}^{m-2}} |f(\sqrt{1-p^2}\,x', p)|\,dx'\,dp \\
&\overset{(29)}{=} \int_{\mathbb{S}^{m-1}} |f(x)|\,dx = \|f\|_{L(\mathbb{S}^{m-1})}.
\end{aligned}$$

Thus, the following inequality holds:

$$|\mathbb{S}^{m-2}|\cdot\|Sf\|_{L_1^\phi(-1,1)} \le \|f\|_{L(\mathbb{S}^{m-1})}, \qquad f\in L(\mathbb{S}^{m-1}). \tag{32}$$

Note that, at least in the following two cases,

1) the function f is nonnegative;

2) the function f is zonal and, more precisely, it depends only on the variable x_m,

the last inequality is, in fact, the equality. Indeed, if f is nonnegative, then the only inequality in the chain of relations given above becomes an equality. But if the function f is zonal, then $Sf = f$, i.e., $f(x) = f(x_m) = g(p)$ and, obviously inequality (32) becomes an equality.

3.2. Let us verify that, for an algebraic polynomial $P_n \in \mathscr{P}_{n,m}$ of degree n in m variables, the function

$$g_n(p) = (SP_n)(p) = \frac{1}{|\mathbb{S}^{m-2}|}\int_{\mathbb{S}^{m-2}} P_n(x'\sqrt{1-p^2}, p)\, dx' \tag{33}$$

is an algebraic polynomial in one variable of the same degree n. We substitute (1) into (33), obtaining

$$\begin{aligned} g_n(p) &= \frac{1}{|\mathbb{S}^{m-2}|}\int_{\mathbb{S}^{m-2}} \sum c_\alpha (x_1\sqrt{1-p^2})^{\alpha_1}\cdots(x_{m-1}\sqrt{1-p^2})^{\alpha_{m-1}} p^{\alpha_m}\, dx' \\ &= \sum c_\alpha (\sqrt{1-p^2})^{\alpha_1+\dots+\alpha_{m-1}} p^{\alpha_m}\cdot \frac{1}{|\mathbb{S}^{m-2}|}\int_{\mathbb{S}^{m-2}} x_1^{\alpha_1}\cdots x_{m-1}^{\alpha_{m-1}}\, dx'. \end{aligned}$$

Thus, we have obtained the following representation for the function g_n:

$$g_n(p) = \sum c_\alpha \eta_{\alpha'} (\sqrt{1-p^2})^{|\alpha'|} p^{\alpha_m}; \tag{34}$$

here

$$\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_+^m, \qquad \alpha' = (\alpha_1, \dots, \alpha_{m-1}) \in \mathbb{Z}_+^{m-1}, \quad |\alpha'| = \sum_{k=1}^{m-1} \alpha_k,$$

$$\eta_{\alpha'} = \frac{1}{|\mathbb{S}^{m-2}|}\int_{\mathbb{S}^{m-2}} (x')^{\alpha'}\, dx'.$$

Replacing the variable x' by $-x'$ in the last integral, we obtain the relation

$$\eta_{\alpha'} = (-1)^{|\alpha'|}\eta_{\alpha'}.$$

Hence $\eta_{\alpha'} = 0$ if the number $|\alpha'|$ is odd. Thus, formula (34) contains only summands whose number $|\alpha'|$ is even. Now it is seen that the function g_n is, indeed, a polynomial (in the variable p) of degree n. Thus, $S\mathscr{P}_{n,m} = \mathscr{P}_n$, $\mathscr{P}_n = \mathscr{P}_{n,1}$.

3.3. Suppose that ζ is a bounded measurable function on the closed interval $[-1, 1]$. In the space of polynomials in one variable $\mathscr{P}_n = \mathscr{P}_{n,1}$, consider the linear functional

$$\Phi g = \int_{-1}^{1} \zeta(t) g(t) \phi(t)\, dt, \qquad g \in \mathscr{P}_n, \tag{35}$$

where ϕ is the ultraspheric weight (31). Denote by $C_{n,m}(\zeta)$ the best constant in the inequality

$$|\Phi g| \le C_{n,m}(\zeta)\|g\|_{L_1^\phi(-1,1)}, \qquad g \in \mathscr{P}_n; \tag{36}$$

the constant $C_{n,m}(\zeta)$ can be regarded as the norm of the functional (35) on the subspace $\mathscr{P}_n$ of the space $L_1^\phi(-1, 1)$ with weight (31).

Lemma 2. *Suppose that $m \ge 3$ and the function ψ in problem* (18), (19) *is zonal; more precisely, let $\psi(x) = \psi(x_1, \dots, x_m) = \zeta(x_m)$, where ζ is a bounded measurable function on the closed interval $[-1, 1]$. Then, for all $n \ge 0$, the best constants in inequalities* (19) *and* (36) *satisfy*

$$c_{n,m}(\psi) = C_{n,m}(\zeta). \tag{37}$$

Proof. Consider a polynomial $P_n \in \mathscr{P}_{n,m}$, and let $g_n \in \mathscr{P}_n$ be a polynomial in one variable constructed from formula (33). By (36), the polynomial g_n satisfies the inequality

$$|\Phi g_n| \le C_{n,m}(\zeta)\|g_n\|_{L_1^\phi(-1,1)}. \tag{38}$$

By formulas (35), (33), and (28)–(30), we have

$$\begin{aligned}|\mathbb{S}^{m-2}|\Phi g_n &\overset{(35)}{=} |\mathbb{S}^{m-2}|\int_{-1}^{1}\zeta(p)g_n(p)(1-p^2)^{(m-3)/2}\,dp\\ &\overset{(33)}{=} |\mathbb{S}^{m-2}|\int_{-1}^{1}\zeta(p)(1-p^2)^{(m-3)/2}\frac{1}{|\mathbb{S}^{m-2}|}\int_{\mathbb{S}^{m-2}}P_n(x'\sqrt{1-p^2},p)\,dx'\,dp\\ &\overset{(28)}{=} \int_{\mathbb{S}^{m-1}}\zeta(x_m)P_n(x)\,dx \overset{(18)}{=} F(\zeta)P_n,\end{aligned} \tag{39}$$

$$\begin{aligned}|\mathbb{S}^{m-2}|\|g_n\|_{L_1^\phi(-1,1)} &= |\mathbb{S}^{m-2}|\int_{-1}^{1}|g_n(p)|(1-p^2)^{(m-3)/2}\,dp\\ &\overset{(33)}{\le} |\mathbb{S}^{m-2}|\int_{-1}^{1}\frac{1}{|\mathbb{S}^{m-2}|}\int_{\mathbb{S}^{m-2}}|P_n(x'\sqrt{1-p^2},p)|\,dx'\,(1-p^2)^{(m-3)/2}\,dp\\ &\overset{(29),(30)}{=} \int_{\mathbb{S}^{m-1}}|P_n(x)|\,dx = \|P_n\|_{L(\mathbb{S}^{m-1})}.\end{aligned} \tag{40}$$

This implies that the functional (18) with the function $\psi(x)=\zeta(x_m)$ satisfies the inequality

$$|F(\zeta)P_n| \le C_{n,m}(\zeta)\|P_n\|_{L(\mathbb{S}^{m-1})}, \qquad P_n \in \mathscr{P}_{n,m}. \tag{41}$$

Indeed, from (39) we obtain

$$|F(\zeta)P_n| = |\mathbb{S}^{m-2}||\Phi g_n| \overset{(38)}{\le} |\mathbb{S}^{m-2}|C_{n,m}(\zeta)\|g_n\|_{L_1^\phi(-1,1)} \overset{(40)}{\le} C_{n,m}(\zeta)\|P_n\|_{L(\mathbb{S}^{m-1})}.$$

Comparing (41) with (19), we find that $c_{n,m}(\psi) \le C_{n,m}(\zeta)$, because $c_{n,m}(\psi)$ is the least constant in such an inequality.

Let us verify the reverse inequality. Suppose that $g_n \in \mathscr{P}_n$ is a polynomial in one variable of degree n. Assume it to be a polynomial in m variables: $G_n(x)=G_n(x_1,\dots,x_m)=g_n(x_m)$. By (18) and (28), we have

$$F(\zeta)G_n = \int_{\mathbb{S}^{m-1}}\zeta(x_m)g_n(x_m)\,dx = \int_{-1}^{1}(1-p^2)^{(m-3)/2}\biggl(\int_{\mathbb{S}^{m-2}}\zeta(p)g_n(p)\,dx'\biggr)\,dp,$$

and since the function $\zeta(p)g_n(p)$ is independent of x', in view of (35), we have

$$F(\zeta)G_n = |\mathbb{S}^{m-2}|\int_{-1}^{1}(1-p^2)^{(m-3)/2}\zeta(p)g_n(p)\,dp = |\mathbb{S}^{m-2}|\Phi g_n. \tag{42}$$

In addition,

$$\begin{aligned}\|G_n\|_{L(\mathbb{S}^{m-1})} &= \int_{\mathbb{S}^{m-1}}|G_n(x)|\,dx = \int_{\mathbb{S}^{m-1}}|g_n(x_m)|\,dx\\ &\overset{(28)}{=} \int_{-1}^{1}(1-p^2)^{(m-3)/2}\biggl(\int_{\mathbb{S}^{m-2}}|g_n(p)|\,dx'\biggr)\,dp,\end{aligned}$$

and since the function $|g_n(p)|$ is independent of x', we finally have

$$\|G_n\|_{L(\mathbb{S}^{m-1})} = |\mathbb{S}^{m-2}|\int_{-1}^{1}(1-p^2)^{(m-3)/2}|g_n(p)|\,dp = |\mathbb{S}^{m-2}|\|g_n\|_{L_1^\phi(-1,1)}. \tag{43}$$

Substituting the polynomial G_n into (19), we find that

$$|\Phi g_n|\cdot|\mathbb{S}^{m-2}| \overset{(42)}{=} |F(\zeta)G_n| \overset{(19)}{\le} c_{n,m}(\psi)\|G_n\|_{L(\mathbb{S}^{m-1})} \overset{(43)}{=} c_{n,m}(\psi)|\mathbb{S}^{m-2}|\cdot\|g_n\|_{L_1^\phi(-1,1)}.$$

As a result, we obtain the inequality

$$|\Phi g_n| \le c_{n,m}(\psi)\|g_n\|_{L_1^\phi(-1,1)}, \qquad g_n \in \mathscr{P}_n.$$

Comparing the last inequality with (36), we find that $C_{n,m}(\zeta) \le c_{n,m}(\psi)$, because $C_{n,m}(\zeta)$ is the least constant in such an inequality. Thus, relation (37) is verified. The lemma is proved. □

Inequality (36) is a particular case of inequality (9). Consider the corresponding dual problem (12). Denote by $\mathscr{P}_n^\perp = \mathscr{P}_n^\perp(\phi)$ the set of functions $\varphi \in L_\infty(-1,1)$ orthogonal to the space of polynomials $\mathscr{P}_n$ with ultraspheric weight (31), i.e., possessing the property

$$\int_{-1}^{1} \varphi(t) g_n(t)(1-t^2)^{(m-3)/2}\, dt = 0, \qquad g_n \in \mathscr{P}_n. \tag{44}$$

Consider the value of the best approximation

$$\Omega_{n,m}(\zeta) = \Omega(\zeta, \mathscr{P}_n^\perp(\phi)) = \inf\{\|\zeta - \varphi\|_{L_\infty(-1,1)} : \varphi \in \mathscr{P}_n^\perp(\phi)\} \tag{45}$$

to the function ζ by the subspace $\mathscr{P}_n^\perp(\phi) \subset L_\infty(-1,1)$ in $L_\infty(-1,1)$.

Lemma 3. *Suppose that $m \ge 3$ and the function ψ in problem (18), (19) is zonal; more precisely, let $\psi(x) = \psi(x_1, \dots, x_m) = \zeta(x_m)$, where ζ is a bounded measurable function on the closed interval $[-1,1]$. Then, for all $n \ge 0$, the deviation values (21) and (45) satisfy*

$$\omega_{n,m}(\psi) = \Omega_{n,m}(\zeta). \tag{46}$$

Proof. Suppose that $\varphi \in \mathscr{P}_{n,m}^\perp$ and z is a function (of one variable $p \in [-1,1]$) constructed for the function φ from formula (33), i.e.,

$$z(p) = \frac{1}{|\mathbb{S}^{m-2}|}\int_{\mathbb{S}^{m-2}} \varphi(x'\sqrt{1-p^2}, p)\, dx'.$$

Let us verify that the function z satisfies the orthogonality condition (44), i.e., $z \in \mathscr{P}_n^\perp(\phi)$. For a polynomial $g_n \in \mathscr{P}_n$, using formula (29), we obtain

$$|\mathbb{S}^{m-2}|\int_{-1}^{1} z(t) g_n(t)(1-t^2)^{(m-3)/2}\, dt = \int_{\mathbb{S}^{m-1}} g_n(x_m)\varphi(x)\, dx.$$

The last integral is zero by the orthogonality condition (20), and hence, indeed, $z \in \mathscr{P}_n^\perp(\phi)$. Thus, we have proved that $S\mathscr{P}_{n,m}^\perp = \mathscr{P}_n^\perp$.

Moreover, we have

$$\zeta(p) - z(p) = \frac{1}{|\mathbb{S}^{m-2}|}\int_{\mathbb{S}^{m-2}} (\zeta(p) - \varphi(x'\sqrt{1-p^2}, p))\, dx'. \tag{47}$$

This implies the estimate

$$\|\zeta - z\|_{L_\infty(-1,1)} \le \|\psi - \varphi\|_{L_\infty(\mathbb{S}^{m-1})}.$$

Indeed, the following chain of relations is valid:

$$\begin{aligned}
\|\zeta - z\|_{L_\infty(-1,1)} &= \operatorname*{ess\,sup}_{p\in(-1,1)} |\zeta(p) - z(p)| \\
&\overset{(47)}{\le} \operatorname*{ess\,sup}_{p\in(-1,1)} \frac{1}{|\mathbb{S}^{m-2}|}\int_{\mathbb{S}^{m-2}} |\zeta(p) - \varphi(x'\sqrt{1-p^2}, p)|\, dx' \\
&\le \operatorname*{ess\,sup}_{p\in(-1,1)} \frac{1}{|\mathbb{S}^{m-2}|}\int_{\mathbb{S}^{m-2}} \operatorname*{ess\,sup}_{x\in\mathbb{S}^{m-1}} |\psi(x) - \varphi(x)|\, dx'.
\end{aligned}$$

Since the integrand is independent of x', we finally obtain the estimate

$$\|\zeta - z\|_{L_\infty(-1,1)} \le \frac{1}{|\mathbb{S}^{m-2}|} \cdot |\mathbb{S}^{m-2}| \cdot \|\psi - \varphi\|_{L_\infty(\mathbb{S}^{m-1})} = \|\psi - \varphi\|_{L_\infty(\mathbb{S}^{m-1})},$$

which implies the inequality $\Omega_{n,m}(\zeta) \le \omega_{n,m}(\psi)$.

Let us verify the reverse inequality. Suppose that $z \in \mathscr{P}_n^\perp(\phi)$. We verify that, in that case, the function $\varphi(x) = z(x_m)$ belongs to the set $\mathscr{P}_{n,m}^\perp$. To an arbitrary polynomial $P_n \in \mathscr{P}_{n,m}$ we assign a polynomial $g_n \in \mathscr{P}_n$ by formula (33). Again, using formulas (29), we obtain

$$\int_{\mathbb{S}^{m-1}} P_n(x) z(x_m)\, dx = |\mathbb{S}^{m-2}| \int_{-1}^{1} z(t) g_n(t)(1-t^2)^{(m-3)/2}\, dt = 0,$$

so that, indeed, $z \in \mathscr{P}_{n,m}^\perp$. Moreover, we have

$$\|\psi - \varphi\|_{L_\infty(\mathbb{S}^{m-1})} = \operatorname*{ess\,sup}_{x \in \mathbb{S}^{m-1}} |\psi(x) - \varphi(x)| = \operatorname*{ess\,sup}_{x_m \in (-1,1)} |\zeta(x_m) - z(x_m)| = \|\zeta - z\|_{L_\infty(-1,1)}.$$

This implies the inequality

$$\omega(\psi, \mathscr{P}_{n,m}^\perp) \le \Omega(\zeta, \mathscr{P}_n^\perp(\phi)).$$

Thus, relation (46) is verified. The lemma is proved. □

Lemma 4. *Suppose that $m \ge 3$ and ζ is a function of one variable bounded and measurable on the closed interval $[-1,1]$. Then, for any $n \ge 0$, the following statements are valid*:

1) *The relation $C_{n,m}(\zeta) = \Omega_{n,m}(\zeta)$ holds.*

2) *In problems* (36) *and* (45)*, there exists an extremal polynomial $f_n^* \in \mathscr{P}_n$ and an extremal function $\varphi_n^* \in \mathscr{P}_n^\perp(\phi)$.*

3) *The polynomial $\widetilde{f}_n \in \mathscr{P}_n$ and the function $\widetilde{\varphi}_n \in \mathscr{P}_n^\perp(\phi)$ are extremal in problems* (36) *and* (45)*, respectively, if and only if there exists a constant A such that the relation*

$$\zeta - \widetilde{\varphi}_n = A \operatorname{sign} \widetilde{f}_n$$

holds almost everywhere on $(-1,1)$; moreover, we must have $|A| = C_{n,m}(\zeta) = \Omega_{n,m}(\zeta)$.

This statement is a particular case of Lemma 1 and is given here only for convenience in view of subsequent references.

4. APPROXIMATION PROBLEM

4.1. Problems (8), (9), and (11) are closely related to the problem of the best approximation

$$e(\psi, \mathscr{F}) = e(\psi, \mathscr{F})_{L(E)} = \inf\{\|\psi - f\|_{L(E)} : f \in \mathscr{F}\}$$

of a functions ψ by the subspace $\mathscr{F}$ itself in the space $L(E)$. Let us discuss these questions with regard to the approximation of algebraic polynomials on the sphere and in the corresponding one-dimensional case.

Consider the best approximation

$$e_{n,m}(\psi) = \inf\{\|\psi - P_n\|_{L(\mathbb{S}^{m-1})} : P_n \in \mathscr{P}_{n,m}\} \tag{48}$$

in $L(\mathbb{S}^{m-1})$ to the function ψ by the subspace $\mathscr{P}_{n,m}$ and the best approximation

$$E_n^\phi(\zeta) = \inf\{\|\zeta - g_n\|_{L_1^\phi(-1,1)} : g_n \in \mathscr{P}_n\} \tag{49}$$

in $L_1^\phi(-1,1)$ to the function ζ by the subspace $\mathscr{P}_n$. The following statement is valid.

Lemma 5. *Suppose that $m \geq 3$ and the function ψ in problem* (18), (19) *is zonal; more precisely, let $\psi(x) = \psi(x_1, \dots, x_m) = \zeta(x_m)$, where ζ is a function (of one variable) from the space $L_1^{\phi}(-1,1)$. Then, for all $n \geq 0$, the deviation values* (48) *and* (49) *are related by*

$$e_{n,m}(\psi) = |\mathbb{S}^{m-2}| E_n^{\phi}(\zeta). \tag{50}$$

Proof. Suppose that $P_n \in \mathscr{P}_{n,m}$ and g_n is a function (of one variable) constructed from formula (33) so that

$$g_n(p) = \frac{1}{|\mathbb{S}^{m-2}|} \int_{\mathbb{S}^{m-2}} P_n(x'\sqrt{1-p^2}, p)\, dx'.$$

As established above, the function g_n is a polynomial in one variable of degree n, i.e., $g_n \in \mathscr{P}_n$. Moreover, we have

$$\zeta(p) - g_n(p) = \frac{1}{|\mathbb{S}^{m-2}|} \int_{\mathbb{S}^{m-2}} (\zeta(p) - P_n(x'\sqrt{1-p^2}, p))\, dx'.$$

This yields the estimate

$$|\mathbb{S}^{m-2}| \cdot \|\zeta - g_n\|_{L_1^{\phi}(-1,1)} \leq \|\psi - P_n\|_{L(\mathbb{S}^{m-1})}.$$

Indeed, the following chain of relations is valid:

$$\begin{aligned}
\|\zeta - g_n\|_{L_1^{\phi}(-1,1)} &= \int_{-1}^{1} |\zeta(p) - g_n(p)|(1-p^2)^{(m-3)/2}\, dp \\
&\leq \int_{-1}^{1} (1-p^2)^{(m-3)/2} \frac{1}{|\mathbb{S}^{m-2}|} \int_{\mathbb{S}^{m-2}} |\zeta(p) - P_n(x'\sqrt{1-p^2}, p)|\, dx'\, dp \\
&\overset{(29)}{=} \frac{1}{|\mathbb{S}^{m-2}|} \int_{\mathbb{S}^{m-1}} |\psi(x) - P_n(x)|\, dx = \frac{1}{|\mathbb{S}^{m-2}|} \|\psi - P_n\|_{L(\mathbb{S}^{m-1})};
\end{aligned}$$

this implies the inequality

$$|\mathbb{S}^{m-2}| E_n^{\phi}(\zeta) \leq e_{n,m}(\psi).$$

Let us verify the reverse inequality. Suppose that $P_n \in \mathscr{P}_{n,m}$ is a zonal polynomial, i.e.,

$$P_n(x) = P_n(x_m) = P_n(p) = g_n(p).$$

Then, applying (29), we obtain

$$\begin{aligned}
\|\psi - P_n\|_{L(\mathbb{S}^{m-1})} &= \int_{\mathbb{S}^{m-1}} |\psi(x) - P_n(x)|\, dx = \int_{\mathbb{S}^{m-1}} |\zeta(p) - g_n(p)|\, dx \\
&= \int_{-1}^{1} (1-p^2)^{(m-3)/2} \int_{\mathbb{S}^{m-2}} |\zeta(p) - g_n(p)|\, dx'\, dp \\
&= |\mathbb{S}^{m-2}| \int_{-1}^{1} (1-p^2)^{(m-3)/2} |\zeta(p) - g_n(p)|\, dp \\
&= |\mathbb{S}^{m-2}| \|\zeta - g_n\|_{L_1^{\phi}(-1,1)};
\end{aligned}$$

this yields the estimate

$$e_{n,m}(\psi) \leq |\mathbb{S}^{m-2}| E_n^{\phi}(\zeta).$$

Thus, relation (50) is verified. The lemma is proved. □

4.2. Consider problem (49) in greater detail. Just as any approximation problem, it is concerned with the evaluation of a certain quantity, namely, (49), and the determination of the extremal polynomial g_n^* on which the lower bound in (49) is attained. Note that, because the space $\mathscr{P}_n$ is finite-dimensional, there exists an extremal polynomial g_n^*. As to problem (49), the following result is well known (see, for example, [8, Theorem 3.3.2], [9, Chap. 1, Sec. 6] or [10, Chap. 3, Sec. 10]).

Lemma 6. *In order that the polynomial $\overline{g}_n \in \mathscr{P}_n$ be extremal in problem* (49)*, it is sufficient, and if the function ζ and the polynomial $\overline{g}_n$ do not coincide almost everywhere on* $(-1,1)$*, it is also necessary that the following relation hold*:

$$\int_{-1}^{1} f_n(t)\phi(t)\operatorname{sign}(\zeta(t)-\overline{g}_n(t))\,dt = 0, \qquad f_n \in \mathscr{P}_n. \tag{51}$$

Moreover,

$$E_n^{\phi}(\zeta)_{L_1^{\phi}(-1,1)} = \left|\int_{-1}^{1} \zeta(t)\phi(t)\operatorname{sign}(\zeta(t)-\overline{g}_n(t))\,dt\right|. \tag{52}$$

4.3. For the particular function $\zeta(t) = t^{n+1}$, problem (49) is the well-known problem of the polynomial of least deviation from zero. In this case, (49) takes the form

$$E_{n+1}^{\phi} = \inf\{\|g_{n+1}\|_{L_1^{\phi}(-1,1)} : g_{n+1} \in \mathscr{P}_{n+1}^1\}, \tag{53}$$

where $\mathscr{P}_{n+1}^1$ is the set of polynomials of degree $n+1$ with leading coefficient equal to 1. Denote by q_{n+1}^* the extremal polynomial (the solution) of problem (53); this polynomial exists and is unique. All $n+1$ zeros of the polynomial q_{n+1}^* are simple and lie in the interval $(-1,1)$. This fact is well known; nevertheless, we present its proof. Suppose that this not so, i.e., on $(-1,1)$, q_{n+1}^* has only m, $m \le n$, changes of sign at points $\{t_k\}_{k=1}^m$. Consider the polynomial

$$\overline{q}_m(t) = \prod_{k=1}^{m}(t-t_k) \in \mathscr{P}_n.$$

Since the degree of the polynomial $\overline{q}_m$ is at most n, according to Lemma 6 (see property (51)), the following relation holds:

$$\int_{-1}^{1} \overline{q}_m(t)\phi(t)\operatorname{sign} q_{n+1}^*(t)\,dt = 0.$$

However, on the other hand,

$$\int_{-1}^{1} \overline{q}_m(t)\phi(t)\operatorname{sign}\overline{q}_m(t)\,dt = \int_{-1}^{1} |\overline{q}_m(t)|\phi(t)\,dt = \|\overline{q}_m\|_{L_1^{\phi}(-1,1)} > 0.$$

The resulting contradiction proves that all the zeros of the polynomial q_{n+1}^* are simple and lie in the interval $(-1,1)$.

Denote by $t(k,m) = t(k,m,n)$, $0 \le k \le n$, the zeros of the polynomial q_{n+1}^* (on $(-1,1)$) numbered in increasing order of the index k. The weight ϕ is an even function; therefore, the zeros $t(k,m)$, $0 \le k \le n$, are located symmetrically with respect to the point 0, i.e., $t(n-k,m) = -t(k,m)$, $0 \le k \le n$.

4.4. The main goal in this section is the study of the best approximation

$$e_{n,m}(h) = e_{n,m}(\chi_h) = \inf\{\|\chi_h - P_n\|_{L(\mathbb{S}^{m-1})} : P_n \in \mathscr{P}_{n,m}\} \tag{54}$$

in the space $L(\mathbb{S}^{m-1})$ of the characteristic function of the cap $\mathbb{C}(h)$ by the subspace $\mathscr{P}_{n,m}$ of algebraic polynomials. Now this quantity is calculated for the values of the parameter h which are the zeros $t(k,m)$, $0 \le k \le n$, of the just constructed polynomial q_{n+1}^*. However, as a preliminary, consider the corresponding one-dimensional problem.

The function (6) is zonal; namely,

$$\chi_h(x) = \zeta_h(x_m), \quad x = (x_1, \dots, x_m), \qquad \text{where} \quad \zeta_h(t) = \begin{cases} 1, & t \in (h, 1), \\ 0, & t \in (-1, h). \end{cases} \tag{55}$$

Consider the best approximation

$$E_n^\phi(h) = E_n^\phi(\zeta_h) = \inf\{\|\zeta_h - P_n\|_{L_1^\phi(-1,1)} : P_n \in \mathscr{P}_n\} \tag{56}$$

of the "step" ζ_h in the space $L_1^\phi(-1,1)$ by the subspace $\mathscr{P}_n$ of algebraic polynomials (in one variable).

Following the arguments of Taikov's papers [1], [2] and of those of Babenko and Kryakin [11], we can write out the solution of problem (56) for the values of the parameter h coinciding with the zeros $t(k,m)$ of the polynomial q_{n+1}^*; the quantity (56) is an even function of h; therefore, it suffices to do this for $n/2 \le k \le n$. For the values $h = t(k,m)$, the function ζ_h is denoted by $\zeta_{k,m}$; this is a "step function" with a discontinuity at the point $t(k,m)$.

First, consider the case $k = n$. We introduce the algebraic polynomial of degree n,

$$g_n^*(t) = g_{n,m}^*(t) = A\frac{q_{n+1}^*(t)}{t - t(n,m)}, \tag{57}$$

where the constant A is chosen from the condition $g_{n,m}^*(1) = 1$.

Lemma 7. *For all $m \ge 2$, $n \ge 1$, $0 \le \lambda \le 1$, the following relations hold*:

$$E_n^\phi(\zeta_{n,m})_{L_1^\phi(-1,1)} = \int_{-1}^1 \phi(t)|\zeta_{n,m}(t) - \lambda g_n^*(t)|\,dt = \int_{t(n,m)}^1 \phi(t)\,dt. \tag{58}$$

The first equality implies that, for all $0 \le \lambda \le 1$, the polynomial λg_n^* is extremal. The second equality in (58) implies that the function $\zeta_{n,m}$ cannot be approximated by polynomials of degree n.

Proof of Lemma 7. For $0 < \lambda \le 1$, the assertion of Lemma 7 follows from Lemma 6. Indeed, for any $f_n \in \mathscr{P}_n$, we have

$$\int_{-1}^1 f_n(t)\phi(t)\,\mathrm{sign}(\zeta_{n,m}(t) - g_n^*(t))\,dt = \int_{-1}^1 f_n(t)\phi(t)\,\mathrm{sign}\,q_{n+1}^*(t)\,dt$$
$$= \int_{-1}^1 f_n(t)\phi(t)\,\mathrm{sign}(t^{n+1} - \overline{q}_n(t))\,dt \overset{(51)}{=} 0,$$

i.e., according to Lemma 6, the polynomial g_n^* is extremal in the problem of the approximation of the "step function," while since

$$\mathrm{sign}(\zeta_{n,m}(t) - \lambda g_n^*(t)) = \mathrm{sign}(\zeta_{n,m}(t) - g_n^*(t)), \qquad 0 < \lambda \le 1, \quad t \in (-1,1),$$

it follows that λg_n^* is also extremal. Further, from (52) we obtain

$$E_n^\phi(\zeta_{n,m})_{L_1^\phi(-1,1)} = \left|\int_{-1}^1 \zeta_{n,m}(t)\phi(t)\,\mathrm{sign}(\zeta_{n,m}(t) - \lambda g_n^*(t))\,dt\right| = \int_{t(n,m)}^1 \phi(t)\,dt.$$

The case $\lambda = 0$ is obtained from (58) by passing to the limit as $\lambda \to +0$. The lemma is proved. □

Now suppose that $2 \le k \le n-1$. Suppose that $g_{k,m}^*$ is an algebraic polynomial of degree $n-1$ interpolating the function $\zeta_{k,m}$ at the points $t(\nu,m)$, $0 \le \nu \le n$, $\nu \ne k$. Following [11], it is easy to verify that (on $[-1,1]$) the sign of the difference $\zeta_{k,m} - g_{k,m}^*$ coincides with the sign of the polynomial q_{n+1}^*:

$$\mathrm{sign}(\zeta_{k,m} - g_{k,m}^*) = \mathrm{sign}\,q_{n+1}^*.$$

Therefore, the following statement is valid.

Lemma 8. *For all $m \ge 2$, $n \ge 2$, $2 \le k \le n-1$, we have*

$$E_n^\phi(\zeta_{k,m})_{L_1^\phi(-1,1)} = \int_{-1}^1 \phi(t)|\zeta_{k,m}(t) - g_{k,m}^*(t)|\,dt = \left|\int_{t(k,m)}^1 \phi(t)\,\mathrm{sign}\,q_{n+1}^*(t)\,dt\right|.$$

4.5. Now we can write out the solution of problem (54) for the values of h which are the zeros $t(k,m)$, $0 \le k \le n$, of the polynomial q^*_{n+1}. As a consequence of Lemmas 5, 7, 8, we obtain the following statements.

Theorem 1. *For all $m \ge 3$, $n \ge 0$, the values of $h = t(n,m)$ satisfy the following assertions*:

1) *the relation $e_{n,m}(h) = |\mathbb{C}(h)|$ holds*;
2) *for any λ, $0 \le \lambda \le 1$, the polynomial $\lambda g^*_{n,m}$, as a zonal polynomial in one variable $t = x_m$, $x = (x_1, \dots, x_m) \in \mathbb{S}^{m-1}$, is extremal in problem* (54).

Theorem 2. *Suppose that $m \ge 2$, $n \ge 2$, and $2 \le k \le n-1$. Then the values of $h = t(k,m)$ satisfy the following assertions*:

1) *the formulas*

$$e_{n,m}(h) = \left| \int_{\mathbb{C}(h)} \operatorname{sign} q^*_{n+1}(x_m)\, dx \right| = |\mathbb{S}^{m-2}| \left| \int_h^1 (1-t^2)^{(m-3)/2} \operatorname{sign} q^*_{n+1}(t)\, dt \right|$$

are valid;

2) *the polynomial $g^*_{k,m}$ (of degree $n-1$) as a zonal polynomial in one variable*

$$t = x_m, \qquad x = (x_1, \dots, x_m) \in \mathbb{S}^{m-1},$$

is extremal in problem (54).

5. THE NORM OF THE TAIKOV FUNCTIONAL AND THE DUAL PROBLEM

In this section, using the results obtained above, we write out the solution of problem (2)–(4) and of the problem dual to it for $h = t(n,m)$, which is the largest zero of the polynomial q^*_{n+1} (in one variable of degree $n+1$) of least deviation from zero in the space L_1^ϕ.

Let us discuss problems (35), (36), and (45) for the function $\zeta_{n,m}$, i.e., for the function (55) for $h = t(n,m)$. We introduce the notation $C_{n,m} = C_{n,m}(\zeta_{n,m})$, $\Omega_{n,m} = \Omega_{n,m}(\zeta_{n,m})$. By Lemma 6, the function

$$\varphi_{n,m} = \frac{1}{2} \operatorname{sign} q^*_{n+1} \tag{59}$$

is orthogonal with weight ϕ to the space $\mathscr{P}_n$, i.e., $\varphi_{n,m} \in \mathscr{P}_n^\perp(\phi)$. It is easy to see that it possesses the property

$$\zeta_{n,m} - \varphi_{n,m} = \frac{1}{2} \operatorname{sign} g_{n,m}, \qquad \text{where} \quad g_{n,m}(t) = \frac{q^*_{n+1}(t)}{t - t(n,m)}. \tag{60}$$

Hence, by Lemma 4, we can prove the following statement.

Theorem 3. *For all $m \ge 2$, $n \ge 0$ the following assertions are valid*:

1) *the relations $C_{n,m} = \Omega_{n,m} = 1/2$ hold*;
2) *the polynomials $a g_{n,m}$, $a \in \mathbb{R}$, and the function $\varphi_{n,m}$ defined by relations* (57) *and* (59) *are extremal in problems* (36) *and* (45)*, respectively, for the function $\zeta_{n,m}$*.

Using the results given above (Lemmas 2, 3, and Theorem 3), we can write out the solution of Taikov's problem (4) for the value of $h = t(n,m)$ and also the solution of problem (21) for the function $\chi_{n,m} = \chi_h$, $h = t(n,m)$.

Theorem 4. *For all $m \ge 3$, $n \ge 0$ the following assertions are valid*:

1) *the relations $\nu_{n,m}(t(n,m)) = \omega_{n,m}(\chi_{n,m}) = 1/2$ hold*;
2) *the polynomials $a g_{n,m}$, $a \in \mathbb{R}$, and the function $\varphi_{n,m}$ defined by relations* (57) *and* (59) *as zonal functions of one variable $t = x_m$, $x = (x_1, \dots, x_m) \in \mathbb{S}^{m-1}$ are extremal in problems* (4) *and* (21)*, respectively, for the function $\chi_{n,m}$*.

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