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**A Tauberian theorem related to approximation theory.**
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As the author has remarked [Bull. Amer. Math. Soc. **74** (1968), 500–504; MR0225074 (37 #671); "Smoothing and approximation of functions" (mimeographed), Matsci. Report No. 55, Inst. Math. Sci., Madras, 1967] his work "completes in some essential respects a program outlined by P. L. Butzer in a series of papers" [e.g., Arch. Rational Mech. Anal. **5** (1960), 390–415; MR0117480 (22 #8259)]. In the present paper he extends his one-dimensional comparison theory (cf. the articles cited above) to functions defined on Euclidean $n$-space $R^n$. For bounded continuous $f$, $f \in C$, and a bounded measure $\sigma$, $\sigma \in M$, the $\sigma$-deviation is defined by $D_\sigma(f;a) = \|\int f(t-au)\,d\sigma_u\|$, $a > 0$, $\|\cdot\|$ denoting the sup-norm. For instance, if $d\sigma = \chi\,du - d\delta$ with $\chi \in L^1$, $\int \chi(u)\,du = 1$ and $\delta$ the Dirac $\delta$-measure, then $D_\sigma(f;a) = \|[f(t-u)-f(t)]\{\rho^n\chi(\rho u)\}\,du\|$, $a = \rho^{-1}$. Thus $D_\sigma$ describes the norm-approximation of $f$ by the convolution integral $\int f(t-u)\{\rho^n\chi(\rho u)\}\,du$ as $\rho \to \infty$. Similarly, (higher) differences and thus Lipschitz conditions can be expressed in terms of $D_\sigma$.

Comparison theorems for deviations corresponding to different measures are proved under divisibility hypotheses of the measures to be compared. Setting $\sigma^\vee(v) = \int e^{-ivu}\,d\sigma_u$, $vu = \sum_1^n v_j u_j$, $\sigma \in M$ is said to divide $\lambda \in M$ globally [locally] if there is $\tau \in M$ such that $\lambda^\vee(v) = \tau^\vee(v)\sigma^\vee(v)$ for all $v \in R^n$ in a neighborhood of $v = 0$]. It is easy to prove that if $\sigma$ divides $\lambda$ globally, then $D_\lambda(f;a) \le A_1 D_\sigma(f;a)$. More delicate is the exciting local result: If $\sigma$ satisfies a "Tauberian condition", namely $\sigma^\vee(v)$ does not vanish identically on each half-ray through the origin, and $\sigma$ divides $\lambda$ locally, then, for any $f \in C$, there are constants $A_2, m, B_j$ (depending only on $\lambda$ and $\sigma$) and $b$ with $0 < b < 1$ (depending only on $\sigma$) such that $D_\lambda(f;a) \le A_2 \sum_{k=0}^\infty \sum_{j=1}^m D_\sigma(f;B_j b^k a)$. A third result is the following: Let $\sigma \in M$ satisfy the Tauberian condition and let $D_\sigma(f;a) = O(a^\alpha)$ for some $f \in C$, $\alpha > 0$ as $a \to 0+$. Suppose that for $\lambda \in M$ there exists a positive-homogeneous $\psi$ of degree $\beta > 0$ such that $\psi(v)$ and $\lambda^\vee(v)/\psi(v)$ are locally Fourier-Stieltjes transforms. Then $D_\lambda(f;a)$ is large $O$ of $a^\alpha$, $a^\alpha|\log a|$, or $a^\beta$ according as $\alpha$ is less than, equal to, or greater than $\beta$. Finally, the results are extended to $L^p$-space, $1 \le p < \infty$.

Comparison theory is a stimulating general framework which has to be supplemented by detailed discussion, particularly on saturation theory. The author has given remarkable one-dimensional applications in his articles cited above [see also the forthcoming second edition of his Madras lecture notes by van Nostrand]; however, the present paper only treats the theory itself. Though a point of explicit checking is involved, which may sometimes be non-trivial, the global result does cover the comparison of the approximation of $f$ by different convolution integrals. For the comparison of the approximation by a convolution integral with Lipschitz conditions, the local result may be used. However, the problem is to find comparable Lipschitz-type conditions for a given convolution integral, particularly if $\beta$ is fractional. Finally, the third result may be applied in the case of "non-optimal" approximation, but this does not cover the saturation case. Here the global result may be applied, if the saturation behaviour of a sufficiently general convolution integral is known. For this purpose one may use a recent contribution by E. Görlich (see *Abstract spaces and approximation*, edited by P. L. Butzer and B. Sz.-Nagy, Birkhäuser, Basel, in press) where distribution-theoretic methods are employed in the solution of concrete saturation problems.

Reviewed by *P. L. Butzer and R. J. Nessel*