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JACKSON TYPE THEOREMS FOR COMPACT RANK 1 SYMMETRIC SPACES

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§ 1. Introduction and Statement of the Main Results

Since recently various problems of approximation theory have been actively studied for functions on the n-dimensional sphere S^n (see [1–3] and the bibliography therein). A broader but natural class of spaces for which we can pose such problems is that of all compact rank 1 symmetric spaces (CROSP in the terminology of the monograph [4]). Some results are already available for these spaces (see [5–9]); however, the main problems still remain open. This article is devoted to proving the direct Jackson-type theorems for an arbitrary CROSP M.

The complete classification of all CROSPs is well known. It comprises the four series: the spheres S^n ($n = 1, 2, \dots$) and the real, complex, and quaternionic projective spaces ($P^n(\mathbb{R})$, $n = 2, 3, \dots$; $P^n(\mathbb{C})$, $n = 4, 6, \dots$; and $P^n(\mathbb{H})$, $n = 8, 12, 16, \dots$) (everywhere the superscript n denotes the dimension of the space) and one special space, the Cayley elliptic plane $P^{16}(Cay)$. Since the problems of harmonic analysis on $P^n(\mathbb{R})$ are easily reduced to the corresponding problems on S^n, we assume that $M \neq P^n(\mathbb{R})$.

A CROSP M is always a Riemannian manifold. Let Δ be the Laplace–Beltrami operator on M. The spectrum of Δ is discrete, real, and nonpositive. We arrange it in decreasing order ($0 = \lambda_0 > \lambda_1 > \lambda_2 > \dots$) and denote the eigensubspace of Δ corresponding to the eigenvalue λ_k by $\mathscr{H}_k$ (it is always finite-dimensional). Put $\mathscr{P}_N(M) := \mathscr{H}_0 + \mathscr{H}_1 + \dots + \mathscr{H}_N$. The functions in $\mathscr{P}_N(M)$ are referred to as *spherical polynomials* on M of degree m (for $M = S^n$ they coincide with the ordinary spherical polynomials).

Given a set X with a measure $d\sigma$, we as usual denote by $L_p(X, d\sigma)$ the Banach space of complex-valued measurable functions $f(x)$ on X with the finite norm

$$\|f\|_p = \|f\|_{L_p(X)} := \left(\int_X |f(x)|^p d\sigma \right)^{1/p}, \quad 1 \le p < \infty.$$

If X is a compact topological space then we let $L_\infty(X) = C(X)$ stand for the space of continuous functions on X with the norm

$$\|f\|_\infty = \|f\|_C := \max_{x \in X} |f(x)|.$$

In particular, the Banach spaces $L_p(M) = L_p(M, dx)$ and $L_\infty(M) = C(M)$ are defined whenever M is a CROSP; in this case dx is the Riemannian measure on M.

The *best approximation* of a function $f(x) \in L_p(M)$ by spherical polynomials of degree N in the L_p metric is the number

$$E_N(f)_p := \inf_{\Phi \in \mathscr{P}_N} \|f - \Phi\|_p.$$

We let T_xM stand for the set of tangent vectors to M at a point x; $S(x)$, the set of the unit tangent vectors (the unit sphere) at x; and U, the isometry group of the CROSP M.

Let B be the manifold of unit tangent vectors to M. The points of B are the pairs (x, ξ), where $x \in M$ and $\xi \in S(x)$. The group U acts naturally on B by $u(x, \xi) = (ux, u_*\xi)$, where u_* is the induced

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mapping of tangent vectors. This action is transitive (i.e., M is isotropic; see [10, Chapter I, § 4; 11, Chapter IX, § 5]). Therefore, there is a U-invariant measure $dv(x,\xi)$ on B which is unique up to multiplication by a number (see, for instance, [10, Chapter I, Theorem 1.9]). Put $L_p(B) := L_p(B, dv)$, $1 \le p < \infty$, and $L_\infty(B) = C(B)$. The space $L_p(M)$ is naturally embedded in $L_p(B)$ on putting $f(x,\xi) = f(x)$ for $f(x) \in L_p(M)$.

For $(x,\xi) \in B$, define $\gamma(x,\xi,t)$ to be a geodesic on M satisfying the conditions

$$\gamma(x,\xi;0) = x, \quad \frac{d}{dt}\gamma(x,\xi;t)\Big|_{t=0} = \xi,$$

where $\frac{d}{dt}\gamma(t)$ is the tangent vector to the curve $\gamma(t)$. Given a function $F(x,\xi) \in L_p(B)$, we put

$$F^t(x,\xi) := F\left(\gamma(x,\xi,t), \frac{d}{dt}\gamma(x,\xi;t)\right). \tag{1.1}$$

We define the kth difference of F with step t by

$$\widetilde{\Delta}_t^k F(x,\xi) := \sum_{j=0}^{k} (-1)^{k-j} \mathrm{C}_k^j F^{jt}(x,\xi),$$

where C_k^j are the binomial coefficients. The continuity modulus of order k of a function $F \in L_p(B)$ is defined by

$$\omega_k(F,\delta)_p := \sup_{0<t\le\delta} \big\|\widetilde{\Delta}_t^k F\big\|_{L_p(B)}. \tag{1.2}$$

In particular, we can consider $\omega_k(f,\delta)_p$ for $f \in L_p(M)$.

Theorem 1. *The following inequality is valid for every function $f \in L_p(M)$:*

$$E_N(f)_p \le c_1 \omega_k\left(f, \frac{\pi}{N}\right)_p, \quad N = 1, 2, \dots,$$

where c_1 is some constant independent of f and N.

We can strengthen Theorem 1 for differentiable functions. We say that a function $F \in L_p(B)$ is *differentiable* in $L_p(B)$ if for some $G(x,\xi) \in L_p(B)$

$$\left\|\frac{1}{t}(F^t - F) - G\right\|_{L_p(B)} \to 0 \quad \text{as} \quad t \to 0.$$

The function G is called the *derivative* of F and denoted by $G = F'$ as usual. A function F is r times differentiable in $L_p(B)$ if $F \in L_p(B)$ and the derivatives $F', F'', \dots, F^{(r)} \in L_p(B)$ exist. Here $F^{(k)} = (F^{(k-1)})'$. In particular, we can consider the rth derivative $f^{(r)}$ of a function $f(x) \in L_p(M)$, but observe that $f^{(r)}$ is now a function on B.

Theorem 2. *Suppose that a function $f(x)$ belongs to $L_p(M)$ and is r times differentiable in $L_p(B)$. Then*

$$E_N(f)_p \le \frac{c_2}{N^r} \omega_k\left(f^{(r)}, \frac{\pi}{N}\right)_p, \quad N = 1, 2, \dots,$$

where the constant c_2 is independent of f and N.

Henceforth $c_1, c_2, c_3, \dots$ are some positive constants independent of f and N.

Theorems 1 and 2 resemble the classical direct Jackson-type theorems of approximation theory (see [12]) and proving them is the main aim of this article. We note that for $M = S^n$ and $p = \infty$

Theorems 1 and 2 were proven by S. M. Nikol′skiĭ and P. I. Lizorkin [1] for odd values of n and by A. P. Terekhin [13] for even values of n. For $M = S^n$, $1 \le p < \infty$, Theorems 1 and 2 were in fact established by P. I. Lizorkin in [14], because as we easily see (from formula (2.17) of the present article) the smoothness modulus ${}^p\omega_k(f,\delta)_p$ in [14] coincides with $\omega_k(f,\delta)_p$ to within a factor. A brief exposition of the results of the article was given in [15]. We also indicate the article [16] which is a natural continuation of the present article. In [16], the author introduced the Nikol′skiĭ–Besov type function space $B^r_{p,q}(M)$ for an arbitrary CROSP M and provided their description in terms of best approximations by spherical polynomials.

§ 2. Auxiliary Results

Lemma 1. *If $F(x,\xi)$ is a continuous function on B then for every $s \in \mathbb{R}$*

$$\int_B F(x,\xi)\,dv = \int_B F^s(x,\xi)\,dv. \tag{2.1}$$

Proof. Given $u \in U$ and $F(x,\xi) \in C(B)$, put

$$(uF)(x,\xi) := F(ux, u_*\xi).$$

Verify that for $t \in \mathbb{R}$ and $u \in U$

$$u(F^t) = (uF)^t. \tag{2.2}$$

Indeed, since u is an isometry, we have

$$u\gamma(x,\xi,t) = \gamma(ux, u_*\xi; t), \quad u_*\frac{d}{dt}\gamma(x,\xi;t) = \frac{d}{dt}\gamma(ux,u_*\xi;t).$$

Then

$$(uF)^t = F\left(u\gamma(x,\xi,t), u_*\frac{d}{dt}\gamma(x,\xi;t)\right) = F\left(\gamma(ux,u_*\xi;t), \frac{d}{dt}\gamma(ux,u_*\xi;t)\right) = u(F^t).$$

It follows from (2.2) that the mapping

$$F \mapsto \int_B F^s(x,\xi)\,dv$$

defines a positive definite U-invariant functional on $C(B)$. Uniqueness of an invariant measure (to within a factor) implies that

$$\int_B F^s(x,\xi)\,dv = c\int_B F(x,\xi)\,dv \tag{2.3}$$

for some constant $c > 0$. Inserting $F \equiv 1$ in (2.3), we conclude that $c = 1$.

Since the set of continuous functions is dense in $L_p(B)$ ($1 \le p < \infty$), it follows from (2.1) that the mapping $F \mapsto F^s$ takes $L_p(B)$ into itself and

$$\|F^s\|_{L_p(B)} = \|F\|_{L_p(B)}. \tag{2.4}$$

Also, (2.4) is obvious for $p = \infty$. Moreover, observe that

$$(F^t)^s = F^{t+s} \quad \text{for all } t,s \in \mathbb{R}. \tag{2.5}$$

Using (2.5), we can easily infer that

$$\widetilde{\Delta}_t^{r+k}F = \widetilde{\Delta}_t^r\big(\widetilde{\Delta}_t^k F\big), \quad k,r = 1,2,\dots. \tag{2.6}$$

Lemma 2 (properties of a continuity modulus). *The following are valid:*

$$\omega_k(F,\delta)_p \le \omega_k(F,\lambda)_p \quad \text{for} \quad \delta \le \lambda; \tag{2.7}$$

$$\omega_k(F,l\delta)_p \le (l+1)^k \omega_k(F,\delta)_p, \tag{2.8}$$

where $l > 0$ is an arbitrary number.

Proof. Property (2.7) is obvious. Given a natural number l, we induct on k and use (2.5) to validate the equality

$$\widetilde{\Delta}_{lt}^k F^s(x,\xi) = \sum_{i_1=0}^{l-1} \cdots \sum_{i_k=0}^{l-1} \widetilde{\Delta}_t^k F^{s+i_1t+\dots+i_kt}(x,\xi). \tag{2.9}$$

Then (2.4) implies

$$\omega_k(F,l\delta)_p \le l^k \omega_k(F,\delta)_p, \tag{2.10}$$

and (2.10) and (2.7) yield (2.8) for every, possibly fractional, $l > 0$.

It is well known (see [11, Chapter IX, § 5]) that on every CROSP M all geodesics are closed and have the same length $2L$. The Riemannian metric on M is defined to within multiplication by a positive number. For simplicity, we normalize the Riemannian metric so that $L = \pi$.

Given $0 < t < \pi$, let $\alpha(t)$ be the total volume of a sphere of radius t in M. It is well known [10] that

$$\alpha(t) = c(\sin t/2)^a (\sin t)^b, \tag{2.11}$$

where c is a constant and the numbers a and b take the following values for various CROSPs:

$$\begin{aligned} &M = S^n: \quad a = 0, \quad b = n-1;\\ &M = P^n(\mathbb{C}): \quad a = n-2, \quad b = 1;\\ &M = P^n(\mathbb{H}): \quad a = n-4, \quad b = 3;\\ &M = P^{16}(Cay): \quad a = 8, \quad b = 7. \end{aligned}$$

Formula (2.11) enables us to extend $\alpha(t)$ to all values $t \in \mathbb{R}$.

If a continuous function $f(x)$ depends only on the distance from some fixed point $o \in M$ to x (i.e., $f(x) = \tilde{f}(|o,x|)$, where $|x,y|$ is the distance between x and y) then, as is well known (see, for instance, [10, Chapter 1, § 5]),

$$\int_M f(x)\,dx = \int_0^\pi \tilde{f}(t)\alpha(t)\,dt. \tag{2.12}$$

Let

$$\mathscr{D}_m(t) = \left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^{2s}, \quad m,s = 1,2,\dots,$$

be the Jackson kernel. Put

$$\theta_m = \int_M \mathscr{D}_m(|o,x|)\,dx = \int_0^\pi \mathscr{D}_m(t)\alpha(t)\,dt; \tag{2.13}$$

$$\mathscr{J}_m(t) = \frac{1}{\theta_m}\mathscr{D}_m(t). \tag{2.14}$$

Observe that

$$\int_0^\pi \mathscr{J}_m(t)\alpha(t)\,dt = 1, \tag{2.15}$$

and also

$$\int_{\delta}^{\pi} \mathscr{J}_m(t)\alpha(t)\,dt \to 0 \tag{2.16}$$

for each $\delta > 0$ as $m \to \infty$.

As above we let $S(x)$ stand for the unit sphere in the tangent space T_xM. Let $d\sigma_x(\xi)$ be the volume element on $S(x)$, and let σ be the total volume of this sphere.

The properties of invariant measures on homogeneous spaces (see [10, Chapter I, § 1, Proposition 1.13]) imply the following formula for some positive constant $A > 0$ and every function $F(x, \xi) \in L_1(B)$:

$$\int_{M}\Bigl(\int_{S(x)} F(x,\xi)\,d\sigma_x(\xi)\Bigr)\,dx = A\int_{B} F(x,\xi)\,dv(x,\xi). \tag{2.17}$$

We normalize the measure $dv(x, \xi)$ (which is defined to within a positive factor) so that the coefficient A in (2.17) equal σ. Then (2.17) implies

$$\int_{B} f(x)\,dv(x,\xi) = \int_{M} f(x)\,dx$$

for $f \in L_1(M)$ and

$$\|f\|_{L_p(M)} = \|f\|_{L_p(B)} \tag{2.18}$$

for $f(x) \in L_p(M)$.

In line with [2] we define the shift operator $S_tf(x)$ of a function $f(x)$ on M by step t by the formula

$$S_tf(x) := \frac{1}{\sigma}\int_{S(x)} f(\gamma(x,\xi;t))\,d\sigma_x(\xi), \quad t \in \mathbb{R}. \tag{2.19}$$

Obviously, the function $S_tf(x)$ is even and 2π-periodic in t.

Lemma 3. *If $f(x) \in L_p(M)$ then $S_tf(x) \in L_p(M)$ and*

$$\|S_tf\|_p \le \|f\|_p. \tag{2.20}$$

Proof. It follows from Hölder's inequality that

$$\begin{aligned} |S_tf(x)| &= \Bigl|\int_{S(x)} f(\gamma(x,\xi,t))\cdot 1\sigma^{-1}\,d\sigma_x(\xi)\Bigr| \\ &\le \Bigl(\int_{S(x)} |f(\gamma(x,\xi,t))|^p\sigma^{-1}\,d\sigma_x(\xi)\Bigr)^{1/p} \\ &= \sigma^{-1/p}\Bigl(\int_{S(x)} |f(\gamma(x,\xi,t))|^p\,d\sigma_x(\xi)\Bigr)^{1/p}. \end{aligned} \tag{2.21}$$

Using (2.17), we infer that

$$\|S_tf(x)\|_p^p \le \frac{1}{\sigma}\int_{M}\int_{S(x)} |f(\gamma(x,\xi,t))|^p\,d\sigma_x(\xi)\,dx = \int_{B} |f(\gamma(x,\xi,t))|^p\,dv(x,\xi) = \|f^t\|_{L_p(B)}^p.$$

From (2.4) and (2.18) we derive the equality

$$\|f^t\|^p_{L_p(B)} = \|f\|^p_{L_p(B)} = \|f\|^p_p$$

which proves (2.20). It is easy to see that (2.20) holds also for $p = \infty$.

Denote by $S_t^- f(x)$ the odd 2π-periodic extension in t of the function $S_t f(x)$ from the interval $[0, \pi]$. More exactly, we put

$$S_t^- f(x) := (-1)^{[t/\pi]} S_t f(x),$$

where $[t]$ is the integral part of t. Observe that the oddness of $S_t^- f(x)$ is violated at the points $t = \pi l$, $l \in \mathbb{Z}$, but this fact plays no role in what follows.

Given a function $\varphi(\tau)$ of the real variable τ, denote by $\Delta_t^k \varphi(\tau)$ the difference of order k with step t, i.e.,

$$\Delta_t^k \varphi(\tau) = \sum_{j=0}^{k} (-1)^{k-j} \mathrm{C}_k^j \varphi(\tau + jt).$$

Lemma 4. *Suppose that $f \in L_p(M)$ and m is an arbitrary natural number divisible by $2k!$. The function $\Phi_N(x)$ defined by*

$$\Phi_N(x) = f(x) - (-1)^k \int_0^\pi \big(\Delta_t^k S_\tau f(x)\big)\big|_{\tau=0} \mathscr{I}_m(t) \alpha(t)\, dt \tag{2.22}$$

if $M = S^n$ and n is odd and by

$$\Phi_N(x) = f(x) - (-1)^k \int_0^\pi \big(\Delta_t^k S_\tau^- f(x)\big)\big|_{\tau=0} \mathscr{I}_m(t) \alpha(t)\, dt \tag{2.23}$$

if $M \neq S^n$ or $M = S^n$ and n is even is a spherical polynomial on M of degree $N = s(m-1)$.

Proof. In the case of $M = S^n$, (2.22) and (2.23) were obtained in [2]. For the other CROSPs M, (2.23) was established in [17, Lemma 2.7].

Lemma 5. *The following inequality is valid for $F \in L_p(B)$ and every $t > 0$:*

$$\omega_k(F, t)_p \leq \frac{k}{2} \omega_{k+1}(F, t)_p + \frac{1}{2^k} \omega_k(F, 2t)_p. \tag{2.24}$$

Proof. The lemma is proven by the scheme of the proof of the classical Marchaud inequality (see [18, Theorem 3.3.1]). To prove (2.24), it suffices to establish the inequality

$$\|\widetilde{\Delta}_t^k F\|_{L_p(B)} \leq \frac{k}{2} \omega_{k+1}(F, t)_p + \frac{1}{2^k} \|\widetilde{\Delta}_{2t}^k F\|_{L_p(B)}. \tag{2.25}$$

From (2.9) we easily infer that

$$\widetilde{\Delta}_{2t}^k F(x, \xi) = \sum_{\nu=0}^{k} \mathrm{C}_k^\nu \widetilde{\Delta}_t^k F^{\nu t}(x, \xi). \tag{2.26}$$

Using (2.4), (2.6), and the relation $\sum_{\nu=1}^{k} \nu \mathrm{C}_k^\nu = k2^{k-1}$, we find that

$$\|\widetilde{\Delta}_{2t}^k F - 2^k \widetilde{\Delta}_t^k F\|_{L_p(B)} = \left\| \sum_{\nu=0}^{k} \mathrm{C}_k^\nu \widetilde{\Delta}_t^k (F^{\nu t} - F) \right\|_{L_p(B)}$$

$$= \left\| \sum_{\nu=0}^{k} \mathrm{C}_k^\nu \widetilde{\Delta}_t^k \sum_{j=0}^{\nu-1} \widetilde{\Delta}_t^1 F^{jt} \right\|_{L_p(B)} = \left\| \sum_{\nu=0}^{k} \mathrm{C}_k^\nu \sum_{j=0}^{\nu-1} \widetilde{\Delta}_t^{k+1} F^{jt} \right\|_{L_p(B)}$$

$$\leq \sum_{\nu=0}^{k} \mathrm{C}_k^\nu \nu \omega_{k+1}(F,t)_p = k2^{k-1}\omega_k(F,t)_p;$$

whence (2.25) ensues.

Since the function $F^s(x,\xi)$ is 2π-periodic in s, it follows that $\omega_k(F,t)_p = \omega_k(F,2\pi)_p$ for $t \geq 2\pi$. If we take $t = 2\pi$ in (2.23) then we see that

$$\left(2 - \frac{1}{2^{k-1}}\right)\omega_k(F,2\pi)_p \leq k\omega_{k+1}(F,2\pi)_p$$

and hence

$$\omega_k(F,2\pi)_p \leq k\omega_{k+1}(F,2\pi)_p. \tag{2.27}$$

Lemma 7. *The following asymptotic equality holds for a fixed s as $m \to \infty$:*

$$\theta_m \asymp m^{2s-a-b-1}, \quad 2s-a-b-1 > 0; \tag{2.28}$$

$$\int_0^\pi t^r \mathscr{J}_m(t)\alpha(t)\,dt \asymp m^{-r}, \quad 2s-a-b-1 > r > 0, \tag{2.29}$$

where $A_m \asymp B_m$ if $A_m, B_m > 0$ and $c_1 B_m \leq A_m \leq c_2 B_m$ for some constants $c_1, c_2 > 0$.

See a proof in [17, Lemma 2.5].

§ 3. Proof of the Theorems

Since Theorem 1 is a particular instance of Theorem 2 for $r = 0$, it suffices to prove Theorem 2. Fix some natural number s satisfying the inequality $2s - a - b - 1 > k + r$. Let m be an arbitrary natural number divisible by $2(k+r)!$ and a function $f(x)$ belong to the space $L_p(M)$ and satisfy the conditions of Theorem 2. Henceforth we deal with functions on the manifolds M and B. Denote by $\|F\|_p$ the $L_p(B)$ norm for $F \in L_p(B)$ or the $L_p(M)$ norm for $F \in L_p(M)$. It follows from (2.18) that this notation is correct.

Suppose that a function $F(x,\xi)$ belongs to $L_p(B)$ and is r times differentiable in $L_p(B)$. It follows from (2.5) that

$$(F')^u(x,\xi) = (F^u)'(x,\xi) \tag{3.1}$$

for every $u \in \mathbb{R}$; therefore, $u \mapsto F^u$ as a vector-function on $\mathbb{R}$ with values in $L_p(B)$ is r times differentiable and

$$\frac{d}{du}F^u = (F')^u. \tag{3.2}$$

Observe also that (3.1) implies the equality

$$\widetilde{\Delta}_t(F') = (\widetilde{\Delta}_t F)' \quad \text{for all } t \in \mathbb{R}. \tag{3.3}$$

Using (3.2) and (2.4), we obtain

$$\big\|\widetilde{\Delta}_t^1 F\big\|_p = \|F^t - F^0\|_p = \left\| \int_0^t (F')^\tau \, d\tau \right\|_p \le |t| \|F'\|_p.$$

Arguing similarly and using (3.3), we easily find that

$$\big\|\widetilde{\Delta}_t^{k+r} F\big\|_p \le |t|^r \big\|\widetilde{\Delta}_t^k F^{(r)}\big\|_p, \quad k, r = 0, 1, 2, \dots . \tag{3.4}$$

Finally, (3.4) implies the following useful relation:

$$\omega_{k+r}(F, \delta)_p \le \delta^r \omega_k(F^{(r)}, \delta)_p \quad \text{for all } \delta > 0. \tag{3.5}$$

(a) Consider the case in which $M = S^n$ and the number n is odd. It is obvious that

$$\big(\Delta_t^{k+r} S_\tau f(x)\big)\big|_{\tau=0} = \frac{1}{\sigma} \int_{S(x)} \big(\widetilde{\Delta}_t^{k+r} f\big)(x, \xi) \, d\sigma_x(\xi). \tag{3.6}$$

Then, from (3.4) we obtain

$$\begin{aligned} \big\|\Delta_t^{k+r} S_\tau f(x)\big|_{\tau=0}\|_p &\le \frac{1}{\sigma} \int_{S(x)} \big\|\widetilde{\Delta}_t^{k+l} f\big\|_p \, d\sigma_x(\xi) \\ &\le \frac{1}{\sigma} \int_{S(x)} |t|^r \big\|\widetilde{\Delta}_t^k f^{(r)}\big\|_p \, d\sigma_x(\xi) \le |t|^r \omega_k(f^{(r)}, t)_p. \end{aligned} \tag{3.7}$$

Put

$$\Phi_N(x) := f(x) - (-1)^{k+r} \int_0^\pi \big(\Delta_t^{k+r} S_\tau f(x)\big)\big|_{\tau=0} \mathscr{I}_m(t) \alpha(t) \, dt. \tag{3.8}$$

By Lemma 4, the functions $\Phi_N(x)$ are spherical polynomials on M of degree $N = s(m-1)$. It follows from (3.8) and (3.7) that

$$\|f - \Phi_N\|_p \le \int_0^\pi \big\|\Delta_t^{k+r} S_\tau f(x)\big|_{\tau=0}\big\|_p \mathscr{I}_m(t) \alpha(t) \, dt \le \int_0^\pi \omega_k(f^{(r)}, t)_p t^r \mathscr{I}_m(t) \alpha(t) \, dt. \tag{3.9}$$

Divide the interval $[0, \pi]$ into the two parts $[0, \pi/m]$ and $[\pi/m, \pi]$. Using the properties (2.7) and (2.8) of a continuity modulus over each part, we obtain the estimates

$$\omega_k(f^{(r)}, t)_p \le \omega_k\left(f^{(r)}, \frac{\pi}{m}\right)_p, \quad 0 \le t \le \frac{\pi}{m}, \tag{3.10}$$

$$\begin{aligned} \omega_k(f^{(r)}, t)_p = \omega_k\left(f^{(r)}, \frac{\pi}{m} \frac{mt}{\pi}\right)_p &\le \left(\frac{mt}{\pi} + 1\right)^k \omega_k\left(f^{(r)}, \frac{\pi}{m}\right)_p \\ &\le \left(\frac{2mt}{\pi}\right)^k \omega_k\left(f^{(r)}, \frac{\pi}{m}\right)_p, \quad \frac{\pi}{m} \le t \le \pi. \end{aligned} \tag{3.11}$$

Applying these estimates together with the upper estimate of (2.28), we infer that

$$\int_0^{\pi} \omega_k(f^{(r)},t)_p t^r \mathscr{I}_m(t)\alpha(t)\,dt = \int_0^{\pi/m} \omega_k(f^{(r)},t)_p t^r \mathscr{I}_m(t)\alpha(t)\,dt$$
$$+\int_{\pi/m}^{\pi} \omega_k(f^{(r)},t)_p t^r \mathscr{I}_m(t)\alpha(t)\,dt \le \omega_k\Big(f^{(r)},\frac{\pi}{m}\Big)_p \int_0^{\pi/m} t^r \mathscr{I}_m(t)\alpha(t)\,dt$$
$$+\Big(\frac{2m}{\pi}\Big)^k \omega_k\Big(f^{(r)},\frac{\pi}{m}\Big)_p \int_{\pi/m}^{\pi} t^{k+r} \mathscr{I}_m(t)\alpha(t)\,dt \le \frac{c_3}{m^r}\omega_k\Big(f^{(r)},\frac{\pi}{m}\Big)_p;$$

i.e., the following inequality is valid:

$$\int_0^{\pi} \omega_k(f^{(r)},t)_p t^r \mathscr{I}_m(t)\alpha(t)\,dt \le c_3 m^{-r}\omega_k(f^{(r)},\pi/m)_p. \tag{3.12}$$

It follows from (3.9) and (3.10) that

$$\|f-\Phi_N\|_p \le c_3 m^{-r}\omega_k\Big(f^{(r)},\frac{\pi}{m}\Big)_p$$
$$= c_3 m^{-r}\omega_k\Big(f^{(r)},\frac{\pi}{N}\frac{s(m-1)}{m}\Big)_p \le c_4 N^{-r}\omega_k\Big(f^{(r)},\frac{\pi}{N}\Big), \tag{3.13}$$

where $N = s(m-1)$.

Now, suppose that N is an arbitrary natural number. Recall that m is a number of the form $m = 2(k+r)!d$, $d = 1,2,\dots$. Without loss of generality we may assume that $N \ge 2(k+r)!$. Choose a number d so that

$$s(m-1) = s(2(k+r)!d-1) \le N \le s(2(k+r)!(d+1)-1).$$

Then

$$E_N(f)_p \le E_{s(m-1)}(f)_p \le \frac{c_4}{s(m-1)^r}\omega_k\Big(f^{(r)},\frac{\pi}{s(m-1)}\Big)_p \le \frac{c_5}{N^r}\omega_k\Big(f^{(r)},\frac{\pi}{N}\Big)_p,$$

which proves Theorem 2 in this case.

(b) Suppose that $M \ne S^n$ or $M = S^n$ and n is even. Define the function

$$G_t f(x) := S_t f(x) - f(x).$$

Let $G_t^- f(x)$ be the odd 2π-periodic extension in t of $G_t f(x)$ from the interval $[0,\pi]$. More exactly, for every $t \in \mathbb{R}$ put

$$G_t^- f(x) = S_t^- f(x) - (-1)^{[t/\pi]} f(x),$$

where $S_t^- f(x) = (-1)^{[t/\pi]} S_t f(x)$. The oddness of $G_t^- f(x)$ may be violated at the points $t = \pi(2l+1)$ and $l \in \mathbb{Z}$, but this is inessential for the sequel.

By Lemma 4, there is a spherical polynomial Φ_N of degree $N = s(m-1)$ such that

$$f(x) - \Phi_N(x) = (-1)^{k+r} \int_0^{\pi} \big(\Delta_t^{k+r} S_\tau^- f(x)\big)\big|_{\tau=0} \mathscr{I}_m(t)\alpha(t)\,dt$$

$$= (-1)^{k+r} \int_0^{\pi} \big(\Delta_t^{k+r} G_\tau^- f(x)\big)\big|_{\tau=0} \mathscr{I}_m(t)\alpha(t)\,dt$$

$$+(-1)^{k+r} f(x) \int_0^{\pi} \big(\Delta_t^{k+r} (-1)^{[\tau/\pi]}\big)\big|_{\tau=0} \mathscr{I}_m(t)\alpha(t)\,dt.$$

Put

$$d_m := (-1)^{k+r} \int_0^{\pi} \big(\Delta_t^{k+r} (-1)^{[\tau/\pi]}\big)\big|_{\tau=0} \mathscr{I}_m(t)\alpha(t)\,dt.$$

Then

$$f(x) - \Psi_N(x) = \frac{(-1)^{k+r}}{1-d_m} (-1)^{k+r} \int_0^{\pi} \big(\Delta_t^{k+r} G_\tau^- f(x)\big)\big|_{\tau=0} \mathscr{I}_m(t)\alpha(t)\,dt, \tag{3.14}$$

where $\Psi_N := (1-d_m)^{-1}\Phi_N$ is a spherical polynomial of degree $N = s(m-1)$, whenever $1 - d_m \neq 0$. Observe that for $0 < t < \pi/(k+r)$

$$\big(\Delta_t^{k+r} (-1)^{[\tau/\pi]}\big)\big|_{\tau=0} = 0;$$

consequently,

$$|d_m| = \left| \int_{\pi/(k+r)}^{\pi} \big(\Delta_t^{k+r} (-1)^{[\tau/\pi]}\big)\big|_{\tau=0} \mathscr{I}_m(t)\alpha(t)\,dt \right| \leq 2^{k+r} \int_{\pi/(k+r)}^{\pi} \mathscr{I}_m(t)\alpha(t)\,dt. \tag{3.15}$$

It follows from (3.15) and (2.16) that d_m vanishes as $m \to \infty$. Hence, the following inequality is valid for a sufficiently large m:

$$|1 - d_m| \geq 1/2. \tag{3.16}$$

Divide the interval of integration on the right-hand side of (3.14) into the parts $[0, \pi/(k+r)]$ and $[\pi/(k+r), \pi]$ and denote the integrals over these subintervals by I_1 and I_2. Estimate $\|I_1\|_p$ and $\|I_2\|_p$ from above. From the definition of $G_t^- f(x)$ we see that for $t \in [0, \pi/(k+r)]$

$$\big(\Delta_t^{k+r} G_\tau^- f(x)\big)\big|_{\tau=0} = \big(\Delta_t^{k+r} S_\tau f(x)\big)\big|_{\tau=0}.$$

Using (3.7) and (3.12) (which are obvious for every CROSP M), we obtain the following estimate:

$$\|I_1\|_p = \left\| \int_0^{\pi/(k+r)} \big(\Delta_t^{k+r} S_\tau f(x)\big)\big|_{\tau=0} \mathscr{I}_m(t)\alpha(t)\,dt \right\|_p$$

$$\leq \int_0^{\pi/(k+r)} \big\|\big(\Delta_t^{k+r} S_\tau f(x)\big)\big|_{\tau=0}\big\|_p \mathscr{I}_m(t)\alpha(t)\,dt \leq \int_0^{\pi/(k+r)} t^r \omega_k(f^{(r)}, t)_p \mathscr{I}_m(t)\alpha(t)\,dt$$

$$\leq \int_0^{\pi} \omega_k(f^{(r)}, t)_p t^r \mathscr{I}_m(t)\alpha(t)\,dt \leq c_6 m^{-r} \omega_k(f^{(r)}, \pi/m)_p;$$

consequently,

$$\|I_1\|_p \le c_6 m^{-r} \omega_k(f^{(r)}, \pi/m)_p. \tag{3.17}$$

Take $t \in [\pi/(k+r), \pi]$. Note that for every $t \in \mathbb{R}$

$$\|G_t^- f(x)\|_p \le \omega_1(f, t)_p \le \omega_1(f, 2\pi)_p.$$

Here we have used the fact that the function $G_t^- f(x)$ is 2π-periodic in t. Hence,

$$\big\|\big(\Delta_t^{k+r} G_\tau^- f(x)\big)\big|_{\tau=0}\|_p \le 2^{k+r}\omega_1(f, 2\pi)_p. \tag{3.18}$$

Using (2.27) and (3.5), we obtain

$$\omega_1(f, 2\pi)_p \le (k+r)!\omega_{k+r}(f, 2\pi)_p \le (2\pi)^r (k+r)!\omega_k(f^{(r)}, 2\pi). \tag{3.19}$$

Moreover, the property (2.8) of a continuity modulus implies that

$$\omega_k(f^{(r)}, 2\pi)_p \le (1+2m)^k \omega_k(f^{(r)}, \pi/m)_p. \tag{3.20}$$

Applying (3.18)–(3.20) and using the definition of $\mathscr{J}_m(t)$ (see (2.14)), we find that

$$\begin{aligned}\|I_2\|_p &= \left\| \int\limits_{\pi/(k+r)}^{\pi} \big(\Delta_t^{k+r} G_\tau^- f(x)\big)\big|_{\tau=0} \mathscr{J}_m(t)\alpha(t)\,dt \right\|_p \\ &\le 2^{k+r}\omega_1(f, 2\pi)_p \int\limits_{\pi/(k+r)}^{\pi} \mathscr{J}_m(t)\alpha(t)\,dt \\ \le \frac{(2\pi)^r 2^{k+r}(k+r)!\omega_k(f^{(r)}, 2\pi)}{\theta_m} &\int\limits_{\pi/(k+r)}^{\pi} \mathscr{D}_m(t)\alpha(t)\,dt \le c_7 \frac{\omega_k(f^{(r)}, \pi/m)_p (1+2m)^k}{m^{2s-a-b-1}}.\end{aligned}$$

Here we have used (2.28) and boundedness of $\mathscr{D}_m(t)$ from above on the interval $[\pi/(k+r), \pi]$. Observe also that

$$\frac{c_7(1+2m)^k}{m^{2s-a-b-1}} \le c_8 m^{-r},$$

since $2s - a - b - 1 > k + r$. Finally we obtain

$$\|I_2\|_p \le c_8 m^{-r} \omega_k(f^{(r)}, \pi/m)_p. \tag{3.21}$$

It follows from (3.14), (3.16), (3.17), and (3.21) that

$$\begin{aligned}\|f - \Psi_N\| &\le (1 - d_m)^{-1}(\|I_p\|_p + \|I_2\|_p) \\ &\le c_9 m^{-r} \omega_k(f^{(r)}, \pi/m)_p \le c_{10} N^{-r} \omega_k(f^{(r)}, \pi/N)\end{aligned} \tag{3.22}$$

for $N = s(m-1)$. Arguing as in the case (a), from (3.22) we now derive

$$E_N(f)_p \le c_{11} N^{-r} \omega_k(f^{(r)}, \pi/N)$$

for every natural N. This completes the proof of Theorem 2.

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